6. Free Lattices

Freedom's just another word for nothing left to lose ....
–Kris Kristofferson

If x, y and z are elements of a lattice, then $x \vee (y \vee (x \wedge z)) = x \vee y$ is always true, while $x \vee y = z$ is usually not true. *Is there an algorithm that, given two lattice expressions p and q, determines whether $p = q$ holds for every substitution of the variables in every lattice?* The answer is yes, and finding this algorithm (Corollary to Theorem 6.2) is our original motivation for studying free lattices.

We say that a lattice $\mathcal{L}$ is *generated* by a set $X \subseteq L$ if no proper sublattice of $\mathcal{L}$ contains X. In terms of the subalgebra closure operator Sg introduced in Chapter 3, this means $\mathrm{Sg}(X) = \mathcal{L}$.

A lattice $\mathcal{F}$ is *freely generated by X* if

(I) $\mathcal{F}$ is a lattice,
(II) X generates $\mathcal{F}$,
(III) for every lattice $\mathcal{L}$, every map $h_0 : X \to L$ can be extended to a homomorphism $h : \mathcal{F} \to \mathcal{L}$.

A *free lattice* is a lattice that is freely generated by one of its subsets.

Condition (I) is sort of redundant, but we include it because it is important when constructing a free lattice to be sure that the algebra constructed is indeed a lattice. In the presence of condition (II), there is only one way to define the homomorphism h in condition (III): for example, if $x, y, z \in X$ then we must have $h(x \vee (y \wedge z)) = h_0(x) \vee (h_0(y) \wedge h_0(z))$. Condition (III) really says that this natural extension is well defined. This in turn says that the only time two lattice terms in the variables X are equal in $\mathcal{F}$ is when they are equal in every lattice.

Now the class of lattices is an *equational class*, i.e., it is the class of all algebras with a fixed set of operation symbols ($\vee$ and $\wedge$) satisfying a given set of equations (the idempotent, commutative, associative and absorption laws). Equational classes are also known as *varieties*, and in Chapter 7 we will take a closer look at varieties of lattices. A fundamental theorem of universal algebra, due to Garrett Birkhoff [3], says that given any nontrivial[1] equational class $\mathbf{V}$ and any set X, there is an algebra in $\mathbf{V}$ freely generated by X. Thus the existence of free groups, free semilattices, and in particular free lattices is guaranteed.[2] Likewise, there are free distributive

[1] A variety $\mathbf{T}$ is *trivial* if it satisfies the equation $x \approx y$, which means that every algebra in $\mathbf{T}$ has exactly one element. This is of course the smallest variety of any type.

[2] However, there is no free lunch.

lattices, free modular lattices, and free Arguesian lattices, since each of these laws can be written as a lattice equation.

Theorem 6.1. *For any nonempty set X, there exists a free lattice generated by X.*

The proof uses three basic principles of universal algebra. The principles correspond for lattices to Theorems 5.1, 5.4, and 5.5 respectively. However, the proofs of these theorems involved nothing special to lattices except the operation symbols $\wedge$ and $\vee$, which can easily be changed to arbitrary operation symbols. Thus, with only minor modification, the proof of this theorem can be adapted to show the existence of free algebras in any nontrivial equational class of algebras.

Basic Principle 1. If $h : \mathcal{A} \twoheadrightarrow \mathcal{B}$ is a surjective homomorphism, then $\mathcal{B} \cong \mathcal{A}/\ker h$.

Basic Principle 2. If $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{A} \twoheadrightarrow \mathcal{C}$ are homomorphism with g surjective, and $\ker g \leq \ker f$, then there exists $h : C \to B$ such that $f = hg$.

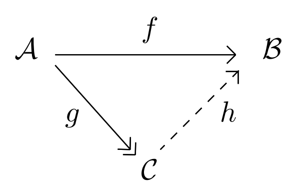

Figure 6.1

Basic Principle 3. If $\psi = \bigwedge_{i \in I} \theta_i$ in **Con** $\mathcal{A}$, then $\mathcal{A}/\psi$ is isomorphic to a subalgebra of the direct product $\Pi_{i \in I} \mathcal{A}/\theta_i$.

With these principles in hand, we proceed with the proof of Theorem 6.1.

Proof of Theorem 6.1. Given the set X, define the *word algebra* $W(X)$ to be the set of all formal expressions (strings of symbols) satisfying the following properties:

(1) $X \subseteq W(X)$,
(2) if $p, q \in W(X)$, then $(p \vee q)$ and $(p \wedge q)$ are in $W(X)$,
(3) only the expressions given by the first two rules are in $W(X)$.

