3. Algebraic Lattices

The more I get, the more I want it seems ....
–King Oliver

In this section we want to focus our attention on the kind of closure operators and lattices that are associated with modern algebra. A closure operator Γ on a set X is said to be *algebraic* if for every $B \subseteq X$,

$$\Gamma(B) = \bigcup\{\Gamma(F) : F \text{ is a finite subset of } B\}.$$

Equivalently, Γ is algebraic if the right hand side RHS of the above expression is closed for every $B \subseteq X$, since $B \subseteq \text{RHS} \subseteq \Gamma(B)$ holds for any closure operator.

A closure rule is said to be *finitary* if it is a rule of the form $x \in S$ or the form $F \subseteq S \implies z \in S$ with F a finite set. Again the first form is a degenerate case of the second, taking $F = \emptyset$. It is not hard to see that a closure operator is algebraic if and only if it is determined by a set of finitary closure rules; see Theorem 3.2(1).

Let us catalogue some important examples of algebraic closure operators.

(1) Let $\mathcal{A}$ be any algebra with only finitary operations – for example, a group, ring, vector space, semilattice or lattice. The closure operator Sg on A such that $\text{Sg}(B)$ is the subalgebra of $\mathcal{A}$ generated by B is algebraic, because we have $a \in \text{Sg}(B)$ if and only if a can be expressed as a term $a = t(b_1, \dots, b_n)$ for some finite subset $\{b_1, \dots, b_n\} \subseteq B$, in which case $a \in \text{Sg}(\{b_1, \dots, b_n\})$. The corresponding complete lattice is of course the subalgebra lattice $\mathbf{Sub}\,\mathcal{A}$.

(2) Looking ahead a bit (to Chapter 5), the closure operator Cg on $A \times A$ such that $\text{Cg}(B)$ is the congruence on $\mathcal{A}$ generated by the set of pairs B is also algebraic. The corresponding complete lattice is the congruence lattice $\mathbf{Con}\,\mathcal{A}$. For groups this is isomorphic to the normal subgroup lattice; for rings, it is isomorphic to the lattice of ideals.

(3) For ordered sets, the order ideal operator O is algebraic. In fact we have

$$O(B) = \bigcup\{O(\{b\}) : b \in B\}$$

for all $B \subseteq P$.

(4) Let $\mathcal{S} = (S; \vee)$ be a join semilattice. A subset J of S is called an *ideal* if

(i) $x, y \in J$ implies $x \vee y \in J$,
(ii) $z \leq x \in J$ implies $z \in J$.

Since ideals are defined by closure rules, the intersection of a set of ideals of $\mathcal{S}$ is again one. Since they are finitary closure rules, the lattice of ideals is algebraic. The closure operator I on S such that $I(B)$ is the ideal of $\mathcal{S}$ generated by B is given by

$$I(B) = \{x \in S : x \leq \bigvee F \text{ for some finite } F \subseteq B\}.$$

The ideal lattice of a join semilattice is denoted by $\mathcal{I}(\mathcal{S})$. Again, ideals of the form $\downarrow x$ are called *principal.* Note that the empty set is an ideal of $(S; \vee)$.

(5) If $\mathcal{S} = (S; \vee, 0)$ is a semilattice with a least element 0, *regarded as a constant of the algebra,* then an ideal J must also satisfy

(iii) $0 \in J$

so that $\{0\}$, rather than the empty set, is the least ideal. The crucial factor is not that there is a least element, but that it is considered to be a constant in the type of the algebra.

(6) An *ideal* of a lattice is defined in the same way as (4), since every lattice is in particular a join semilattice. The ideal lattice of a lattice $\mathcal{L}$ is likewise denoted by $\mathcal{I}(\mathcal{L})$. The dual of an ideal in a lattice is called a *filter.* (See Exercise 4.)

On the other hand, it is not hard to see that the closure operators associated with the closed sets of a topological space are usually *not* algebraic, since the closure depends on infinite sequences. The closure operator M associated with the MacNeille completion is not in general algebraic, as is seen by considering the ordered set $\mathcal{P}$ consisting of an infinite set X and all of its finite subsets, ordered by set containment. This ordered set is already a complete lattice, and hence its own MacNeille completion. For any subset $Y \subseteq X$, let $\widehat{Y} = \{S \in P : S \subseteq Y\}$. If Y is an infinite proper subset of X, then $M(\widehat{Y}) = \widehat{X}$. On the other hand, for any finite collection $F = \{Z_1, \dots, Z_k\}$ of finite subsets, $M(F) = Z_1 \cup \cdots \cup Z_k$. Thus for an infinite proper subset $Y \subset X$ we have $M(\widehat{Y}) = \widehat{X} \supset \widehat{Y} = \bigcup\{M(F) : F \text{ is a finite subset of } \widehat{Y}\}$.

