## 4. Representation by Equivalence Relations

*No taxation without representation!*

So far we have no analogue for lattices of the Cayley theorem for groups, that every group is isomorphic to a group of permutations. The corresponding representation theorem for lattices, that every lattice is isomorphic to a lattice of equivalence relations, turns out to be considerably deeper. Its proof uses a recursive construction technique that has become a standard tool of lattice theory and universal algebra.

An *equivalence relation* on a set $X$ is a binary relation $E$ satisfying, for all $x, y, z \in X$,

(1) $x\ E\ x$,
(2) $x\ E\ y$ implies $y\ E\ x$,
(3) if $x\ E\ y$ and $y\ E\ z$, then $x\ E\ z$.

We think of an equivalence relation as partitioning the set $X$ into blocks of $E$-related elements, called equivalence classes. Conversely, any partition of $X$ into a disjoint union of blocks induces an equivalence relation on $X$: $x\ E\ y$ iff $x$ and $y$ are in the same block. As usual with relations, we write $x\ E\ y$ and $(x, y) \in E$ interchangeably.

The most important equivalence relations are those induced by maps. If $Y$ is another set, and $f : X \to Y$ is any function, then

$$\ker f = \{(x, y) \in X^2 : f(x) = f(y)\}$$

is an equivalence relation, called the *kernel* of $f$. If $X$ and $Y$ are algebras and $f : X \to Y$ is a homomorphism, then $\ker f$ is a *congruence relation*.

Thinking of binary relations as subsets of $X^2$, the axioms (1)–(3) for an equivalence relation are finitary closure rules. Thus the collection of all equivalence relations on $X$ forms an algebraic lattice $\mathbf{Eq}\ X$. The order on $\mathbf{Eq}\ X$ is given by set containment, i.e.,

$$\begin{aligned} R \le S \quad &\text{iff} \quad R \subseteq S \text{ in } \mathfrak{P}(X^2) \\ &\text{iff} \quad (x, y) \in R \implies (x, y) \in S. \end{aligned}$$

The greatest element of $\mathbf{Eq}\ X$ is the universal relation $X^2$, and its least element is the equality relation $=$. The meet operation in $\mathbf{Eq}\ X$ is of course set intersection, which means that $(x, y) \in \bigwedge_{i \in I} E_i$ if and only if $x\ E_i\ y$ for all $i \in I$. The join

$\bigvee_{i\in I} E_i$ is the transitive closure of the set union $\bigcup_{i\in I} E_i$. Thus $(x,y) \in \bigvee E_i$ if and only if there exists a finite sequence of elements $x_j$ and indices $i_j$ such that

$$x = x_0 \ E_{i_1} \ x_1 \ E_{i_2} \ x_2 \dots x_{k-1} \ E_{i_k} \ x_k = y.$$

The lattice $\mathbf{Eq}\, X$ has many nice properties: it is algebraic, strongly atomic, semimodular, relatively complemented and simple [8]; see Chapter 12 of Crawley and Dilworth [1].[1] The proofs of these facts are exercises in this chapter and Chapter 11.

If $R$ and $S$ are relations on $X$, define the *relative product* $R \circ S$ to be the set of all pairs $(x,y) \in X^2$ for which there exists a $z \in X$ with $x \ R \ z \ S \ y$. If $R$ and $S$ are equivalence relations, then because $x \ R \ x$ we have $S \subseteq R \circ S$; similarly $R \subseteq R \circ S$. Thus

$$R \circ S \ \subseteq \ R \circ S \circ R \ \subseteq \ R \circ S \circ R \circ S \ \subseteq \ \cdots$$

and it is not hard to see that $R \vee S$ is the union of this chain. It is possible, however, that $R \vee S$ is in fact equal to some term in the chain; for example, this is always the case when $X$ is finite. Our proof will yield a representation in which this is always the case, for any two equivalence relations that represent elements of the given lattice.

To be precise, a *representation* (by equivalence relations) of a lattice $\mathcal{L}$ is an ordered pair $(X, F)$ where $X$ is a set and $F : \mathcal{L} \to \mathbf{Eq}\, X$ is a lattice embedding. We say that the representation is

(1) *of type* 1 if $F(x) \vee F(y) = F(x) \circ F(y)$ for all $x, y \in L$,
(2) *of type* 2 if $F(x) \vee F(y) = F(x) \circ F(y) \circ F(x)$ for all $x, y \in L$,
(3) *of type* 3 if $F(x) \vee F(y) = F(x) \circ F(y) \circ F(x) \circ F(y)$ for all $x, y \in L$.

