MINIMAL BIG LATTICES

1. Reduction

This is a separate document to temporarily input the results on minimal big lattices. The reduction goes as follows.

Theorem 1. *If* $\mathbf{L}$ *is a finite lattice which is big and not both semidistributive and breadth 2, then* $\mathbf{L}$ *contains a sublattice isomorphic to one of the big lattices* $\mathbf{E}_1$, $\mathbf{E}_1^d$, $\mathbf{E}_2$, $\mathbf{E}_3$, $\mathbf{E}_3^d$, $\mathbf{E}_4$, $\mathbf{E}_5$, $\mathbf{E}_5^d$, $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_7^d$, $\mathbf{E}_8$, $\mathbf{E}_9$, $\mathbf{E}_9^d$, $\mathbf{E}_{10}$, $\mathbf{E}_{10}^d$, $\mathbf{E}_{11}$, $\mathbf{E}_{11}^d$, $\mathbf{E}_{12}$, $\mathbf{E}_{12}^d$, $\mathbf{E}_{15}$, $\mathbf{E}_{15}^d$, $\mathbf{E}_{16}$, $\mathbf{E}_{16}^d$, $\mathbf{E}_{17}$, $\mathbf{E}_{17}^d$, $\mathbf{E}_{18}$, $\mathbf{E}_{18}^d$, *or a sublattice which is big, semidistributive and breadth 2.*

Theorem 2. *If* $\mathbf{L}$ *is a finite, semidistributive, breadth 2, big lattice, then* $\mathbf{L}$ *contains a sublattice of the form* $(p \vee k)/p \,\cup\, k/(p \wedge k)$ *such that* $\mathbf{FQ}(p,k)$ *is infinite.*

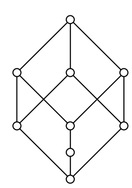

Figure 1. $\mathbf{E}_1$

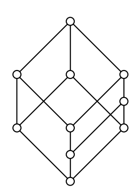

Figure 2. $\mathbf{E}_2$

Date: November 10, 1997.

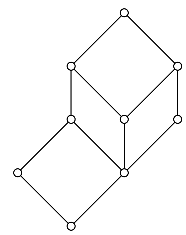

Figure 3. $\mathbf{E}_3$

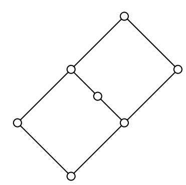

Figure 4. $\mathbf{E}_4$

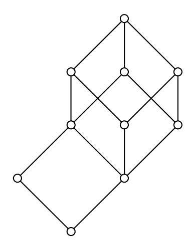

Figure 5. $\mathbf{E}_5$

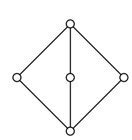

Figure 6. $\mathbf{E}_6$

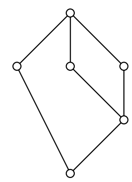

FIGURE 7. $\mathbf{E}_7$

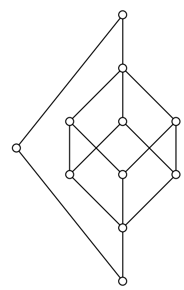

FIGURE 8. $\mathbf{E}_8$

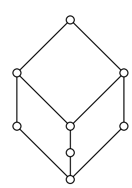

FIGURE 9. $\mathbf{E}_9$

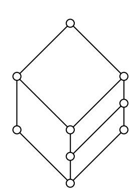

FIGURE 10. $\mathbf{E}_{10}$

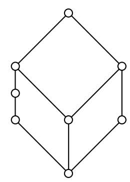

Figure 11. $\mathbf{E}_{11}$

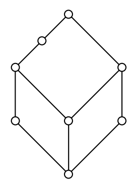

Figure 12. $\mathbf{E}_{12}$

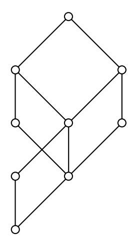

Figure 13. $\mathbf{E}_{15}$

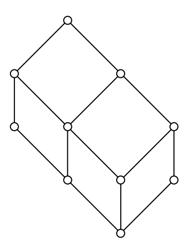

Figure 14. $\mathbf{E}_{16}$

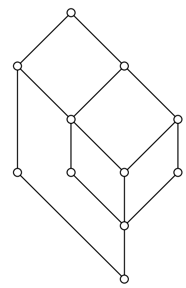

Figure 15. $\mathbf{E}_{17}$

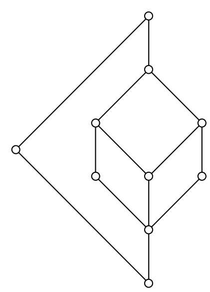

Figure 16. $\mathbf{E}_{18}$

2. When $\mathbf{FQ}(p,k)$ is infinite

Throughout this section $\mathbf{L}$ will be a finite, breadth 2, semidistributive lattice containing elements p, k such that $p \vee k = 1_{\mathbf{L}}$, $p \wedge q = 0_{\mathbf{L}}$ and $L = 1/p \cup k/0$.