A subset S of an ordered set $\mathcal{P}$ is said to be *up-directed* if for every x, $y \in S$ there exists $z \in S$ with $x \leq z$ and $y \leq z$. Thus every chain, or more generally every join semilattice, forms an up-directed set.

The following observation can be useful.

Theorem 3.1. *Let Γ be a closure operator on a set X. The following are equivalent.*

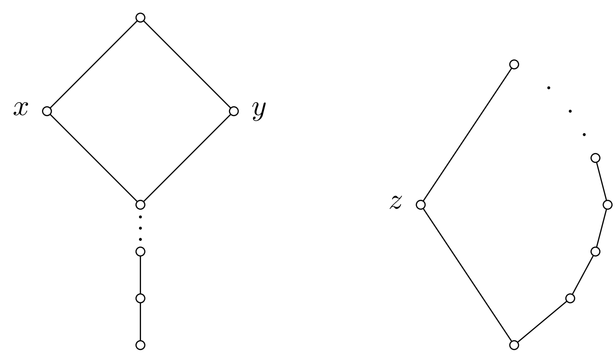
(1) *Γ is an algebraic closure operator.*
(2) *The union of any up-directed set of Γ-closed sets is Γ-closed.*
(3) *The union of any chain of Γ-closed sets is Γ-closed.*

The equivalence of (1) and (2), and the implication from (2) to (3), are straightforward. The implication from (3) to (1) can be done by induction, mimicking the proof of the corresponding step in Theorem 3.8. This is exercise 13.

We need to translate these ideas into the language of lattices. Let $\mathcal{L}$ be a complete lattice. An element $x \in L$ is *compact* if whenever $x \leq \bigvee A$, then there exists a finite subset $F \subseteq A$ such that $x \leq \bigvee F$. The set of all compact elements of $\mathcal{L}$ is denoted by $\mathcal{L}^c$. An elementary argument shows that $\mathcal{L}^c$ is closed under finite joins and contains 0, so it is a join semilattice with a least element. However, $\mathcal{L}^c$ is usually not closed under meets; see Figure 3.1(a), wherein x and y are compact but $x \wedge y$ is not.

A lattice $\mathcal{L}$ is said to be *algebraic*, or *compactly generated*, if it is complete and $\mathcal{L}^c$ is join dense in $\mathcal{L}$, i.e., $x = \bigvee(\downarrow x \ \cap \ L^c)$ for every $x \in L$. Clearly every finite lattice is algebraic. More generally, every element of a complete lattice $\mathcal{L}$ is compact, i.e., $\mathcal{L} = \mathcal{L}^c$ if and only if $\mathcal{L}$ satisfies the ACC.

For an example of a complete lattice that is not algebraic, let $\mathcal{K}$ denote the interval $[0, 1]$ in the real numbers with the usual order. Then $\mathcal{K}^c = \{0\}$, so $\mathcal{K}$ is not algebraic. The non-algebraic lattice in Figure 3.1(b) is another good example to keep in mind. The element z is not compact, and hence in this case not a join of compact elements.



(*a*) (*b*)

FIGURE 3.1

Historically, the role of algebraic closure operators arose in Birkhoff and Frink [3], with the modern definition of a compactly generated lattice following closely in Nachbin [12].

Theorem 3.2. (*1*) *A closure operator* Γ *is algebraic if and only if* $\Gamma = \Gamma_\Sigma$ *for some set* Σ *of finitary closure rules.*

(*2*) *Let* Γ *be an algebraic closure operator on a set* X. *Then* $\mathcal{C}_\Gamma$ *is an algebraic lattice whose compact elements are* $\{\Gamma(F) : F$ *is a finite subset of* $X\}$.

Proof. If Γ is an algebraic closure operator on a set X, then a set $S \subseteq X$ is closed if and only if $\Gamma(F) \subseteq S$ for every finite subset $F \subseteq S$. Thus the collection of all rules $F \subseteq S \implies z \in S$, with F a finite subset of X and $z \in \Gamma(F)$, determines closure for Γ.[1] Conversely, if Σ is a collection of finitary closure rules, then $z \in \Gamma_\Sigma(B)$ if and only if $z \in \Gamma_\Sigma(F)$ for some finite $F \subseteq B$, making Γ_Σ algebraic.