P. M. Whitman [11] proved in 1946 that every lattice has a representation. In 1953 Bjarni Jónsson [7] found a simpler proof that gives a slightly stronger result.

**Theorem 4.1.** *Every lattice has a type 3 representation.*

*Proof.* Given a lattice $\mathcal{L}$, we will use transfinite recursion to construct a type 3 representation of $\mathcal{L}$.

A *weak representation* of $\mathcal{L}$ is a pair $(U, F)$ where $U$ is a set and $F : \mathcal{L} \to \mathbf{Eq}\, U$ is a one-to-one meet homomorphism. Let us order the weak representations of $\mathcal{L}$ by

$$(U, F) \sqsubseteq (V, G) \text{ if } U \subseteq V \text{ and } G(x) \cap U^2 = F(x) \text{ for all } x \in L.$$

We want to construct a (transfinite) sequence $(U_\xi, F_\xi)_{\xi<\lambda}$ of weak representations of $\mathcal{L}$, with $(U_\alpha, F_\alpha) \sqsubseteq (U_\beta, F_\beta)$ whenever $\alpha \leq \beta$, whose limit (union) will be a lattice embedding of $L$ into $\mathbf{Eq} \bigcup_{\xi<\lambda} U_\xi$. We can begin our construction by letting $(U_0, F_0)$ be the weak representation with $U_0 = L$ and $(y, z) \in F_0(x)$ iff $y = z$ or $y \vee z \leq x$. The crucial step is where we fix up the joins one at a time.

[1] The terms *relatively complemented* and *simple* are defined in Chapter 10; we include them here for the sake of completeness.

**Sublemma 1.** *If $(U, F)$ is a weak representation of $\mathcal{L}$ and $(p, q) \in F(x \vee y)$, then there exists $(V, G) \sqsupseteq (U, F)$ with $(p, q) \in G(x) \circ G(y) \circ G(x) \circ G(y)$.*

*Proof of Sublemma 1.* Form $V$ by adding three new points to $U$, say $V = U \,\dot\cup\, \{r, s, t\}$, as in Figure 4.1. We want to make

$$p\ G(x)\ r\ G(y)\ s\ G(x)\ t\ G(y)\ q.$$

Accordingly, for $z \in L$ we define $G(z)$ to be the reflexive, symmetric relation on $V$ satisfying, for $u$, $v \in U$,

(1) $u\ G(z)\ v$ iff $u\ F(z)\ v$,
(2) $u\ G(z)\ r$ iff $z \geq x$ and $u\ F(z)\ p$,
(3) $u\ G(z)\ s$ iff $z \geq x \vee y$ and $u\ F(z)\ p$,
(4) $u\ G(z)\ t$ iff $z \geq y$ and $u\ F(z)\ q$,
(5) $r\ G(z)\ s$ iff $z \geq y$,
(6) $s\ G(z)\ t$ iff $z \geq x$,
(7) $r\ G(z)\ t$ iff $z \geq x \vee y$.

You must check that each $G(z)$ defined thusly really is an equivalence relation, i.e., that it is transitive. This is routine but a bit tedious to write down, so we leave it to the reader. There are four cases, depending on whether or not $z \geq x$ and on whether or not $z \geq y$. Straightforward though it is, this verification would not work if we had only added one or two new elements between $p$ and $q$; see Theorems 4.5 and 4.6.

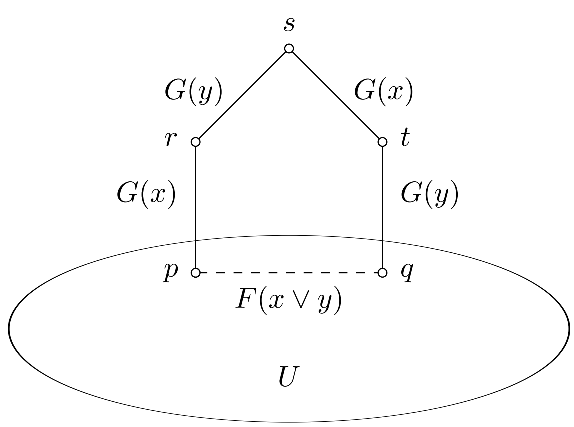


FIGURE 4.1

Now (1) says that $G(z) \cap U^2 = F(z)$. Since $F$ is one-to-one, this implies $G$ is also. Note that for $z$, $z' \in L$ we have $z \wedge z' \geq x$ iff $z \geq x$ and $z' \geq x$, and symmetrically for $y$. Using this with conditions (1)–(7), it is not hard to check that $G(z \wedge z') = G(z) \cap G(z')$. Hence, $G$ is a weak representation of $\mathcal{L}$, and clearly $(U, F) \sqsubseteq (V, G)$. $\square$