Theorem 3. *If $\mathbf{L} - \{p,k\}$ contains a 3 element antichain, then $\mathbf{L}$ has a sublattice isomorphic to one of* $\mathbf{E}_{19}$, $\mathbf{E}_{20}$, $\mathbf{E}_{20}^d$, $\mathbf{E}_{21}$, $\mathbf{E}_{22}^*$, $\mathbf{E}_{22}^{*d}$, $\mathbf{E}_{23}$, $\mathbf{E}_{23}^d$, $\mathbf{E}_{24}$, $\mathbf{E}_{24}^d$, $\mathbf{E}_{25}$, $\mathbf{E}_{25}^d$, $\mathbf{E}_{26}$, $\mathbf{E}_{26}^d$, $\mathbf{E}_{27}$, $\mathbf{E}_{27}^d$, $\mathbf{T}_4$, $\mathbf{T}_4^d$, $\mathbf{T}_5$, $\mathbf{T}_6$, $\mathbf{T}_{16}$, $\mathbf{T}_{16}^d$, $\mathbf{T}_{22}$, $\mathbf{T}_{22}^d$, $\mathbf{T}_{23}$, $\mathbf{T}_{23}^d$, $\mathbf{T}_{24}$, $\mathbf{T}_{24}^d$, $\mathbf{T}_{25}$, $\mathbf{T}_{25}^d$, $\mathbf{T}_{26}$, $\mathbf{T}_{26}^d$, $\mathbf{T}_{27}$, $\mathbf{T}_{27}^d$, $\mathbf{T}_{29}$, $\mathbf{T}_{29}^d$, $\mathbf{T}_{30}$, $\mathbf{T}_{30}^d$.

The proof divides into a couple of cases and their duals.