For (2), let us first observe that for any closure operator Γ on X, and for any collection of subsets A_i of X, we have $\Gamma(\bigcup A_i) = \bigvee \Gamma(A_i)$ where the join is computed in the lattice $\mathcal{C}_\Gamma$. The inclusion $\Gamma(\bigcup A_i) \supseteq \bigvee \Gamma(A_i)$ is immediate, while $\bigcup A_i \subseteq \bigcup \Gamma(A_i) \subseteq \bigvee \Gamma(A_i)$ implies $\Gamma(\bigcup A_i) \subseteq \Gamma(\bigvee \Gamma(A_i)) = \bigvee \Gamma(A_i)$.

In particular, for all $B \subseteq X$,

$$\Gamma(B) = \bigvee\{\Gamma(F) : F \text{ is a finite subset of } B\}.$$

Thus $\mathcal{C}_\Gamma$ will be an algebraic lattice, when Γ is an algebraic closure operator, if we can show that the closures of finite sets are compact.

So assume that Γ is algebraic, and let F be a finite subset of X. If $\Gamma(F) \leq \bigvee A_i$ in $\mathcal{C}_\Gamma$, then

$$F \subseteq \bigvee A_i = \Gamma(\bigcup A_i) = \bigcup\{\Gamma(G) : G \text{ finite } \subseteq \bigcup A_i\}.$$

Consequently each $x \in F$ is in some $\Gamma(G_x)$, where G_x is in turn contained in the union of finitely many A_i's. Therefore $\Gamma(F) \subseteq \Gamma(\bigcup_{x \in F} \Gamma(G_x)) \subseteq \bigvee_{j \in J} A_j$ for some finite subset $J \subseteq I$. We conclude that $\Gamma(F)$ is compact in $\mathcal{C}_\Gamma$.

Conversely, let C be compact in $\mathcal{C}_\Gamma$. Since C is closed and Γ is algebraic, $C = \bigvee\{\Gamma(F) : F \text{ finite } \subseteq C\}$. Since C is compact, there exist finitely many finite subsets of C, say $F_1, \dots, F_n$, such that $C = \Gamma(F_1) \vee \dots \vee \Gamma(F_n) = \Gamma(F_1 \cup \dots \cup F_n)$. Thus C is the closure of a finite set. $\square$

Thus in a subalgebra lattice **Sub** $\mathcal{A}$, the compact elements are the finitely generated subalgebras. In a congruence lattice **Con** $\mathcal{A}$, the compact elements are the finitely generated congruences.

It is not true that $\mathcal{C}_\Gamma$ being algebraic implies that Γ is algebraic. For example, let X be the disjoint union of a one element set $\{b\}$ and an infinite set Y, and let Γ be the closure operator on X such that $\Gamma(A) = A$ if A is a proper subset of Y, $\Gamma(Y) = X$ and $\Gamma(B) = X$ if $b \in B$. The Γ-closed sets are all proper subsets of Y, and $X = Y \cup \{b\}$ itself. Thus $\mathcal{C}_\Gamma$ is isomorphic to the lattice of all subsets of Y. But $b \in \Gamma(Y)$, while b is not in the union of the closures of the finite subsets of Y, so Γ is not algebraic.

The following theorem includes a representation of any algebraic lattice as the lattice of closed sets of an algebraic closure operator.

Theorem 3.3. *If $\mathcal{S} = \langle S; \vee, 0\rangle$ is a join semilattice with 0, then the ideal lattice $\mathcal{I}(\mathcal{S})$ is algebraic. The compact elements of $\mathcal{I}(\mathcal{S})$ are the principal ideals $\downarrow x$ with*

[1] In general there are also valid infinitary closure rules for Γ, but for an algebraic closure operator these are redundant.

$x \in S$. Conversely, if $\mathcal{L}$ is an algebraic lattice, then $\mathcal{L}^c$ is a join semilattice with 0, and $\mathcal{L} \cong \mathcal{I}(\mathcal{L}^c)$.

Proof. Let $\mathcal{S}$ be a join semilattice with 0. I is an algebraic closure operator, so $\mathcal{I}(\mathcal{S})$ is an algebraic lattice. If $F \subseteq S$ is finite, then $I(F) = (\bigvee F)/0$, so compact ideals are principal.