**Sublemma 2.** *Let $\lambda$ be a limit ordinal, and for $\xi < \lambda$ let $(U_\xi, F_\xi)$ be weak representations of $\mathcal{L}$ such that $\alpha < \beta < \lambda$ implies $(U_\alpha, F_\alpha) \sqsubseteq (U_\beta, F_\beta)$. Let $V = \bigcup_{\xi<\lambda} U_\xi$ and $G(x) = \bigcup_{\xi<\lambda} F_\xi(x)$ for all $x \in L$. Then $(V, G)$ is a weak representation of $\mathcal{L}$ with $(U_\xi, F_\xi) \sqsubseteq (V, G)$ for each $\xi < \lambda$.*

*Proof.* Let $\xi < \lambda$. Since $F_\alpha(x) = F_\xi(x) \cap U_\alpha^2 \subseteq F_\xi(x)$ whenever $\alpha < \xi$ and $F_\xi(x) = F_\beta(x) \cap U_\xi^2$ whenever $\beta \geq \xi$, for all $x \in L$ we have

$$\begin{aligned} G(x) \cap U_\xi^2 &= \big( \bigcup_{\gamma<\lambda} F_\gamma(x) \big) \cap U_\xi^2 \\ &= \bigcup_{\gamma<\lambda} (F_\gamma(x) \cap U_\xi^2) \\ &= F_\xi(x). \end{aligned}$$

Thus $(U_\xi, F_\xi) \sqsubseteq (V, G)$. Since $F_0$ is one-to-one, this implies that $G$ is also.

It remains to show that $G$ is a meet homomorphism. Clearly $G$ preserves order, so for any $x,\ y \in L$ we have $G(x \wedge y) \subseteq G(x) \cap G(y)$. On the other hand, if $(u, v) \in G(x) \cap G(y)$, then there exists $\alpha < \lambda$ such that $(u, v) \in F_\alpha(x)$, and there exists $\beta < \lambda$ such that $(u, v) \in F_\beta(y)$. If $\gamma$ is the larger of $\alpha$ and $\beta$, then $(u, v) \in F_\gamma(x) \cap F_\gamma(y) = F_\gamma(x \wedge y) \subseteq G(x \wedge y)$. Thus $G(x) \cap G(y) \subseteq G(x \wedge y)$. Combining the two inclusions gives equality. $\square$

Now we want to use these two sublemmas to construct a type 3 representation of $\mathcal{L}$, i.e., a weak representation that also satisfies $G(x \vee y) = G(x) \circ G(y) \circ G(x) \circ G(y)$.

Start with an arbitrary weak representation $(U_0, F_0)$, and consider the set of all quadruples $(p, q, x, y)$ such that $p, q \in U_0$ and $x, y \in L$ and $(p, q) \in F_0(x \vee y)$. Arrange these into a well ordered sequence $(p_\xi, q_\xi, x_\xi, y_\xi)$ for $\xi < \eta$. Applying the sublemmas repeatedly, we can obtain a sequence of weak representations $(U_\xi, F_\xi)$ for $\xi \leq \eta$ such that

1. (1) if $\xi < \eta$, then $(U_\xi, F_\xi) \sqsubseteq (U_{\xi+1}, F_{\xi+1})$ and $(p_\xi, q_\xi) \in F_{\xi+1}(x_\xi) \circ F_{\xi+1}(y_\xi) \circ F_{\xi+1}(x_\xi) \circ F_{\xi+1}(y_\xi)$;
2. (2) if $\lambda \leq \eta$ is a limit ordinal, then $U_\lambda = \bigcup_{\xi<\lambda} U_\xi$ and $F_\lambda(x) = \bigcup_{\xi<\lambda} F_\xi(x)$ for all $x \in L$.

