SPERNER'S LEMMA: SUMMARY

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Let P be a finite ordered set and let L be the set of all maximal-sized antichains.

Theorem 1 (Dilworth)**.** *L is a lattice.*

Let $G = \mathrm{Aut}(P)$ be the group of automorphisms of P. If $A \in L$ and $\sigma \in G$ then $\sigma(A) \in L$; so G acts on L. Since the greatest element of L is clearly fixed by every element of G, we have the following.

Theorem 2 (Kleitman, et al.)**.** *There is an $A \in L$ invariant under every $\sigma \in G$.*

Let X be a finite set and consider three P's:

- P is all subsets of X.
- P is all partitions of X.
- P is all subspaces of a finite dimensional vector space over a finite field.

Each of these P's has "levels." (In the second example, the levels are all partitions with k blocks, for a fixed k.) In the first and third example $\mathrm{Aut}(P)$ acts transitively on the levels. So in these cases, if $a \in A$, where A is a maximal-sized antichain invariant under every σ, then all of the elements in the level containing a are contained in A; so A is just that level since the levels are maximal antichains. So the size of the largest antichain in the first example is the number of subsets of size $\lceil |X|/2 \rceil$ and the size of the largest antichain of the lattice of subspaces on an n-dimensional vectors is the number of subspaces of dimension $\lceil n/2 \rceil$.

But since the automorphisms of the partition lattice are not transitive, the argument doesn't work in this case. (And in fact the size of a maximal-sized antichain can exceed the size of every level.)

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