Now let $\mathcal{L}$ be an algebraic lattice. There are two natural maps: $f : \mathcal{L} \to \mathcal{I}(\mathcal{L}^c)$ by $f(x) = \downarrow x \cap L^c$, and $g : \mathcal{I}(\mathcal{L}^c) \to \mathcal{L}$ by $g(J) = \bigvee J$. Both maps are clearly order preserving, and they are mutually inverse: $fg(J) = (\bigvee J)/0 \cap L^c = J$ by the definition of compactness, and $gf(x) = \bigvee(\downarrow x \cap L^c) = x$ by the definition of algebraic. Hence they are both isomorphisms, and $\mathcal{L} \cong \mathcal{I}(\mathcal{L}^c)$. □

Let us digress for a moment into universal algebra. A classic result of Birkhoff and Frink gives a concrete representation of algebraic closure operators [3].

Theorem 3.4. *Let Γ be an algebraic closure operator on a set X. Then there is an algebra $\mathcal{A}$ on the set X such that the subalgebras of $\mathcal{A}$ are precisely the closed sets of Γ.*

Corollary. *Every algebraic lattice is isomorphic to the lattice of all subalgebras of an algebra.*

Proof. An algebra in general is described by $\mathcal{A} = \langle A; F, C \rangle$ where A is a set, $F = \{f_i : i \in I\}$ a collection of operations on A (so $f_i : A^{n_i} \to A$), and C is a set of constants in A. The third section of Appendix 1 reviews the basic definitions of universal algebra.

The carrier set for our algebra must of course be X. For each nonempty finite set $G \subseteq X$ and element $x \in \Gamma(G)$, we have an operation $f_{G,x} : X^{|G|} \to X$ given by

$$f_{G,x}(a_1, \ldots, a_n) = \begin{cases} x & \text{if } \{a_1, \ldots, a_n\} = G \\ a_1 & \text{otherwise.} \end{cases}$$

Our constants are $C = \Gamma(\emptyset)$, the elements of the least closed set (which may be empty).

Note that since Γ is algebraic, a set $B \subseteq X$ is closed if and only if $\Gamma(G) \subseteq B$ for every finite $G \subseteq B$. Using this, it is very easy to check that the subalgebras of $\mathcal{A}$ are precisely the closed sets of $\mathcal{C}_\Gamma$. □

A direct proof of the Corollary is also of interest. Given an algebraic lattice $\mathcal{L}$, define an algebra $\mathcal{B}$ as follows. The carrier set of $\mathcal{B}$ is the set of compact elements $\mathcal{L}^c$, the join $\vee$ is one operation, and the least element 0 is a constant. For each $a \in \mathcal{L}^c$, define a unary operation f_a by

$$f_a(x) = \begin{cases} a & \text{if } a \leq x \\ x & \text{otherwise.} \end{cases}$$

The construction insures that the subalgebras of $\mathcal{B}$ are exactly the ideals of $\mathcal{L}^c$, whence $\mathbf{Sub}\,\mathcal{B} = \mathcal{I}(\mathcal{L}^c) \cong \mathcal{L}$.

However, the algebra constructed in the proof of Theorem 3.4 will have $|X|$ operations when X is infinite, and that in the second construction has $|\mathcal{L}^c| + 1$ operations. Having lots of operations is not necessarily a bad thing: vector spaces are respectable algebras, and a vector space over a field F has basic operations $f_r : \mathcal{V} \to \mathcal{V}$ where $f_r(v) = rv$ for every $r \in F$. Nonetheless, we like algebras to have few operations, like groups and lattices. A theorem due to Bill Hanf tells us when we can get by with a small number of operations.[2]

Theorem 3.5. *For any nontrivial algebraic lattice $\mathcal{L}$ the following conditions are equivalent.*

(1) *Each compact element of $\mathcal{L}$ contains only countably many compact elements.*
(2) *There exists an algebra $\mathcal{A}$ with only countably many operations and constants such that $\mathcal{L}$ is isomorphic to the subalgebra lattice of $\mathcal{A}$.*
(3) *There exists an algebra $\mathcal{B}$ with one binary operation (and no constants) such that $\mathcal{L}$ is isomorphic to the subalgebra lattice of $\mathcal{B}$.*

Proof. Of course (3) implies (2).

In general, if an algebra $\mathcal{A}$ has κ basic operations, λ constants and γ generators, then it is a homomorphic image of the absolutely free algebra $W(X)$ generated by a set X with $|X| = \gamma$ and the same κ operation symbols and λ constants. This free algebra can be constructed recursively: with F denoting the set of operation symbols and C the constant symbols, let $W_0 = X \cup C$ and $W_{k+1} = \{f(t_1, \ldots, t_n) : f \in F,\ t_1, \ldots, t_n \in W_k\}$. Then $W(X) = \bigcup_{k \geq 0} W_k$. Using this, it is easy to count that $|W(X)| \leq \max(\gamma, \kappa, \lambda, \aleph_0)$, with equality unless $\kappa = 0$. Moreover, $|\mathcal{A}| \leq |W(X)|$ since the former is a homomorphic image of the latter.