Let $V_1 = U_\eta$ and $G_1 = F_\eta$. If $p, q \in U_0$, and $x, y \in L$ and $p\ F_0(x \vee y)\ q$, then $(p, q, x, y) = (p_\xi, q_\xi, x_\xi, y_\xi)$ for some $\xi < \eta$, so that $(p, q) \in F_{\xi+1}(x) \circ F_{\xi+1}(y) \circ F_{\xi+1}(x) \circ F_{\xi+1}(y)$. Consequently,

$$F_0(x \vee y) \subseteq G_1(x) \circ G_1(y) \circ G_1(x) \circ G_1(y).$$

Note $(U_0, F_0) \sqsubseteq (V_1, G_1)$.

Of course, along the way we have probably introduced lots of new failures of the join property that need to be fixed up. So repeat this whole process $\omega$ times, obtaining a sequence

$$(U_0, F_0) = (V_0, G_0) \sqsubseteq (V_1, G_1) \sqsubseteq (V_2, G_2) \sqsubseteq \cdots$$

such that $G_n(x \vee y) \subseteq G_{n+1}(x) \circ G_{n+1}(y) \circ G_{n+1}(x) \circ G_{n+1}(y)$ for all $n \in w$, $x, y \in L$.

Finally, let $W = \bigcup_{n \in w} V_n$ and $H(x) = \bigcup_{n \in \omega} G_n(x)$ for all $x \in L$, and you get a type 3 representation of $\mathcal{L}$. $\square$

Since the proof involves transfinite recursion, it produces a representation $(X, F)$ with $X$ infinite, even when $\mathcal{L}$ is finite. For a long time one of the outstanding questions of lattice theory was whether every finite lattice can be embedded into the lattice of equivalence relations on a finite set. In 1980, Pavel Pudlák and Jíři Tůma showed that the answer is *yes* [10]. The proof is quite difficult.

**Theorem 4.2.** *Every finite lattice has a representation $(Y, G)$ with $Y$ finite.*

One of the motivations for Whitman's theorem was Garrett Birkhoff's observation, made in the 1930's, that a representation of a lattice $\mathcal{L}$ by equivalence relations induces an embedding of $\mathcal{L}$ into the lattice of subgroups of a group. Given a representation $(X, F)$ of $\mathcal{L}$, let $\mathcal{G}$ be the group of all permutations on $X$ that move only finitely many elements, and let **Sub** $\mathcal{G}$ denote the lattice of subgroups of $\mathcal{G}$. Let $h : \mathcal{L} \to \textbf{Sub}\, \mathcal{G}$ by

$$h(a) = \{\pi \in G : x \; F(a) \; \pi(x) \text{ for all } x \in X\}.$$

Then it is not too hard to check that $h$ is an embedding.

**Theorem 4.3.** *Every lattice can be embedded into the lattice of subgroups of a group.*

Not all lattices have representations of type 1 or 2, so it is natural to ask which ones do. First we consider sublattices of **Eq** $X$ with type 2 joins.

**Lemma 4.4.** *Let $\mathcal{L}$ be a sublattice of* **Eq** *$X$ with the property that $R \vee S = R \circ S \circ R$ for all $R$, $S \in L$. Then $\mathcal{L}$ satisfies*

$$(M) \qquad x \geq y \quad \textit{implies} \quad x \wedge (y \vee z) = y \vee (x \wedge z).$$

The implication $(M)$ is known as the *modular law.*

*Proof.* Assume that $\mathcal{L}$ is a sublattice of **Eq** $X$ with type 2 joins, and let $A, B, C \in L$ with $A \geq B$. If $p, q \in X$ and $(p, q) \in A \wedge (B \vee C)$, then

$$p \; A \; q$$
$$p \; B \; r \; C \; s \; B \; q$$

for some $r, s \in X$ (see Figure 4.2). Since

$$r \; B \; p \; A \; q \; B \; s$$

and $B \leq A$, we have $(r, s) \in A \wedge C$. It follows that $(p, q) \in B \vee (A \wedge C)$. Thus $A \wedge (B \vee C) \leq B \vee (A \wedge C)$. The reverse inclusion is trivial, so we have equality. □

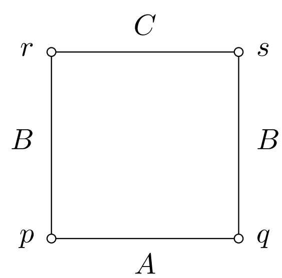


FIGURE 4.2

On the other hand, Jónsson gave a slight variation of the proof of Theorem 4.1 that shows that every modular lattice has a type 2 representation [7], [1]. Combining this with Lemma 4.4, we obtain the following.