Thus $W(X)$ is the absolutely free algebra with operation symbols $\vee$ and $\wedge$ generated by X. The elements of $W(X)$, which are called *terms*, are all well-formed expressions in the variables X and the operation symbols $\wedge$ and $\vee$. Clearly $W(X)$ is an algebra generated by X, which is property (II) from the definition of a free lattice. Because two terms are equal if and only if they are identical, $W(X)$ has the mapping property (III). On the other hand, it is definitely not a lattice. We need to identify those pairs $p, q \in W(X)$ that evaluate the same in every lattice, e.g., x and $(x \wedge (x \vee y))$. The point of the proof is that when this is done, properties (II) and (III) still hold.

Let $\Lambda = \{\theta \in \textbf{Con } W(X) : W(X)/\theta \text{ is a lattice}\}$, and let $\lambda = \bigwedge \Lambda$. We claim that $W(X)/\lambda$ is a lattice freely generated by $\{x\lambda : x \in X\}$.

By Basic Principle 3, $W(X)/\lambda$ is isomorphic to a subalgebra of a direct product of lattices, so it is a lattice.[3] Clearly $W(X)/\lambda$ is generated by $\{x\lambda : x \in X\}$, and because there exist nontrivial lattices (more than one element) for X to be mapped to in different ways, $x \neq y$ implies $x\lambda \neq y\lambda$ for $x, y \in X$.

Now let $\mathcal{L}$ be a lattice and let $f_0 : X \to L$ be any map. By the preceding observation, the corresponding map $h_0 : X/\lambda \to L$ defined by $h_0(x\lambda) = f_0(x)$ is well defined. Now f_0 can be extended to a homomorphism $f : W(X) \to \mathcal{L}$, whose range is some sublattice $\mathcal{S}$ of $\mathcal{L}$. By Basic Principle 1, $W(X)/\ker f \cong \mathcal{S}$ so $\ker f \in \Lambda$, and hence $\ker f \geq \lambda$. If we use ε to denote the standard homomorphism $W(X) \twoheadrightarrow W(X)/\lambda$ with $\varepsilon(u) = u\lambda$ for all $u \in W(X)$, then $\ker f \geq \ker \varepsilon = \lambda$. Thus by Basic Principle 2 there exists a homomorphism $h : W(X)/\lambda \to \mathcal{L}$ with $h\varepsilon = f$ (see Figure 6.2). This means $h(u\lambda) = f(u)$ for all $u \in W(X)$; in particular, h extends h_0 as required. □

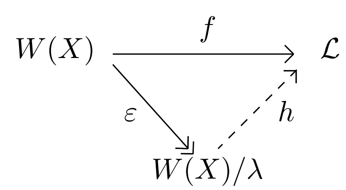

FIGURE 6.2

It is easy to see, using the mapping property (III), that if $\mathcal{F}$ is a lattice freely generated by X, $\mathcal{G}$ is a lattice freely generated by Y, and $|X| = |Y|$, then $\mathcal{F} \cong \mathcal{G}$. Thus we can speak of *the* free lattice generated by X, which we will denote by FL(X). If $|X| = n$, then we also denote this lattice by FL(n). The lattice FL(2) has four elements, so there is not much to say about it. But FL(n) is infinite for $n \geq 3$, and we want to investigate its structure.

The advantage of the general construction we used is that it gives us the existence of free algebras in any variety; the disadvantage is that it does not, indeed cannot, tell us anything about the arithmetic of free lattices. For this we need a result due to Thoralf Skolem [22] (reprinted in [23]), and independently, P. M. Whitman [25].[4]

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[3] This is where we use that lattices are equationally defined, since we need closure under subalgebras and direct products. For example, the class of integral domains is not equationally defined, and the direct product of two or more integral domains is not one.

[4] The history here is rather interesting. Skolem, as part of his 1920 paper which proves the Lowenheim-Skolem Theorem, solved the word problem not only for free lattices, but for finitely presented lattices as well. But by the time the great awakening of lattice theory occurred in the 1930's, his solution had been forgotten. Thus Whitman's 1941 construction of free lattices became the standard reference on the subject. It was not until 1992 that Stan Burris rediscovered Skolem's solution.

Theorem 6.2. *Every free lattice* FL(X) *satisfies the following conditions, where* $x, y \in X$ *and* $p, q, p_1, p_2, q_1, q_2 \in \mathrm{FL}(X)$.

(1) $x \leq y$ *iff* $x = y$.
(2) $x \leq q_1 \vee q_2$ *iff* $x \leq q_1$ *or* $x \leq q_2$.
(3) $p_1 \wedge p_2 \leq x$ *iff* $p_1 \leq x$ *or* $p_2 \leq x$.
(4) $p_1 \vee p_2 \leq q$ *iff* $p_1 \leq q$ *and* $p_2 \leq q$.
(5) $p \leq q_1 \wedge q_2$ *iff* $p \leq q_1$ *and* $p \leq q_2$.
(6) $p = p_1 \wedge p_2 \leq q_1 \vee q_2 = q$ *iff* $p_1 \leq q$ *or* $p_2 \leq q$ *or* $p \leq q_1$ *or* $p \leq q_2$.