In particular then, if $\mathcal{C}$ is compact (i.e., finitely generated) in $\mathbf{Sub}\,\mathcal{A}$, and $\mathcal{A}$ has only countably many basic operations and constants, then $|\mathcal{C}| \leq \aleph_0$. Therefore $\mathcal{C}$ has only countably many finite subsets, and so there are only countably many finitely generated subalgebras $\mathcal{D}$ contained in $\mathcal{C}$. Thus (2) implies (1).

To show (1) implies (3), let $\mathcal{L}$ be a nontrivial algebraic lattice such that for each $x \in L^c$, $|{\downarrow} x \cap L^c| \leq \aleph_0$. We will construct an algebra $\mathcal{B}$ whose universe is $L^c - \{0\}$, with one binary operation $*$, whose subalgebras are precisely the ideals of $\mathcal{L}^c$ with 0 removed. This makes $\mathbf{Sub}\,\mathcal{B} \cong \mathcal{I}(\mathcal{L}^c) \cong \mathcal{L}$, as desired.

For each $c \in L^c - \{0\}$, we make a sequence $\langle c_i \rangle_{i \in \omega}$ as follows. If $2 \leq |{\downarrow} c \cap L^c| = n + 1 < \infty$, arrange ${\downarrow} c \cap L^c - \{0\}$ into a cyclically repeating sequence: $c_i = c_j$ iff $i \equiv j \bmod n$. If ${\downarrow} c \cap L^c$ is infinite (and hence countable), arrange ${\downarrow} c \cap L^c - \{0\}$ into a non-repeating sequence $\langle c_i \rangle$. In both cases start the sequence with $c_0 = c$.

[2]This result is unpublished but well known.

Define the binary operation $*$ for $c, d \in L^c - \{0\}$ by

$$c * d = c \vee d \text{ if } c \text{ and } d \text{ are incomparable,}$$
$$c * d = d * c = c_{i+1} \text{ if } d = c_i \leq c.$$

You can now check that $*$ is well defined, and that the algebra $\mathcal{B} = \langle L^c; * \rangle$ has exactly the sets of nonzero elements of ideals of $\mathcal{L}^c$ as subalgebras. □

The situation with respect to congruence lattices is considerably more complicated. Nonetheless, the basic facts are the same: George Grätzer and E. T. Schmidt proved that every algebraic lattice is isomorphic to the congruence lattice of some algebra [10], and Bill Lampe showed that uncountably many operations may be required [9].

Ralph Freese and Walter Taylor modified Lampe's original example to obtain a very natural one. Let $\mathcal{V}$ be a vector space of countably infinite dimension over a field F with $|F| = \kappa > \aleph_0$. Let $\mathcal{L}$ be the congruence lattice **Con** $\mathcal{V}$, which for vector spaces is isomorphic to the subspace lattice **Sub** $\mathcal{V}$ (since homomorphisms on vector spaces are linear transformations, and any subspace of $\mathcal{V}$ is the kernel of a linear transformation). The representation we have just given for $\mathcal{L}$ involves κ operations f_r $(r \in F)$. In fact, one can show that *any* algebra $\mathcal{A}$ with **Con** $\mathcal{A} \cong \mathcal{L}$ must have at least κ operations.

We now turn our attention to the structure of algebraic lattices. The lattice $\mathcal{L}$ is said to be *weakly atomic* if whenever $a > b$ in $\mathcal{L}$, there exist elements $u, v \in L$ such that $a \geq u \succ v \geq b$.

Theorem 3.6. *Every algebraic lattice is weakly atomic.*

Proof. Let $a > b$ in an algebraic lattice $\mathcal{L}$. Then there is a compact element $c \in L^c$ with $c \leq a$ and $c \nleq b$. Let $\mathcal{P} = \{x \in a/b : c \nleq x\}$. Note $b \in P$, and since c is compact the join of a chain in $\mathcal{P}$ is again in $\mathcal{P}$. Hence by Zorn's Lemma, $\mathcal{P}$ contains a maximal element v, and the element $u = c \vee v$ covers v. Thus $b \leq v \prec u \leq a$. □

A lattice $\mathcal{L}$ is said to be *upper continuous* if whenever D is an up-directed set having a least upper bound $\bigvee D$, then for any element $a \in L$, the join $\bigvee_{d \in D}(a \wedge d)$ exists, and

$$a \wedge \bigvee D = \bigvee_{d \in D} (a \wedge d).$$

The property of being *lower continuous* is defined dually.