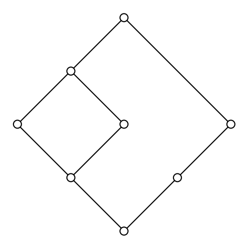

Figure 17. $\mathbf{E}_{19}$

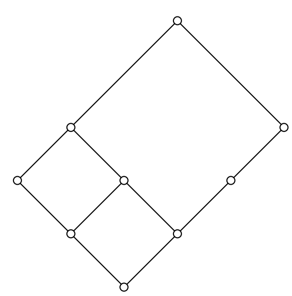

Figure 18. $\mathbf{E}_{20}$

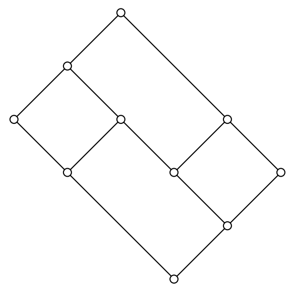

FIGURE 19. $\mathbf{E}_{21}$

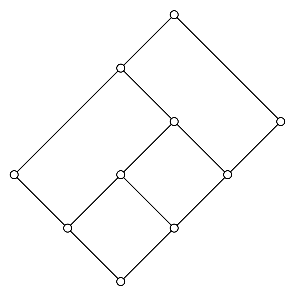

FIGURE 20. $\mathbf{E}_{22}$

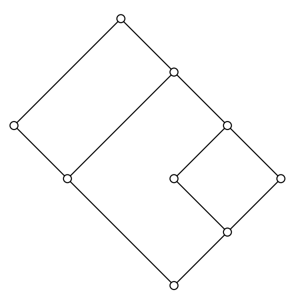

FIGURE 21. $\mathbf{E}_{23}$

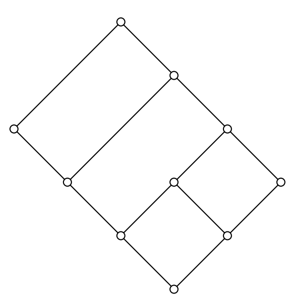

FIGURE 22. $\mathbf{E}_{24}$

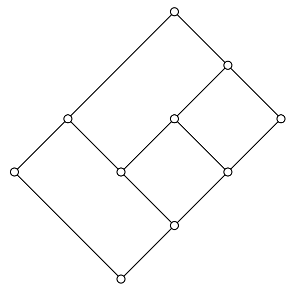

FIGURE 23. $\mathbf{E}_{25}$

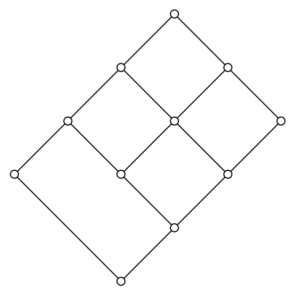

FIGURE 24. $\mathbf{E}_{26}$

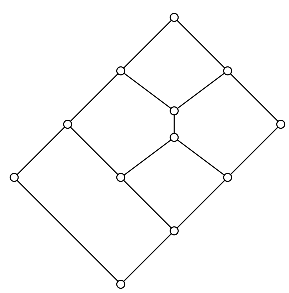

FIGURE 25. $\mathbf{E}_{27}$

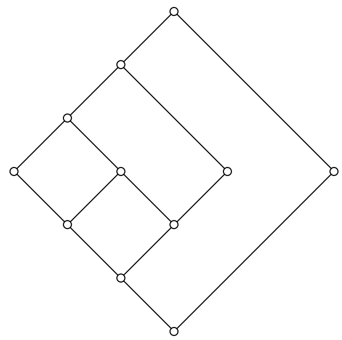

Figure 26. $\mathbf{T}_4$

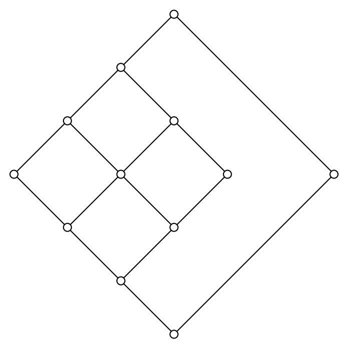

Figure 27. $\mathbf{T}_5$

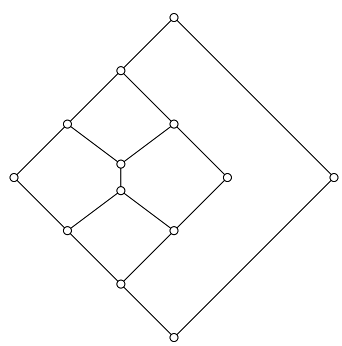

Figure 28. $\mathbf{T}_6$

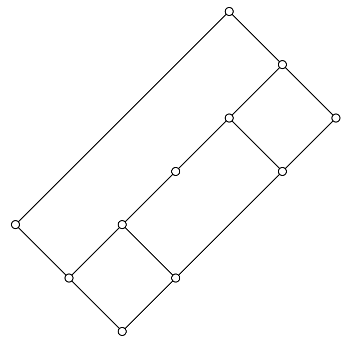

Figure 29. $\mathbf{T}_{16}$

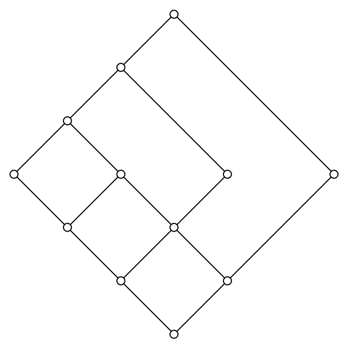

Figure 30. $\mathbf{T}_{22}$

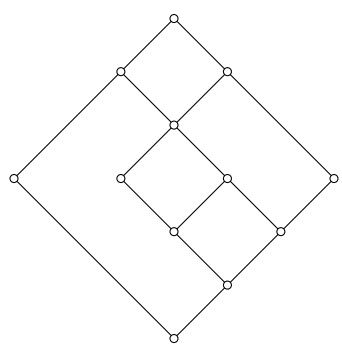

Figure 31. $\mathbf{T}_{23}$

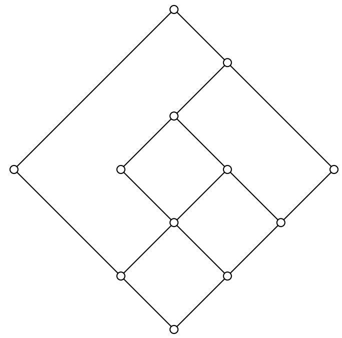

FIGURE 32. $\mathbf{T}_{24}$

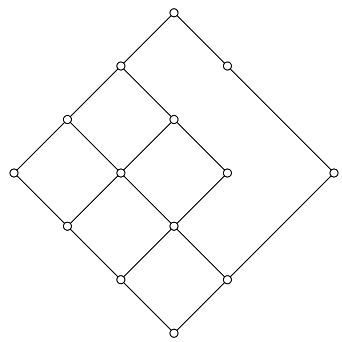

FIGURE 33. $\mathbf{T}_{25}$

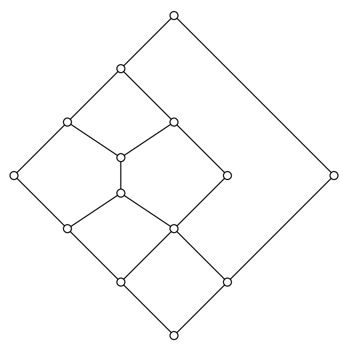

FIGURE 34. $\mathbf{T}_{26}$

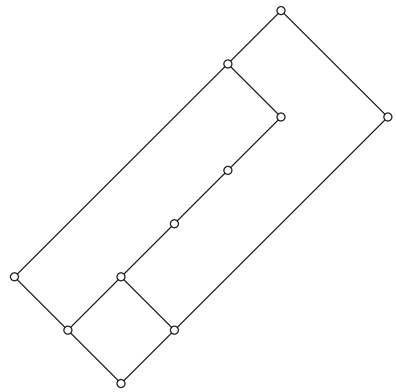

FIGURE 35. $\mathbf{T}_{27}$

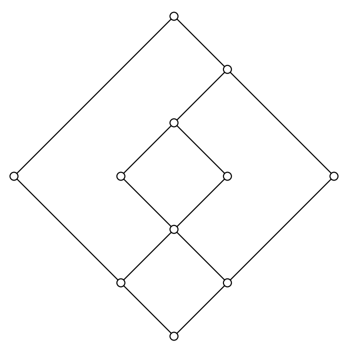

FIGURE 36. $\mathbf{T}_{29}$

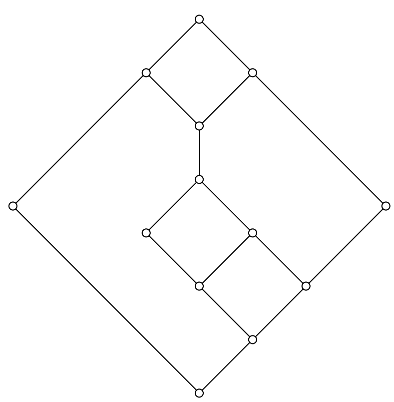

FIGURE 37. $\mathbf{T}_{30}$

Theorem 4. *If* $\mathbf{L}-\{p,k\}$ *contains elements* $x_0 < x_1$, $y_0 < y_1$ *with* $x_0 \nleq y_1$, $y_0 \nleq x_1$, *and satisfying*