Finally, $p = q$ *iff* $p \leq q$ *and* $q \leq p$.

Condition (6) in Theorem 6.2 is known as *Whitman's condition*, and it is usually denoted by (W).

Proof of Theorem 6.2. Properties (4) and (5) hold in every lattice, by the definition of least upper bound and greatest lower bound, respectively. Likewise, the "if" parts of the remaining conditions hold in every lattice.

We can take care of (1) and (2) simultaneously. Fixing $x \in X$, let

$$G_x = \{w \in \mathrm{FL}(X) : w \geq x \text{ or } w \leq \bigvee F \text{ for some finite } F \subseteq X - \{x\}\}.$$

Then $X \subseteq G_x$, and G_x is closed under joins and meets, so $G_x = \mathrm{FL}(X)$. Thus every $w \in \mathrm{FL}(X)$ is either above x or below $\bigvee F$ for some finite $F \subseteq X - \{x\}$. Properties (1) and (2) will follow if we can show that this "or" is exclusive: $x \not\leq \bigvee F$ for all finite $F \subseteq X - \{x\}$. So let $h_0 : X \to \mathbf{2}$ (the two element chain) be defined by $h_0(x) = 1$, and $h_0(y) = 0$ for $y \in X - \{x\}$. This map extends to a homomorphism $h : \mathrm{FL}(X) \to \mathbf{2}$. For every finite $F \subseteq X - \{x\}$ we have $h(x) = 1 \not\leq 0 = h(\bigvee F)$, whence $x \not\leq \bigvee F$.

Condition (3) is the dual of (2). Note that the proof shows $x \not\geq \bigwedge H$ for all finite $H \subseteq X - \{x\}$.

Whitman's condition (6), or (W), can be proved using a slick construction due to Alan Day [5]. This construction can be motivated by a simple example. In the lattice of Figure 6.3(a), the elements a, b, c, d fail (W); in Figure 6.3(b) we have "fixed" this failure by making $a \wedge b \not\leq c \vee d$. Day's method provides a formal way of doing this for any (W)-failure.

Let $I = u/v$ be an interval in a lattice $\mathcal{L}$. We define a new lattice $\mathcal{L}[I]$ as follows. The universe of $\mathcal{L}[I]$ is $(L - I) \cup (I \times \mathbf{2})$. Thus the elements of $\mathcal{L}[I]$ are of the form x with $x \notin I$, and (y, i) with $i \in \{0, 1\}$ and $y \in I$. The order on $\mathcal{L}[I]$ is defined by:

$$\begin{aligned} x \leq y &\text{ if } x \leq_{\mathcal{L}} y \\ (x, i) \leq y &\text{ if } x \leq_{\mathcal{L}} y \\ x \leq (y, j) &\text{ if } x \leq_{\mathcal{L}} y \\ (x, i) \leq (y, j) &\text{ if } x \leq_{\mathcal{L}} y \text{ and } i \leq j. \end{aligned}$$

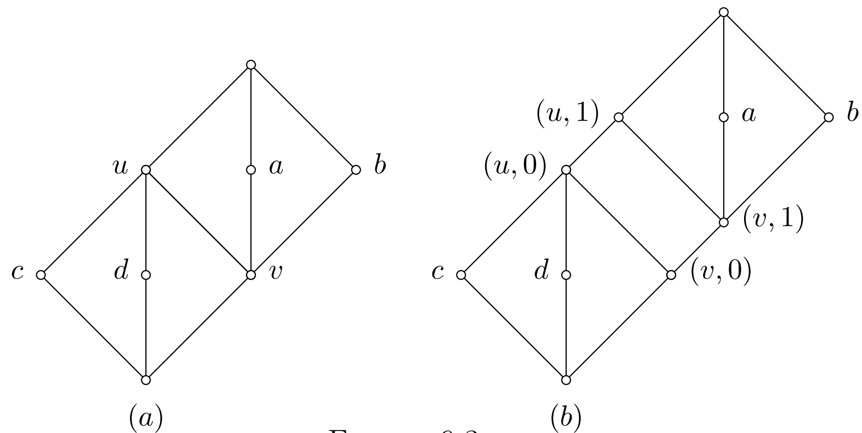


FIGURE 6.3

It is not hard to check the various cases to show that each pair of elements in $L[I]$ has a meet and join, so that $\mathcal{L}[I]$ is indeed a lattice.[5] Moreover, the natural map $\kappa : \mathcal{L}[I] \twoheadrightarrow \mathcal{L}$ with $\kappa(x) = x$ and $\kappa((y, i)) = y$ is a homomorphism. Figure 6.4 gives another example of the doubling construction, where the doubled interval consists of a single element $\{u\}$.