Upper continuity applies most naturally to complete lattices, but it also arises in non-complete lattices. See Section II.2 of Freese, Ježek and Nation [8] and for a more general setting Adaricheva, Gorbunov and Semenova [1].

Theorem 3.7. *Every algebraic lattice is upper continuous.*

Proof. Let $\mathcal{L}$ be algebraic and D an up-directed subset of $\mathcal{L}$. Of course $\bigvee_{d \in D}(a \wedge d) \leq a \wedge \bigvee D$. Let $r = a \wedge \bigvee D$. For each compact element $c \in {\downarrow} r \cap L^c$, we have $c \leq a$

and $c \leq \bigvee D$. The compactness of c implies $c \leq \bigvee F$ for some finite subset F of D. By the up-directed property, we can choose $e \in D$ with $\bigvee F \leq e$, so that $c \leq a \wedge e$. Consequently,

$$r = \bigvee({\downarrow} r \cap L^c) \leq \bigvee_{d \in D} (a \wedge d),$$

and equality follows. □

Two alternative forms of join continuity are often useful.

Theorem 3.8. *For a complete lattice $\mathcal{L}$, the following are equivalent.*

(1) *$\mathcal{L}$ is upper continuous.*
(2) *For every $a \in L$ and chain $C \subseteq L$, we have $a \wedge \bigvee C = \bigvee_{c \in C} a \wedge c$.*
(3) *For every $a \in L$ and $S \subseteq L$,*

$$a \wedge \bigvee S = \bigvee_{F \text{ finite } \subseteq S} (a \wedge \bigvee F).$$

Proof. It is straightforward that (3) implies (1) implies (2), so this is left to the reader. Following Crawley and Dilworth [7], we will show that (2) implies (3) by induction on $|S|$. Property (3) is trivial if $|S|$ is finite, so assume it is infinite, and let λ be the least ordinal with $|S| = |\lambda|$. Arrange the elements of S into a sequence $\langle x_\xi : \xi < \lambda \rangle$. Put $S_\xi = \{x_\nu : \nu < \xi\}$. Then $|S_\xi| < |S|$ for each $\xi < \lambda$, and the elements of the form $\bigvee S_\xi$ are a chain in $\mathcal{L}$. Thus, using (1), we can calculate

$$\begin{aligned} a \wedge \bigvee S &= a \wedge \bigvee_{\xi < \lambda} \bigvee S_\xi \\ &= \bigvee_{\xi < \lambda} (a \wedge \bigvee S_\xi) \\ &= \bigvee_{\xi < \lambda} (\bigvee_{F \text{ finite } \subseteq S_\xi} (a \wedge \bigvee F)) \\ &= \bigvee_{F \text{ finite } \subseteq S} (a \wedge \bigvee F), \end{aligned}$$

as desired. □

An element $a \in L$ is called an *atom* if $a \succ 0$, and a *coatom* if $1 \succ a$. Theorem 3.8 shows that every atom in an upper continuous lattice is compact. More generally, if ${\downarrow} a$ satisfies the ACC in an upper continuous lattice, then a is compact.

We know that every element x in an algebraic lattice can be expressed as the join of ${\downarrow} x \cap L^c$ (by definition). It turns out to be at least as important to know how x can be expressed as a meet of other elements. We say that an element q in a complete lattice $\mathcal{L}$ is *completely meet irreducible* if, for every subset S of L, $q = \bigwedge S$ implies $q \in S$. These are of course the elements that cannot be expressed as the proper meet of other elements. Let $M^*(\mathcal{L})$ denote the set of all completely meet irreducible elements of $\mathcal{L}$. Note that $1 \notin M^*(\mathcal{L})$ (since $\bigwedge \emptyset = 1$ and $1 \notin \emptyset$).

Theorem 3.9. *Let $q \in L$ where $\mathcal{L}$ is a complete lattice. The following are equivalent.*

(1) $q \in M^*(\mathcal{L})$.
(2) $\bigwedge\{x \in L : x > q\} > q$.
(3) *There exists $q^* \in L$ such that $q^* \succ q$ and for all $x \in L$, $x > q$ implies $x \geq q^*$.*

The connection between (2) and (3) is of course $q^* = \bigwedge\{x \in L : x > q\}$. In a finite lattice, $q \in M^*(\mathcal{L})$ iff there is a unique element q^* covering q, but in general we need the stronger property (3).