**Theorem 4.5.** *A lattice has a type 2 representation if and only if it is modular.*

The modular law $(M)$ plays an important role in lattice theory, and we will see it often. It was invented in the 1890's by Richard Dedekind, who showed that the lattice of normal subgroups of a group is modular [2], [3]. Note that $(M)$ fails in the pentagon $\mathcal{N}_5$. In fact, Dedekind proved that a lattice is modular if and only if it does not contain the pentagon as a sublattice; see Theorem 9.1.

The modular law is equivalent to the equation,

$$(M') \qquad x \wedge ((x \wedge y) \vee z) = (x \wedge y) \vee (x \wedge z).$$

It is easily seen to be a special case of (and hence weaker than) the distributive law,

$$(D) \qquad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

*viz.*, $(M)$ says that $(D)$ should hold for $x \geq y$.

Note that the normal subgroup lattice of a group has a natural representation $(X, F)$: take $X = G$ and $F(N) = \{(x, y) \in G^2 : xy^{-1} \in N\}$. This representation is in fact type 1 (Exercise 3), and Jónsson showed that lattices with a type 1 representation, or equivalently sublattices of **Eq** $X$ in which $R \vee S = R \circ S$, satisfy an implication stronger than the modular law. A lattice is said to be *Arguesian* if it satisfies

$$(A) \qquad (a_0 \vee b_0) \wedge (a_1 \vee b_1) \leq a_2 \vee b_2 \text{ implies } c_2 \leq c_0 \vee c_1$$

where

$$c_i = (a_j \vee a_k) \wedge (b_j \vee b_k)$$

for $\{i, j, k\} = \{0, 1, 2\}$. The Arguesian law is (less obviously) equivalent to a lattice inclusion,

$$(A') \qquad (a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2) \leq a_0 \vee (b_0 \wedge (c \vee b_1))$$

where

$$c = c_2 \wedge (c_0 \vee c_1).$$

These are two of several equivalent forms of this law, which is stronger than modularity and weaker than distributivity. The Arguesian law is modelled after Desargues' Law in projective geometry.

**Theorem 4.6.** *If $\mathcal{L}$ is a sublattice of* $\mathbf{Eq}\ X$ *with the property that* $R \vee S = R \circ S$ *for all* $R,\ S \in L$*, then* $\mathcal{L}$ *satisfies the Arguesian law.*

**Corollary.** *Every lattice that has a type 1 representation is Arguesian.*

*Proof.* Let $\mathcal{L}$ be a sublattice of $\mathbf{Eq}\ X$ with type 1 joins. Assume $(A_0 \vee B_0) \wedge (A_1 \vee B_1) \leq A_2 \vee B_2$, and suppose $(p, q) \in C_2 = (A_0 \vee A_1) \wedge (B_0 \vee B_1)$. Then there exist $r, s$ such that

$$p\ A_0\ r\ A_1\ q$$
$$p\ B_0\ s\ B_1\ q.$$

Since $(r, s) \in (A_0 \vee B_0) \wedge (A_1 \vee B_1) \leq A_2 \vee B_2$, there exists $t$ such that $r\ A_2\ t\ B_2\ s$. Now you can check that

$$(p, t) \in (A_0 \vee A_2) \wedge (B_0 \vee B_2) = C_1$$
$$(t, q) \in (A_1 \vee A_2) \wedge (B_1 \vee B_2) = C_0$$

and hence $(p, q) \in C_0 \vee C_1$. Thus $C_2 \leq C_0 \vee C_1$, as desired. (This argument is diagrammed in Figure 4.3.) $\square$

It follows that the lattice of normal subgroups of a group is not only modular, but Arguesian; see exercise 3. All these types of lattices are Arguesian: the lattice of subgroups of an abelian group, the lattice of ideals of a ring, the lattice of subspaces of a vector space, the lattice of submodules of a module. More generally, Ralph

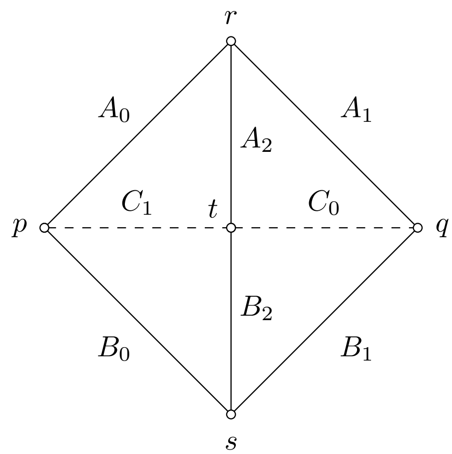


FIGURE 4.3

Freese and Bjarni Jónsson proved that $\mathcal{V}$ is a variety of algebras, all of whose congruence lattices are modular (such as groups or rings), then the congruence lattices of algebras in $\mathcal{V}$ are Arguesian [4].