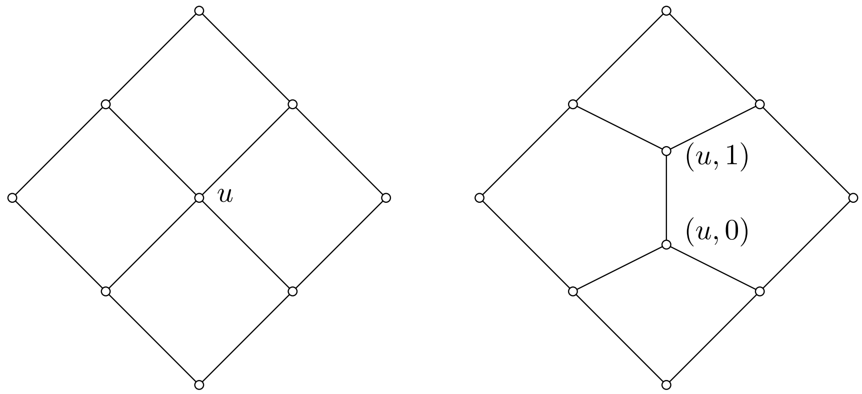
Now suppose a, b, c, d witness a failure of (W) in $\mathrm{FL}(X)$. Let $u = c \vee d$, $v = a \wedge b$ and $I = u/v$. Let $h_0 : X \to \mathrm{FL}(X)[I]$ with $h_0(x) = x$ if $x \notin I$, $h_0(y) = (y, 0)$ if $y \in I$, and extend this map to a homomorphism h. Now $\kappa h : \mathrm{FL}(X) \to \mathrm{FL}(X)$ is also a homomorphism, and since $\kappa h(x) = x$ for all $x \in X$, it is in fact the identity. Therefore $h(w) \in \kappa^{-1}(w)$ for all $w \in \mathrm{FL}(X)$, i.e., $h(w)$ is one of w, $(w, 0)$ or $(w, 1)$. Since $a, b, c, d \notin I$, this means $h(t) = t$ for $t \in \{a, b, c, d\}$. Now $v = a \wedge b \leq c \vee d = u$ in $\mathrm{FL}(X)$, so $h(v) \leq h(u)$. But we can calculate

$$h(v) = h(a) \wedge h(b) = a \wedge b = (v, 1) \nleq (u, 0) = c \vee d = h(c) \vee h(d) = h(u)$$

in $\mathrm{FL}(X)[I]$, a contradiction. Thus (W) holds in $\mathrm{FL}(X)$. $\square$

Theorem 6.2 gives us a solution to the *word problem* for free lattices, i.e., an algorithm for deciding whether two lattice terms $p, q \in W(X)$ evaluate to the same element in $\mathrm{FL}(X)$ (and hence in all lattices). Strictly speaking, we have an evaluation map $\varepsilon : W(X) \to \mathrm{FL}(X)$ with $\varepsilon(x) = x$ for all $x \in X$, and we want to decide whether $\varepsilon(p) = \varepsilon(q)$. Following tradition, however, we suppress the ε and ask whether $p = q$ in $\mathrm{FL}(X)$.

[5]This construction yields a lattice if, instead of requiring that I be an interval, we only ask that it be convex, i.e., if $x, z \in I$ and $x \leq y \leq z$, then $y \in I$. This generalized construction has also proved very useful; see Section II.3 of [12], which is based on Day [6], [7] and [8].



(a) (b)

FIGURE 6.4

Corollary. *Let $p, q \in W(X)$. To decide whether $p \leq q$ in* $\mathrm{FL}(X)$*, apply the conditions of Theorem 6.2 recursively. To test whether $p = q$ in* $\mathrm{FL}(X)$*, check both $p \leq q$ and $q \leq p$.*

The algorithm works because it eventually reduces $p \leq q$ to a statement involving the conjunction and disjunction of a number of inclusions of the form $x \leq y$, each of which holds if and only if $x = y$. Using the algorithm requires a little practice; you should try showing that $x \wedge (y \vee z) \nleq (x \wedge y) \vee (x \wedge z)$ in $\mathrm{FL}(X)$, which is equivalent to the statement that not every lattice is distributive.[6] To appreciate its significance, you should know that it is not always possible to solve the word problem for free algebras. For example, the word problem for a free modular lattice $\mathcal{F}_{\mathbf{M}}(X)$ is not solvable if $|X| \geq 4$ (see Chapter 7).

By isolating the properties that do not hold in every lattice, we can rephrase Theorem 6.2 in the following useful form.