A *decomposition* of an element $a \in L$ is a representation $a = \bigwedge Q$ where Q is a set of completely meet irreducible elements of $\mathcal{L}$. An element in an arbitrary lattice may have any number of decompositions, including none. A theorem due to Garrett Birkhoff says that every element in an algebraic lattice has at least one decomposition [2].

Theorem 3.10. *If $\mathcal{L}$ is an algebraic lattice, then $M^*(\mathcal{L})$ is meet dense in $\mathcal{L}$. Thus for every $x \in L$, $x = \bigwedge(\uparrow x \cap M^*(\mathcal{L}))$.*

Proof. Let $m = \bigwedge(\uparrow x \cap M^*(\mathcal{L}))$, and suppose $x < m$. Then there exists a $c \in L^c$ with $c \leq m$ and $c \not\leq x$. Since c is compact, we can use Zorn's Lemma to find an element q that is maximal with respect to $q \geq x$, $q \not\geq c$. For any $y \in L$, $y > q$ implies $y \geq q \vee c$, so q is completely meet irreducible with $q^* = q \vee c$. Then $q \in \uparrow x \cap M^*(\mathcal{L})$ implies $q \geq m \geq c$, a contradiction. Hence $x = m$. $\square$

Note that the preceding argument is closely akin to that of Theorem 3.6.

It is rare for an element in an algebraic lattice to have a unique decomposition. A somewhat weaker property is for an element to have an *irredundant* decomposition, meaning $a = \bigwedge Q$ but $a < \bigwedge(Q - \{q\})$ for all $q \in Q$, where Q is a set of completely meet irreducible elements. An element in an algebraic lattice need not have an irredundant decomposition either. Let $\mathcal{L}$ be the lattice consisting of the empty set and all cofinite subsets of an infinite set X, ordered by set inclusion. This satisfies the ACC so it is algebraic. The completely meet irreducible elements of $\mathcal{L}$ are its coatoms, the complements of one element subsets of X. The meet of any infinite collection of coatoms is 0 (the empty set), but no such decomposition is irredundant. Clearly also these are the only decompositions of 0, so 0 has no irredundant decomposition.

A lattice is *strongly atomic* if $a > b$ in $\mathcal{L}$ implies there exists $u \in L$ such that $a \geq u \succ b$. A beautiful result of Peter Crawley guarantees the existence of irredundant decompositions in strongly atomic algebraic lattices [5].

Theorem 3.11. *If an algebraic lattice $\mathcal{L}$ is strongly atomic, then every element of $\mathcal{L}$ has an irredundant decomposition.*

If $\mathcal{L}$ is also distributive, we obtain the uniqueness of irredundant decompositions.

Theorem 3.12. *If $\mathcal{L}$ is a distributive, strongly atomic, algebraic lattice, then every element of $\mathcal{L}$ has a unique irredundant decomposition.*

The finite case of Theorem 3.12 is the dual of Theorem 8.6(c), which we will prove later.

The theory of decompositions was studied extensively by Dilworth and Crawley, and their book [7] contains most of the principal results. For refinements since then, see the section on decompositions in The Dilworth Theorems [4], Semenova [13], and the references in [13] to papers of Erné, Gorbunov, Richter, Semenova and Walendziak.

Exercises for Chapter 3

1. Prove that an upper continuous distributive lattice satisfies the infinite distributive law $a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I}(a \wedge b_i)$.

2. Describe the complete sublattices of the real numbers $\Re$ that are algebraic.

3. Show that the natural map from a lattice to its ideal lattice, $\varphi : \mathcal{L} \to \mathcal{I}(\mathcal{L})$ by $\varphi(x) = \downarrow x$, is a lattice embedding. Show that $(\mathcal{I}(\mathcal{L}), \varphi)$ is a join dense completion of $\mathcal{L}$, and that it may differ from the MacNeille completion.

4. Recall that a *filter* is a dual ideal. The filter lattice $\mathcal{F}(\mathcal{L})$ of a lattice $\mathcal{L}$ is ordered by reverse set inclusion: $F \leq G$ iff $F \supseteq G$. Prove that $\mathcal{L}$ is naturally embedded in $\mathcal{F}(\mathcal{L})$, and that $\mathcal{F}(\mathcal{L})$ is dually compactly generated.