1. $p < x_0$ *implies* $x_1 \wedge (x_0 \vee y_1) > x_0$,
2. $p < y_0$ *implies* $y_1 \wedge (y_0 \vee x_1) > y_0$,
3. $x_1 < k$ *implies* $x_0 \vee (x_1 \wedge y_0) < x_1$,
4. $y_1 < k$ *implies* $y_0 \vee (y_1 \wedge x_0) < y_1$,

then $\mathbf{L}$ *has a sublattice isomorphic to one of* $\mathbf{E}_{28}$, $\mathbf{E}_{28}^d$, $\mathbf{E}_{29}$, $\mathbf{E}_{30}$, $\mathbf{E}_{30}^d$, $\mathbf{E}_{31}$, $\mathbf{E}_{31}^d$, $\mathbf{E}_{32}$, $\mathbf{E}_{32}^d$, $\mathbf{E}_{33}$, $\mathbf{E}_{33}^d$, $\mathbf{E}_{34}$, $\mathbf{E}_{35}$, $\mathbf{E}_{36}$, $\mathbf{E}_{36}^d$, $\mathbf{E}_{37}$, $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_2^d$, $\mathbf{T}_{15}^*$, $\mathbf{T}_{15}^{*d}$, $\mathbf{T}_{17}$, $\mathbf{T}_{17}^d$, $\mathbf{T}_{18}$, $\mathbf{T}_{18}^d$, *or one of the previous lattices.*

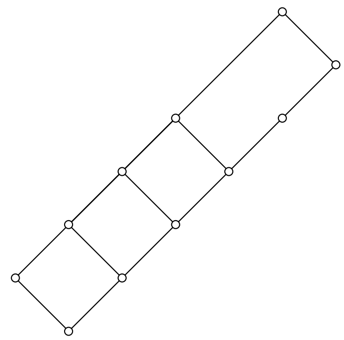

FIGURE 38. $\mathbf{E}_{28}$

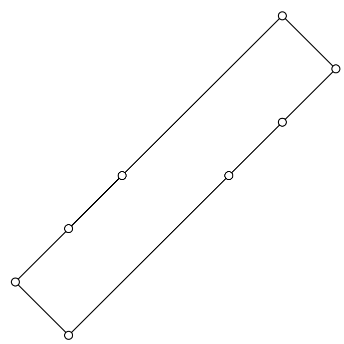

FIGURE 39. $\mathbf{E}_{29}$

Theorem 5. *If* $\mathbf{L} - \{p, k\}$ *contains elements* x *and* $y_0 < y_1 < y_2 < y_3$ *with* $x \nleq y_3$, $y_0 \nleq x$, *and satisfying*

1. $y_3 > p$ *implies* $y_3 \wedge (x \vee p) \nleq y_2$,
2. $(x \vee p) \wedge (y_3 \vee p) > p$,
3. $y_0 < k$ *implies* $y_0 \vee (x \wedge k) \ngeq y_1$,
4. $(x \wedge k) \vee (y_0 \wedge k) < k$,