Mark Haiman has shown that the converse of Theorem 4.6 is false: there are Arguesian lattices that do not have a type 1 representation [5], [6]. In fact, his proof shows that lattices with a type 1 representation must satisfy equations that are strictly stronger than the Arguesian law. It follows, in particular, that the lattice of normal subgroups of a group also satisfies these stronger equations, as do the other types of lattices mentioned in the preceding paragraph. Interestingly, P. P. Pálfy and Laszlo Szabó have shown that subgroup lattices of abelian groups satisfy an equation that does not hold in all normal subgroup lattices [9].

The question remains: *Does there exist a set of equations $\Sigma$ such that a lattice has a type 1 representation if and only if it satisfies all the equations of $\Sigma$?* Haiman proved that if such a $\Sigma$ exists, it must contain infinitely many equations. In Chapter 7 we will see that a class of lattices is characterized by a set of equations if and only if it is closed with respect to direct products, sublattices, and homomorphic images. The class of lattices having a type 1 representation is easily seen to be closed under sublattices and direct products, so the question is equivalent to: *Is the class of all lattices having a type 1 representation closed under homomorphic images?*

## EXERCISES FOR CHAPTER 4

1. Draw $\mathbf{Eq}\,X$ for $|X| = 3, 4$.
2. Find representations in $\mathbf{Eq}\,X$ for
   (a) $\mathfrak{P}(Y)$, $Y$ a set,
   (b) $\mathcal{N}_5$,

(c) $\mathcal{M}_n$, $n < \infty$.

3. Let $\mathcal{G}$ be a group. Let $F : \mathbf{Sub}\,\mathcal{G} \to \mathbf{Eq}\,G$ be the standard representation by cosets: $F(H) = \{(x, y) \in G^2 : xy^{-1} \in H\}$.

(a) Verify that $F(H)$ is indeed an equivalence relation.
(b) Verify that $F$ is a lattice embedding.
(c) Show that $F(H) \vee F(K) = F(H) \circ F(K)$ iff $HK = KH$ $(= H \vee K)$.
(d) Conclude that the restriction of $F$ to the normal subgroup lattice $\mathcal{N}(\mathcal{G})$ is a type 1 representation.

4. Show that for $R, S \in \mathbf{Eq}\,X$, $R \vee S = R \circ S$ iff $S \circ R \subseteq R \circ S$ iff $R \circ S = S \circ R$. (For this reason, such equivalence relations are said to *permute*.)

5. Recall from Exercise 6 of Chapter 3 that a complete sublattice of an algebraic lattice is algebraic.

(a) Let $\mathcal{S}$ be a join semilattice with 0. Assume that $\varphi : \mathcal{S} \to \mathbf{Eq}\ X$ is a join homomorphism with the properties

(i) for each pair $a$, $b \in X$ there exists $\sigma(a, b) \in S$ such that $(a, b) \in \varphi(s)$ iff $s \geq \sigma(a, b)$, and
(ii) for each $s \in S$, there exists a pair $(x_s, y_s)$ such that $(x_s, y_s) \in \varphi(t)$ iff $s \leq t$.

Show that $\varphi$ induces a complete representation $\overline{\varphi} : \mathcal{I}(\mathcal{S}) \to \mathbf{Eq}\,X$.

(b) Indicate how to modify the proof of Theorem 4.1 to obtain, for an arbitrary join semilattice $\mathcal{S}$ with 0, a set $X$ and a join homomorphism $\varphi : \mathcal{S} \to \mathbf{Eq}\ X$ satisfying (i) and (ii).

(c) Conclude that a complete lattice $\mathcal{L}$ has a complete representation by equivalence relations if and only if $\mathcal{L}$ is algebraic.

6. Prove that $\mathbf{Eq}\,X$ is a strongly atomic, semimodular, algebraic lattice.

7. Prove that a lattice with a type 1 representation satisfies the Arguesian inclusion $(A')$.

8. Mimicking the proof of Theorem 4.6, find an 8-variable implication, like the Arguesian law, that holds in every lattice of permuting equivalence relations. (The hard part of Haiman's work was to show that such laws are not a consequence of the Arguesian law, nor of each other as more variables are added.)

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