Theorem 6.3. *A lattice $\mathcal{F}$ is freely generated by its subset X if and only if $\mathcal{F}$ is generated by X, $\mathcal{F}$ satisfies (W), and the following two conditions hold for each $x \in X$:*

(1) *if $x \leq \bigvee G$ for some finite $G \subseteq X$, then $x \in G$;*
(2) *if $x \geq \bigwedge H$ for some finite $H \subseteq X$, then $x \in H$.*

It is worthwhile to compare the roles of $\mathbf{Eq}\ X$ and $\mathrm{FL}(X)$: every lattice can be embedded into a lattice of equivalence relations, while every lattice is a homomorphic image of a free lattice.

[6]The algorithm for the word problem, and other free lattice algorithms, can be efficiently programmed; see Chapter XI of [12]. These programs have proved to be a useful tool in the investigation of the structure of free lattices.

Note that it follows from (W) that no element of FL(X) is properly both a meet and a join, i.e., every element is either meet irreducible or join irreducible. Moreover, the generators are the only elements that are both meet and join irreducible. Thus *the generating set of* FL(X) *is unique.* This is very different from the situation say in free groups: the free group on $\{x, y\}$ is also generated (freely) by $\{x, xy\}$.

Each element $w \in \mathrm{FL}(X)$ corresponds to an equivalence class of terms in $W(X)$. Among the terms that evaluate to w, there may be several of minimal length (total number of symbols), e.g., $(x \vee (y \vee z))$, $((y \vee x) \vee z)$, etc. Note that if a term p can be obtained from a term q by applications of the associative and commutative laws only, then p and q have the same length. This allows us to speak of the length of a term $t = \bigvee t_i$ without specifying the order or parenthesization of the joinands, and likewise for meets. We want to show that a minimal length term for w is unique up to associativity and commutativity. This is true for generators, so by duality it suffices to consider the case when w is a join.

Lemma 6.4. *Let $t = \bigvee t_i$ in $W(X)$, where each t_i is either a generator or a meet. Assume that $\varepsilon(t) = w$ and $\varepsilon(t_i) = w_i$ under the evaluation map $\varepsilon : W(X) \to \mathrm{FL}(X)$. If t is a minimal length term representing w, then the following are true.*

(1) *Each t_i is of minimal length.*
(2) *The w_i's are pairwise incomparable.*
(3) *If t_i is not a generator, so $t_i = \bigwedge_j t_{ij}$, then $\varepsilon(t_{ij}) = w_{ij} \nleq w$ for all j.*

Proof. Only (3) requires explanation. Suppose $w_i = \bigwedge w_{ij}$ in FL(X), corresponding to $t_i = \bigwedge t_{ij}$ in $W(X)$. Note that $w_i \leq w_{ij}$ for all j. If for some j_0 we also had $w_{ij_0} \leq w$, then

$$w = \bigvee w_i \leq w_{ij_0} \vee \bigvee_{k \neq i} w_k \leq w,$$

whence $w = w_{ij_0} \vee \bigvee_{k \neq i} w_k$. But then replacing t_i by t_{ij_0} would yield a shorter term representing w, a contradiction. $\square$

If A and B are finite subsets of a lattice, we say that A *refines* B, written $A \ll B$, if for each $a \in A$ there exists $b \in B$ with $a \leq b$. We define dual refinement by $C \gg D$ if for each $c \in C$ there exists $d \in D$ with $c \geq d$; note that because of the reversed order of the quantification in the two statements, $A \ll B$ is not the same as $B \gg A$. The elementary properties of refinement can be set out as follows, with the proofs left as an exercise.

Lemma 6.5. *The refinement relation on finite subsets of a lattice $\mathcal{L}$ has the following properties.*

(1) *$A \ll B$ implies $\bigvee A \leq \bigvee B$.*
(2) *The relation $\ll$ is a quasiorder on the finite subsets of L.*
(3) *If $A \subseteq B$ then $A \ll B$.*
(4) *If A is an antichain, $A \ll B$ and $B \ll A$, then $A \subseteq B$.*

(5) *If A and B are antichains with $A \ll B$ and $B \ll A$, then $A = B$.*
(6) *If $A \ll B$ and $B \ll A$, then A and B have the same set of maximal elements.*

The preceding two lemmas are connected as follows.

Lemma 6.6. *Let $w = \bigvee_{1 \leq i \leq m} w_i = \bigvee_{1 \leq k \leq n} u_k$ in $\mathrm{FL}(X)$. If each w_i is either a generator or a meet $w_i = \bigwedge_j w_{ij}$ with $w_{ij} \nleq w$ for all j, then*

$$\{w_1, \ldots, w_m\} \ll \{u_1, \ldots, u_n\}.$$

Proof. For each i we have $w_i \leq \bigvee u_k$. If w_i is a generator, this implies $w_i \leq u_s$ for some s by Theorem 6.2(2). If $w_i = \bigwedge w_{ij}$, we apply Whitman's condition (W) to the inclusion $w_i = \bigwedge w_{ij} \leq \bigvee u_k = w$. Since we are given that $w_{ij} \nleq w$ for all j, it must be that $w_i \leq u_t$ for some t. Hence $\{w_1, \ldots, w_m\} \ll \{u_1, \ldots, u_n\}$. □