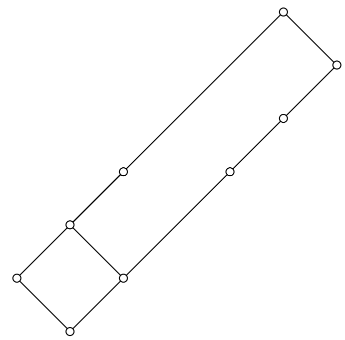

FIGURE 40. $\mathbf{E}_{30}$

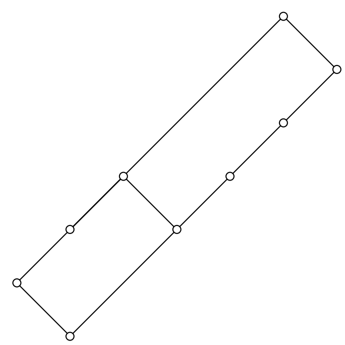

FIGURE 41. $\mathbf{E}_{31}$

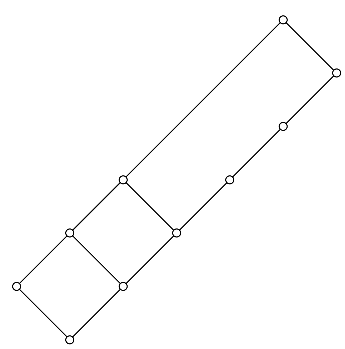

FIGURE 42. $\mathbf{E}_{32}$

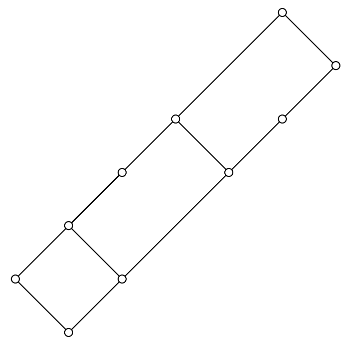

FIGURE 43. $\mathbf{E}_{33}$

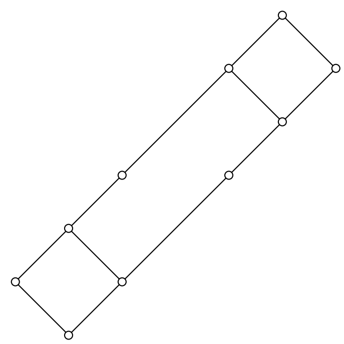

FIGURE 44. $\mathbf{E}_{34}$

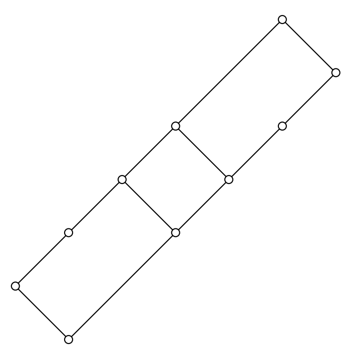

FIGURE 45. $\mathbf{E}_{35}$

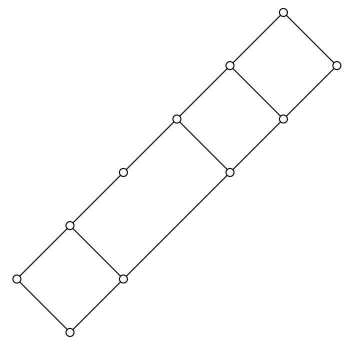

Figure 46. $\mathbf{E}_{36}$

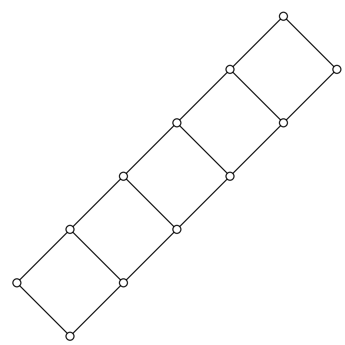

Figure 47. $\mathbf{E}_{37}$

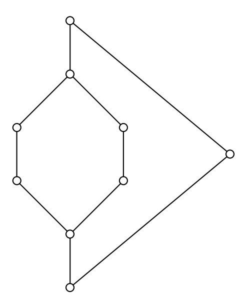

Figure 48. $\mathbf{T}_1$

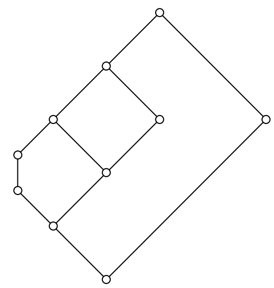

FIGURE 49. $\mathbf{T}_2$

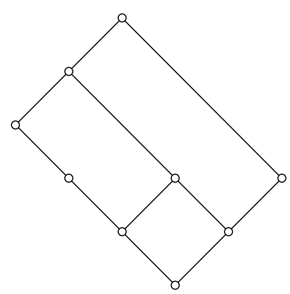

FIGURE 50. $\mathbf{T}_{15}$

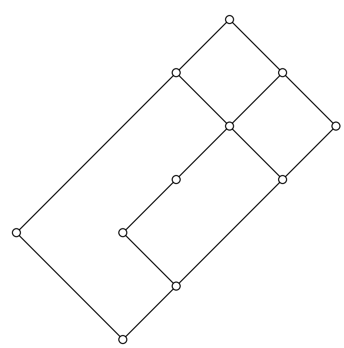

FIGURE 51. $\mathbf{T}_{17}$

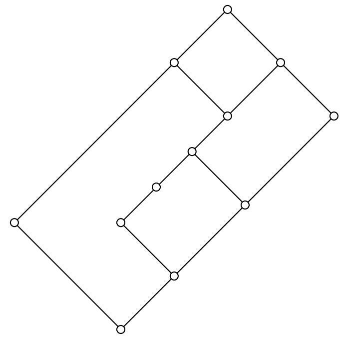

FIGURE 52. $\mathbf{T}_{18}$

then $\mathbf{L}$ *has a sublattice isomorphic to one of* $\mathbf{E}_{38}$, $\mathbf{E}_{39}$, $\mathbf{E}_{39}^d$, $\mathbf{E}_{40}$, $\mathbf{E}_{40}^d$, $\mathbf{E}_{41}$, $\mathbf{E}_{41}^d$, $\mathbf{E}_{42}$, $\mathbf{E}_{42}^d$, $\mathbf{E}_{43}$, $\mathbf{E}_{43}^d$, $\mathbf{E}_{44}$, $\mathbf{E}_{44}^d$, $\mathbf{E}_{45}$, $\mathbf{E}_{45}^d$, $\mathbf{T}_{21}$, $\mathbf{T}_{21}^d$, $\mathbf{T}_{51}$, $\mathbf{T}_{51}^d$, $\mathbf{T}_{52}$, $\mathbf{T}_{52}^d$, $\mathbf{T}_{53}$, $\mathbf{T}_{53}^d$, $\mathbf{T}_{54}$, $\mathbf{T}_{54}^d$, $\mathbf{T}_{55}$, $\mathbf{T}_{55}^d$, $\mathbf{T}_{56}$, $\mathbf{T}_{56}^d$, *or one of the previous lattices.*

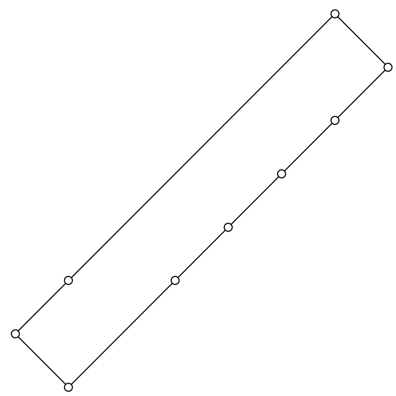

FIGURE 53. $\mathbf{E}_{38}$

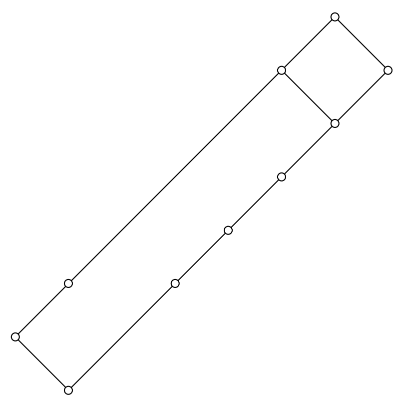

FIGURE 54. $\mathbf{E}_{39}$

Theorem 6. *If* $\mathbf{L} - \{p, k\}$ *contains elements* x *and* $y_0 < y_1 < y_2 < y_3 < y_4$ *with* $x \not\leq y_4$, $y_0 \not\leq x$, *and satisfying*

