THREE EASY EXERCISES

1. Flats in Affine Geometry

We want to describe the flat containing points $\mathbf{a}_1, \ldots, \mathbf{a}_n$ in the affine geometry F^n for a field F. This will be a flat of geometric dimension at most $k-1$. Show that the following are equivalent for a subset $S \subseteq F^n$.

(1) $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\} \subseteq S$, and if $\mathbf{p}$, $\mathbf{q} \in S$ then $s\mathbf{p} + t\mathbf{q} \in S$ whenever $s + t = 1$.
(2) $S = \{c_1\mathbf{a}_1 + \cdots + c_k\mathbf{a}_k : \sum_{i=1}^{k} c_i = 1\}$
(3) $S = \mathbf{a}_1 + \mathrm{Span}(\mathbf{a}_2 - a_1, \ldots, a_k - a_1)$, a translate (coset) of a $(k-1)$-dimensional linear subspace of F^n.

2. Closure Operators

This exercise explores the theme of M. Ward that closure operators are most naturally dealt with in the context of operators on complete lattices (including, but not limited to, $\mathrm{Pow}(X)$). To recall for a complete lattice L, $\varphi : L \to L$ is a *closure operator* if

- $x \leq \varphi(x)$
- $x \leq y$ implies $\varphi(x) \leq \varphi(y)$
- $\varphi(\varphi(x)) = \varphi(x)$.

A closure operator φ is *continuous* if

- for every up-directed set $D \subseteq L$, $\varphi(\bigvee D) = \bigvee\{\varphi(d) : d \in D\}$.

If L is algebraic and K denotes the semilattice of compact elements of L, then φ is *algebraic* if

- $\varphi(x) = \bigvee\{c \in K : c \leq x\}$.

Prove the following claims.

(1) Suppose that $\rho : L \to L$ is an isotone increasing function, i.e., it satisfies the first two properties of being a closure operator. For $x \in L$, define $\tau(x) = \bigwedge\{s \in L : \rho(s) = s \text{ and } s \geq x\}$. Then τ is a closure operator.
(2) If φ is a closure operator on L, then $\{s \in L : \varphi(s) = s\}$ is a complete meet subsemilattice of L.
(3) Let S be a complete meet subsemilattice of L. For $x \in L$, define $\sigma(x) = \bigwedge\{s \in S : s \geq x\}$. Then σ is a closure operator.

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(4) Moreover, σ is continuous iff S is an algebraic subset of L, i.e., also closed under non-empty directed joins.
(5) Assume that L is algebraic. Then a closure operator φ is continuous iff it is algebraic.

3. Order Ideals

Let $R \subseteq A \times B$ be a binary relation. For $a \in A$, $b \in B$ let

$$Ra = \{b \in B : aRb\}$$
$$R^{-1}b = \{a \in A : aRb\} \ .$$

A subset $I \subseteq A$ is an *ideal* if

$$(\exists i \in I \quad Ri \subseteq Ra) \implies a \in I \ .$$

Likewise, $F \subseteq B$ is a *filter* if

$$(\exists f \in F \quad R^{-1}f \subseteq R^{-1}b) \implies b \in F \ .$$

Show that ideals and filters are closure systems on their respective sets, and that a set union of ideals or filters is again one. Give an example to show that the lattices of ideals and filters need not be dually isomorphic.

(For comparison, the MacNeille completion on A is given by the condition that $X \subseteq A$ is *closed* if

$$(\bigcap_{x \in X} Rx \subseteq Ra) \implies a \in X$$

and similarly for subsets of B using R^{-1}.)