Now let $t = \bigvee t_i$ and $s = \bigvee s_j$ be two minimal length terms that evaluate to w in $\mathrm{FL}(X)$. Let $\varepsilon(t_i) = w_i$ and $\varepsilon(s_j) = u_j$, so that $w = \bigvee w_i = \bigvee u_j$ in $\mathrm{FL}(X)$. By Lemma 6.4(1) each t_i is a minimal length term for w_i, and each s_j is a minimal length term for u_j. By induction, these are unique up to associativity and commutativity. Hence we may assume that $t_i = s_j$ whenever $w_i = u_j$. By Lemma 6.4(2), the sets $\{w_1, \ldots, w_m\}$ and $\{u_1, \ldots, u_n\}$ are antichains in $\mathrm{FL}(X)$. By Lemma 6.4(3), the elements w_i satisfy the hypothesis of Lemma 6.6, so $\{w_1, \ldots, w_m\} \ll \{u_1, \ldots, u_n\}$. Symmetrically, $\{u_1, \ldots, u_n\} \ll \{w_1, \ldots, w_m\}$. Applying Lemma 6.5(5) yields $\{w_1, \ldots, w_m\} = \{u_1, \ldots, u_n\}$, whence by our assumption above $\{t_1, \ldots, t_m\} = \{s_1, \ldots, s_n\}$. Thus we obtain the desired uniqueness result.

Theorem 6.7. *The minimal length term representing an element $w \in \mathrm{FL}(X)$ is unique up to associativity and commutativity.*

This minimal length term is called the *canonical form* of w. The canonical form of a generator is just x. The proof of the theorem has shown that if w is a proper join, then its canonical form is determined by the conditions of Lemma 6.4. If w is a proper meet, then of course its canonical form must satisfy the dual conditions.

The proof of Lemma 6.4 gives us an algorithm for finding the canonical form of a lattice term. Let $t = \bigvee t_i$ in $W(X)$, where each t_i is either a generator or a meet, and suppose that we have already put each t_i into canonical form, which we can do inductively. This will guarantee that condition (1) of Lemma 6.4 holds when we are done. For each t_i that is not a generator, say $t_i = \bigwedge t_{ij}$, check whether any $t_{ij} \leq t$ in $\mathrm{FL}(X)$; if so, replace t_i by t_{ij}. Continue this process until you have an expression $u = \bigvee u_i$ which satisfies condition (3). Finally, check whether $u_i \leq u_j$ in $\mathrm{FL}(X)$ for any pair $i \neq j$; if so, delete u_i. The resulting expression $v = \bigvee v_i$ evaluates to the same element as t in $\mathrm{FL}(X)$, and v satisfies (1), (2) and (3). Hence v is the canonical form of t.

If $w = \bigvee w_i$ canonically in $\mathrm{FL}(X)$, then the elements w_i are called the *canonical joinands* of w (dually, *canonical meetands*). It is important to note that these elements satisfy the refinement property of Lemma 6.6.

Corollary. *If w is a proper join in* $\mathrm{FL}(X)$ *and* $w = \bigvee U$*, then the set of canonical joinands of w refines U.*

This has an important structural consequence, observed by Bjarni Jónsson [16].

Theorem 6.8. *Free lattices satisfy the following implications, for all* $u, v, a, b, c \in \mathrm{FL}(X)$*:*

$$(SD_\vee) \quad \text{if } u = a \vee b = a \vee c \text{ then } u = a \vee (b \wedge c),$$

$$(SD_\wedge) \quad \text{if } v = a \wedge b = a \wedge c \text{ then } v = a \wedge (b \vee c).$$

The implications ($\mathrm{SD}_\vee$) and ($\mathrm{SD}_\wedge$) are known as the *semidistributive laws.* Exercises 5 and 6 concern properties of finite join semidistributive lattices.

Proof. We will prove that $\mathrm{FL}(X)$ satisfies ($\mathrm{SD}_\vee$); then ($\mathrm{SD}_\wedge$) follows by duality. We may assume that u is a proper join, for otherwise u is join irreducible and the implication is trivial. So let $u = u_1 \vee \ldots \vee u_n$ be the canonical join decomposition. By the Corollary above, $\{u_1, \ldots, u_n\}$ refines both $\{a, b\}$ and $\{a, c\}$. Any u_i that is not below a must be below both b and c, so in fact $\{u_1, \ldots, u_n\} \ll \{a, b \wedge c\}$. Hence

$$u = \bigvee u_i \leq a \vee (b \wedge c) \leq u \ ,$$

whence $u = a \vee (b \wedge c)$, as desired. □

Now let us recall some basic facts about free groups, so we can ask about their analogs for free lattices. Every subgroup of a free group is free, and the countably generated free group $FG(\omega)$ is isomorphic to a subgroup of $FG(2)$. Every identity which does not hold in all groups fails in some finite group.