1. $y_3 > p$ *implies* $y_4 \wedge (x \vee y_3) > y_3$,
2. $y_1 < k$ *implies* $y_0 \vee (x \wedge y_1) < y_1$,

then $\mathbf{L}$ *has a sublattice isomorphic to one of* $\mathbf{T}_7$, $\mathbf{T}_{61}$, $\mathbf{T}_{61}^d$, $\mathbf{T}_{62}$, $\mathbf{T}_{62}^d$, $\mathbf{T}_{63}$, $\mathbf{T}_{63}^d$, $\mathbf{T}_{64}$, $\mathbf{T}_{64}^d$, $\mathbf{T}_{65}$, $\mathbf{T}_{65}^d$, $\mathbf{T}_{66}$, $\mathbf{T}_{66}^d$, $\mathbf{T}_{67}$, $\mathbf{T}_{67}^d$, *or one of the previous lattices.*

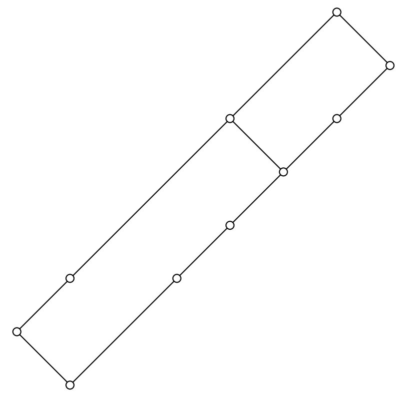

Figure 55. $\mathbf{E}_{40}$

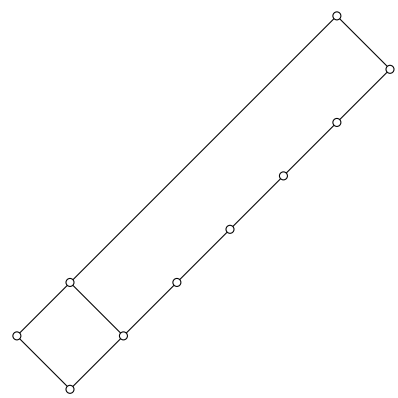

Figure 56. $\mathbf{E}_{41}$

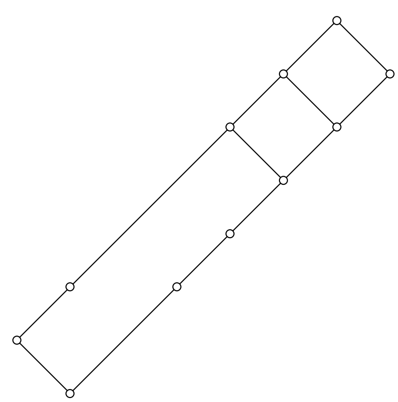

Figure 57. $\mathbf{E}_{42}$

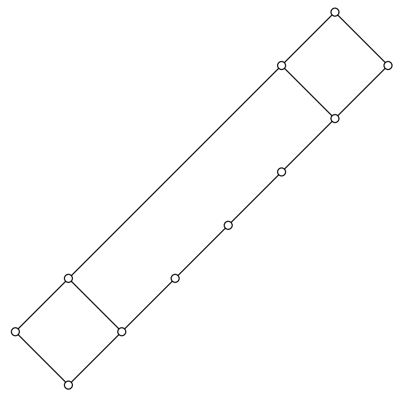

FIGURE 58. $\mathbf{E}_{43}$

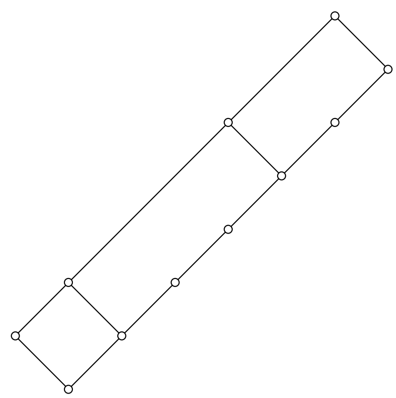

FIGURE 59. $\mathbf{E}_{44}$

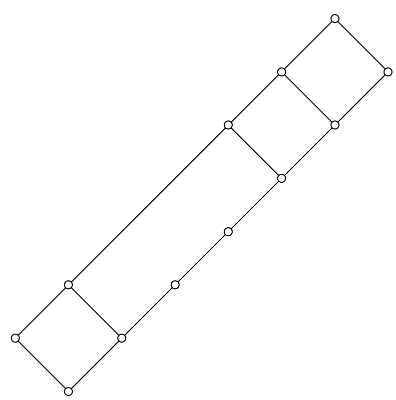

FIGURE 60. $\mathbf{E}_{45}$

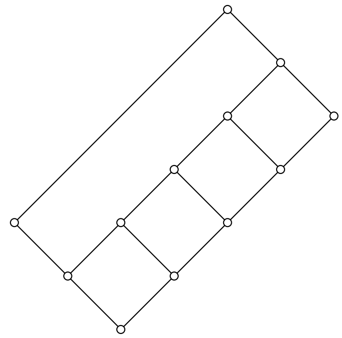

FIGURE 61. $\mathbf{T}_{21}$

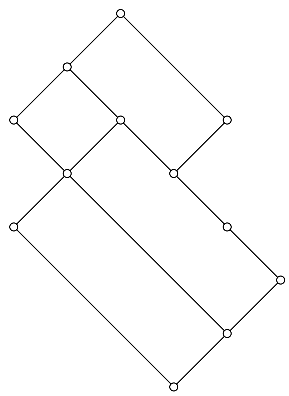

FIGURE 62. $\mathbf{T}_{51}$