5. Prove that every element of a complete lattice $\mathcal{L}$ is compact if and only if $\mathcal{L}$ satisfies the ACC. (Cf. Exercise 2.2.)

6. A nonempty subset S of a complete lattice $\mathcal{L}$ is a *complete sublattice* if $\bigvee A \in S$ and $\bigwedge A \in S$ for every nonempty subset $A \subseteq S$. Prove that a complete sublattice of an algebraic lattice is algebraic.

7. (a) Represent the lattices $\mathcal{M}_3$ and $\mathcal{N}_5$ as $\mathbf{Sub}\,\mathcal{A}$ for a finite algebra $\mathcal{A}$.

(b) Show that $\mathcal{M}_3 \cong \mathbf{Sub}\,\mathcal{G}$ for a (finite) group $\mathcal{G}$, but that $\mathcal{N}_5$ cannot be so represented.

(c) For which values of n is $\mathcal{M}_n$ the lattice of subgroups of an abelian group $\mathcal{G}$?

8. A closure rule is *nullary* if it has the form $x \in S$, and *unary* if it is of the form $y \in S \implies z \in S$. Prove that if Σ is a collection of nullary and unary closure rules, then nonempty unions of closed sets are closed, and hence the lattice of closed sets $\mathcal{C}_\Sigma$ is distributive. Conclude that the subalgebra lattice of an algebra with only constants and unary operations is distributive.

9. Let $\mathcal{L}$ be a complete lattice, J a join dense subset of L and M a meet dense subset of L. Define maps $\sigma : \mathfrak{P}(J) \to \mathfrak{P}(M)$ and $\pi : \mathfrak{P}(M) \to \mathfrak{P}(J)$ by

$$\sigma(X) = X^u \cap M$$

$$\pi(Y) = Y^\ell \cap J.$$

By Exercise 2.13, with R the restriction of $\leq$ to $J \times M$, $\pi\sigma$ is a closure operator on J and $\sigma\pi$ is a closure operator on M. Prove that $\mathcal{C}_{\pi\sigma} \cong \mathcal{L}$ and that $\mathcal{C}_{\sigma\pi}$ is dually isomorphic to $\mathcal{L}$.

10. A lattice is *semimodular* if $a \succ a \wedge b$ implies $a \vee b \succ b$. Prove that if every element of a finite lattice $\mathcal{L}$ has a unique irredundant decomposition, then $\mathcal{L}$ is semimodular. (Morgan Ward)

11. A decomposition $a = \bigwedge Q$ is *strongly irredundant* if $a < q^* \wedge \bigwedge(Q - \{q\})$ for all $q \in Q$. Prove that every irredundant decomposition in a strongly atomic semimodular lattice is strongly irredundant. (Keith Kearnes)

12. Let $\mathcal{L}$ be the lattice of ideals of the ring of integers Z. Find $M^*(\mathcal{L})$ and all decompositions of 0.

13. Complete the proof of Theorem 3.1. That is, let Γ be a closure operator on a set X, and assume that the union of any chain of Γ-closed sets is closed. Prove by induction on $|S|$ that, for any $S \subseteq X$,

$$\Gamma(S) = \bigcup\{\Gamma(F) : F \text{ is a finite subset of } S\}.$$

14. Let κ be an uncountable cardinal. Show that the following are equivalent.

(1) Each compact element of $\mathcal{L}$ contains at most κ compact elements.
(2) There exists an algebra $\mathcal{A}$ with at most κ operations and constants such that $\mathcal{L}$ is isomorphic to the subalgebra lattice of $\mathcal{A}$.

A complete lattice is said to be *spatial* if every element is a join of completely join irreducible elements, i.e., $x = \bigvee(\downarrow x \cap J^*(\mathcal{L}))$. For applications and further references, see for example Santocanale and Wehrung [11]. The next exercises develop a generalization of Theorem 2.9.

15. Show that in a complete, upper continuous lattice, an element is completely join irreducible if and only if it is (finitely) join irreducible and compact.

16. Let $\mathcal{L}$ be a complete, upper continuous, spatial lattice. Show that $\mathcal{L}$ is algebraic, and moreover, $\mathcal{L}$ is isomorphic to the lattice of Σ-closed sets for the closure system Σ on $J^*(\mathcal{L})$ determined by the rules

$$F \subseteq S \implies q \in S$$

where q is completely join irreducible, F is a finite subset of $J^*(\mathcal{L})$, and $q \leq \bigvee F$.

17. Prove that if $\mathcal{L}$ is an algebraic lattice satisfying the DCC, then L is spatial.

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