Whitman used Theorem 6.3 and a clever construction to show that $\mathrm{FL}(\omega)$ can be embedded in $\mathrm{FL}(3)$. It is not known exactly which lattices are isomorphic to a sublattice of a free lattice, but certainly they are not all free. The simplest result (to state, not to prove) along these lines is due to the author [18].

Theorem 6.9. *A finite lattice can be embedded in a free lattice if and only if it satisfies* (W), ($SD_\vee$) *and* ($SD_\wedge$).

We can weaken the question somewhat and ask which ordered sets can be embedded in free lattices. A characterization of sorts for these ordered sets was found by Freese and Nation ([13] and [19]), but unfortunately it is not particularly enlightening. We obtain a better picture of the structure of free lattices by considering the following collection of results due to P. Crawley and R. A. Dean [4], B. Jónsson [16], and J. B. Nation and J. Schmerl [20], respectively.

Theorem 6.10. *Every countable ordered set can be embedded in* $\mathrm{FL}(3)$*. On the other hand, every chain in a free lattice is countable, so no uncountable chain can be embedded in a free lattice. If $\mathcal{P}$ is an infinite ordered set that can be embedded in* *a free lattice, then the dimension* $d(\mathcal{P}) \leq \mathfrak{m}$*, where* $\mathfrak{m}$ *is the smallest cardinal such that* $|\mathcal{P}| \leq 2^{\mathfrak{m}}$*.*

R. A. Dean showed that every equation that does not hold in all lattices fails in some finite lattice [9] (see Exercise 7.5). It turns out (though this is not obvious) that this is related to a beautiful structural result of Alan Day ([6], using [17]).

Theorem 6.11. *If* X *is finite, then* $\mathrm{FL}(X)$ *is weakly atomic.*

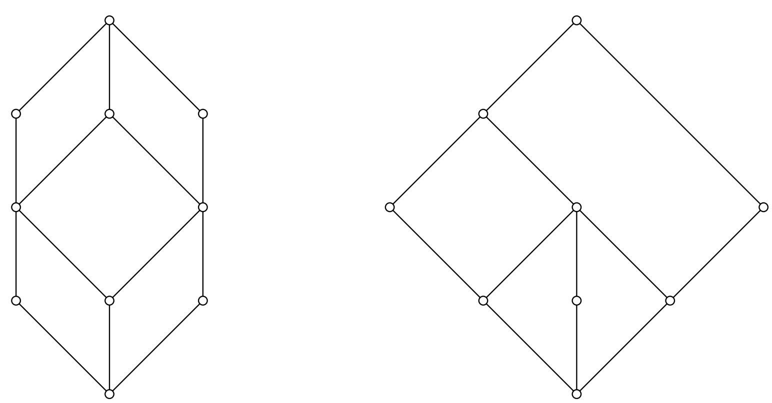
The book *Free Lattices* by Freese, Ježek and Nation [12] contains more information about the surprisingly rich structure of free lattices. Two papers of Ralph Freese contain analagous structure theory for finitely presented lattices [10], [11].

Chapter 2 of the *Free Lattice* book contains an introduction to upper and lower bounded lattices, a topic only hinted at in Exercise 11. These ideas grew from the work of Bjarni Jónsson and Ralph McKenzie; the paper [17] is a classic. For more recent results in this area, see Kira Adaricheva *et. al.* [1], [2] and the references therein.

Exercises for Chapter 6

1. Verify that if $\mathcal{L}$ is a lattice and I is an interval in $\mathcal{L}$, then $\mathcal{L}[I]$ is a lattice.

2. Use the doubling construction to repair the (W)-failures in the lattices in Figure 6.5. (Don't forget to double elements that are both join and meet reducible.) Then repeat the process until you either obtain a lattice satisfying (W), or else prove that you never will get one in finitely many steps.



(*a*) (*b*)

Figure 6.5

3. (a) Show that $x \wedge ((x \wedge y) \vee z) \not\leq y \vee (z \wedge (x \vee y))$ in $\mathrm{FL}(X)$.
(b) Find the canonical form of $x \wedge ((x \wedge y) \vee (x \wedge z))$.
(c) Find the canonical form of $(x \wedge ((x \wedge y) \vee (x \wedge z) \vee (y \wedge z))) \vee (y \wedge z)$.

4. There are five small lattices that fail $\mathrm{SD}_\vee$, but have no proper sublattice failing $\mathrm{SD}_\vee$. Find them.