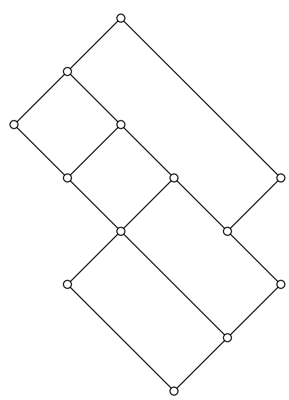

FIGURE 63. $\mathbf{T}_{52}$

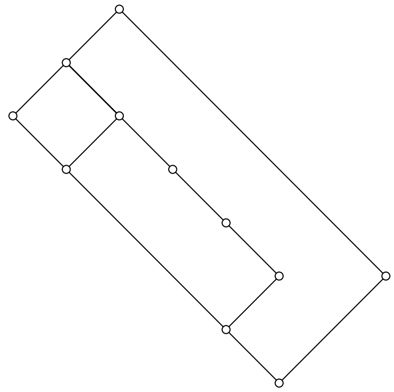

FIGURE 64. $\mathbf{T}_{53}$

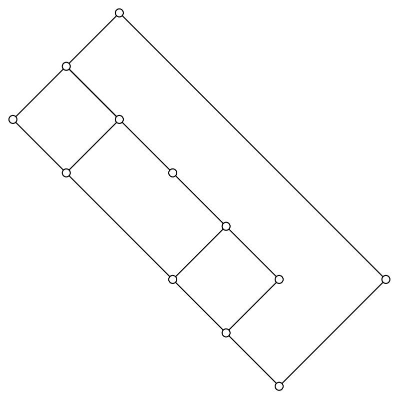

FIGURE 65. $\mathbf{T}_{54}$

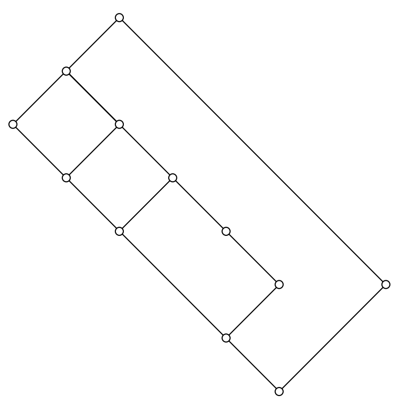

FIGURE 66. $\mathbf{T}_{55}$

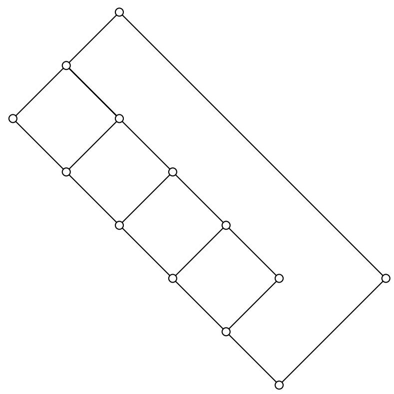

FIGURE 67. $\mathbf{T}_{56}$

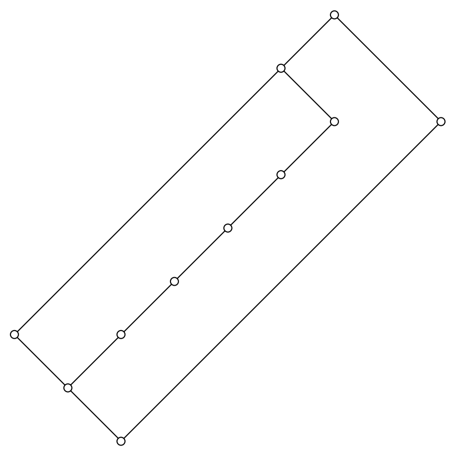

FIGURE 68. $\mathbf{T}_{7}$

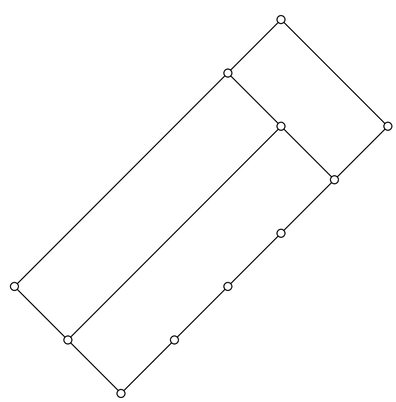

FIGURE 69. $\mathbf{T}_{61}$

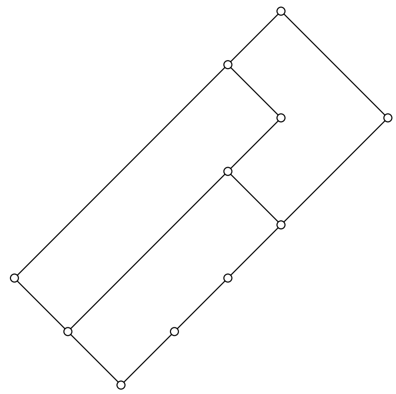

FIGURE 70. $\mathbf{T}_{62}$

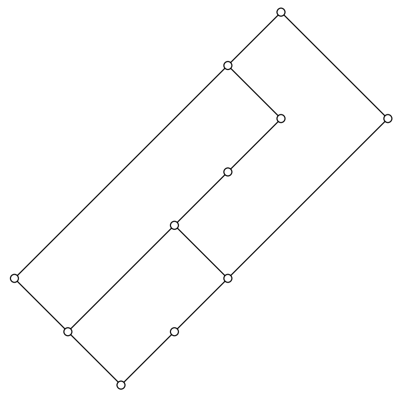

FIGURE 71. $\mathbf{T}_{63}$

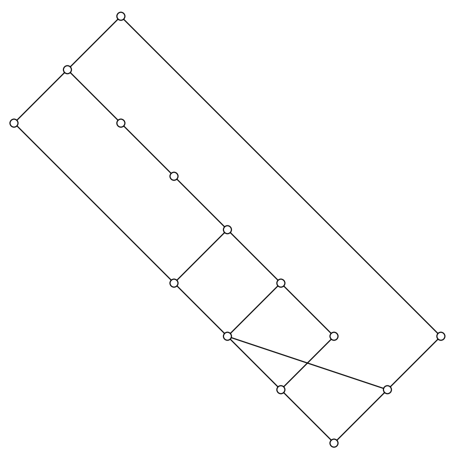

FIGURE 72. $\mathbf{T}_{64}$

Theorem 7. *Let* $\mathbf{L}$ *be a finite, semidistributive, breadth 2 lattice satisfying the following conditions.*

1. $L = 1/p \cup k/0$.
2. $\mathbf{L} - \{p, k\}$ *contains no 3 element antichain.*
3. *If* $\mathbf{L} - \{p, k\}$ *contains elements* $x_0 < x_1$, $y_0 < y_1$ *with* $x_0 \not\leq y_1$, $y_0 \not\leq x_1$, *then either*
   (a) $p < x_0$ *and* $x_1 \wedge (x_0 \vee y_1) = x_0$, *or*
   (b) $p < y_0$ *and* $y_1 \wedge (y_0 \vee x_1) = y_0$, *or*
   (c) $x_1 < k$ *and* $x_0 \vee (x_1 \wedge y_0) = x_1$, *or*
   (d) $y_1 < k$ *and* $y_0 \vee (y_1 \wedge x_0) = y_1$.
4. *If* $\mathbf{L} - \{p, k\}$ *contains elements* x *and* $y_0 < y_1 < y_2 < y_3$ *with* $x \not\leq y_3$, $y_0 \not\leq x$, *then either*
   (a) $y_3 > p$ *and* $y_3 \wedge (x \vee p) \leq y_2$, *or*

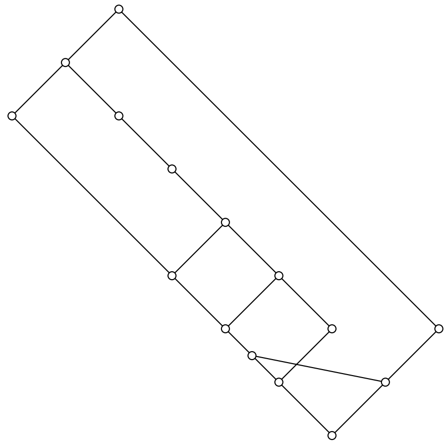

FIGURE 73. $\mathbf{T}_{65}$

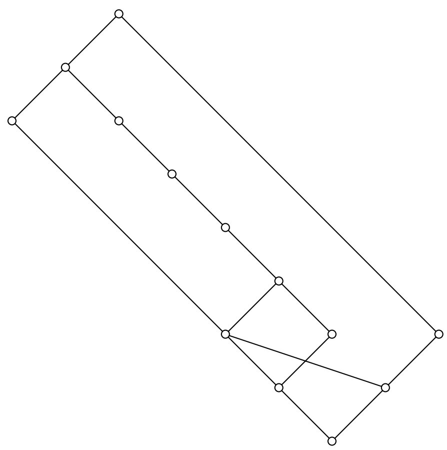

FIGURE 74. $\mathbf{T}_{66}$

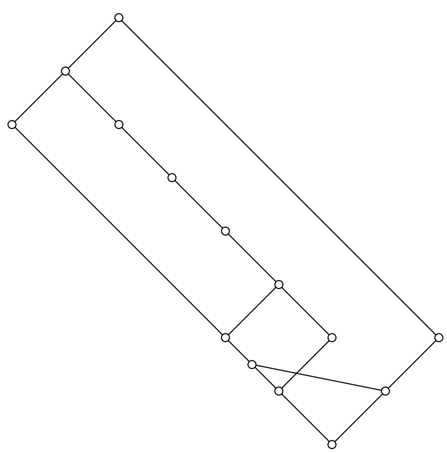

FIGURE 75. $\mathbf{T}_{67}$

(b) $(x \vee p) \wedge (y_3 \vee p) = p$, *or*
(c) $y_0 < k$ *and* $y_0 \vee (x \wedge k) \geq y_1$, *or*
(d) $(x \wedge k) \vee (y_0 \wedge k) = k$.

5. *If* $\mathbf{L} - \{p, k\}$ *contains elements* x *and* $y_0 < y_1 < y_2 < y_3 < y_4$ *with* $x \not\leq y_4$, $y_0 \not\leq x$, *then either*
   (a) $y_3 > p$ *and* $y_4 \wedge (x \vee y_3) = y_3$, *or*
   (b) $y_1 < k$ *and* $y_0 \vee (x \wedge y_1) = y_1$.

Then the finitely presented lattice $\mathbf{FQ}(p, k)$ *is finite.*