5. Show that the following conditions are equivalent (to $\mathrm{SD}_\vee$) in a finite lattice.

(a) $u = a \vee b = a \vee c$ implies $u = a \vee (b \wedge c)$.
(b) For each $m \in M(\mathcal{L})$ there is a unique $j \in J(\mathcal{L})$ such that for all $x \in L$, $m^* \wedge x \not\leq m$ iff $x \geq j$.
(c) For each $a \in L$, there is a set $C \subseteq J(\mathcal{L})$ such that $a = \bigvee C$, and for every subset $B \subseteq L$, $a = \bigvee B$ implies $C \ll B$.
(d) $u = \bigvee_i u_i = \bigvee_j v_j$ implies $u = \bigvee_{i,j}(u_i \wedge v_j)$.

In a finite lattice satisfying these conditions, the elements of the set C given by part (c) are called the *canonical joinands* of a. See Jónsson and Kiefer [15].

6. An element $p \in L$ is *join prime* if $p \leq x \vee y$ implies $p \leq x$ or $p \leq y$; *meet prime* is defined dually. Let $\mathrm{JP}(\mathcal{L})$ denote the set of all join prime elements of $\mathcal{L}$, and let $\mathrm{MP}(\mathcal{L})$ denote the set of all meet prime elements of $\mathcal{L}$. Consider a finite lattice $\mathcal{L}$ satisfying $\mathrm{SD}_\vee$.

(a) Prove that the canonical joinands of 1 are join prime.
(b) Prove that the coatoms of $\mathcal{L}$ are meet prime.
(c) Show that for each $q \in \mathrm{MP}(\mathcal{L})$ there exists a unique element $\eta(q) \in \mathrm{JP}(\mathcal{L})$ such that L is the disjoint union of $\downarrow q$ and $\uparrow \eta(q)$.

7. Prove Lemma 6.5.

8. Let $\mathcal{A}$ and $\mathcal{B}$ be lattices, and let $X \subseteq A$ generate $\mathcal{A}$. Prove that a map $h_0 : X \to \mathcal{B}$ can be extended to a homomorphism $h : \mathcal{A} \to \mathcal{B}$ if and only if, for every pair of lattice terms p and q, and all $x_1, \ldots, x_n \in X$,

$$p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n) \text{ implies } p(h_0(x_1), \ldots, h_0(x_n)) = q(h_0(x_1), \ldots, h_0(x_n)).$$

9. A complete lattice $\mathcal{L}$ has *canonical decompositions* if for each $a \in L$ there exists a set C of completely meet irreducible elements such that $a = \bigwedge C$ irredundantly, and $a = \bigwedge B$ implies $C \gg B$. Prove that an upper continuous lattice has canonical decompositions if and only if it is strongly atomic and satisfies $\mathrm{SD}_\wedge$ (Viktor Gorbunov [14]).

For any ordered set $\mathcal{P}$, a lattice $\mathcal{F}$ is said to be *freely generated by* $\mathcal{P}$ if $\mathcal{F}$ contains a subset P such that

(1) P with the order it inherits from $\mathcal{F}$ is isomorphic to $\mathcal{P}$,
(2) P generates $\mathcal{F}$,
(3) for every lattice $\mathcal{L}$, every order preserving map $h_0 : P \to \mathcal{L}$ can be extended to a homomorphism $h : \mathcal{F} \to \mathcal{L}$.

In much the same way as with free lattices, we can show that there is a unique (up to isomorphism) lattice $\mathrm{FL}(\mathcal{P})$ generated by any ordered set $\mathcal{P}$. Indeed, free lattices $\mathrm{FL}(X)$ are just the case when $\mathcal{P}$ is an antichain.

10. (a) Find the lattice freely generated by $\{x, y, z\}$ with $x \geq y$.
(b) Find FL($\mathcal{P}$) for $\mathcal{P} = \{x_0, x_1, x_2, z\}$ with $x_0 \leq x_1 \leq x_2$.

The lattice freely generated by $\mathcal{Q} = \{x_0, x_1, x_2, x_3, z\}$ with $x_0 \leq x_1 \leq x_2 \leq x_3$ is infinite, as is that generated by $\mathcal{R} = \{x_0, x_1, y_0, y_1\}$ with $x_0 \leq x_1$ and $y_0 \leq y_1$ (Yu. I. Sorkin [24], see [21]).

11. A homomorphism $h : \mathcal{L} \to \mathcal{K}$ is *lower bounded* if for each $a \in K$, $\{x \in L : h(x) \geq a\}$ is either empty or has a least element $\beta(a)$. For example, if $\mathcal{L}$ satisfies the DCC, then h is lower bounded. We regard β as a partial map from $\mathcal{K}$ to $\mathcal{L}$. Let $h : \mathcal{L} \to \mathcal{K}$ be a lower bounded homomorphism.

(a) Show that the domain of β is an ideal of $\mathcal{K}$.
(b) Prove that β preserves finite joins.
(c) Show that if h is onto and $\mathcal{L}$ satisfies $\mathrm{SD}_\vee$, then so does $\mathcal{K}$.

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