A Simple Semidistributive Lattice

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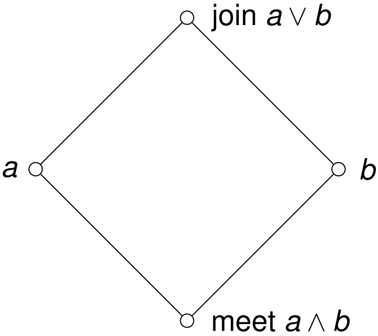
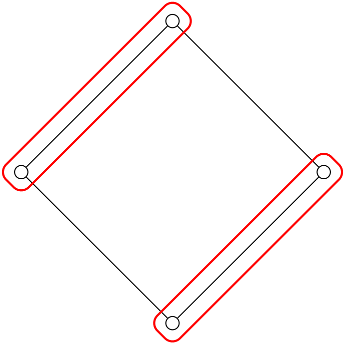
This is a lattice



This is a 2-element lattice

This is a lattice homomorphism onto 2



Prime ideals and filters

If $h : L \twoheadrightarrow 2$, then $h^{-1}(0)$ is a **prime ideal**

- $x \leq y \in I \to x \in I$
- $x, y \in I \to x \vee y \in I$
- $x \wedge y \in I \to x \in I$ or $y \in I$.

Its complement $h^{-1}(1)$ is a **prime filter**

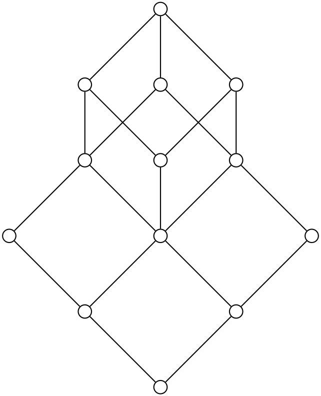
- $x \leq y \in F \to x \in F$
- $x, y \in F \to x \wedge y \in F$
- $x \vee y \in F \to x \in F$ or $y \in F$.

There is a bijection between proper prime ideals of L and homomorphisms $h : L \to 2$.

Distributive lattices

A lattice is **distributive** if it satisfies

$$x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$$

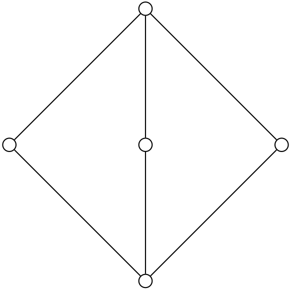


Prime ideals in distributive lattices

Distributive lattices get **choke** prime ideals:

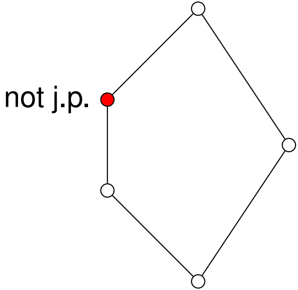
If $x \neq 0$, then every ideal that is maximal with respect to $x \notin I$ is prime.

On the other hand, the non-distributive lattice M_3 has none.
It is simple and has NO proper, nontrivial homomorphic image.



Join prime elements

An element $p \in L$ is **join prime** if $x \vee y \geq p \rightarrow x \geq p$ or $y \geq p$.
That makes $\{x : x \not\geq p\}$ a prime ideal.

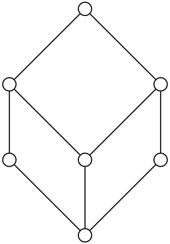


A finite lattice is distributive iff every join irreducible element is join prime.

Semidistributive lattices

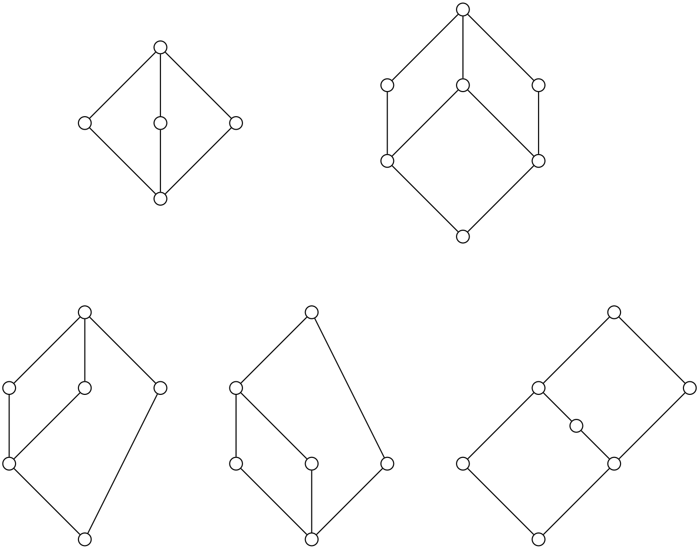
A lattice is **join semidistributive** if it satisfies

$$(\mathrm{SD}_\vee) \qquad x \vee y \approx x \vee z \to x \vee y \approx x \vee (y \wedge z)$$



Meet semidistributive is the dual. **Semidistributive** is both.

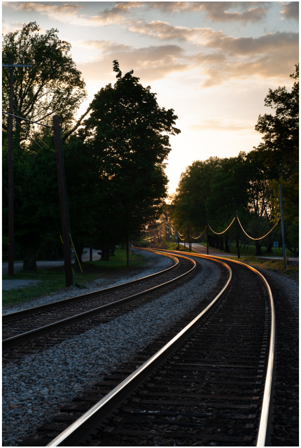
Jónsson and Rival: minimal $(\mathrm{SD}_\vee)$ failures



If L fails $(\mathrm{SD}_\vee)$, then one of these embeds into $\mathrm{Fil}(\mathrm{Id}(L))$.

McKenzie's question

Is there a simple semidistributive lattice with more than 2 elements?



Classical theorem

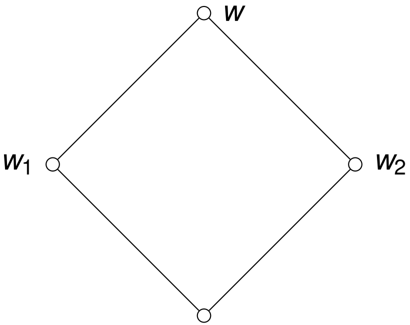
If a join semidistributive lattice has a largest element 1, then it has a prime ideal.

Proof: Let I be an ideal maximal with respect to $1 \notin I$.
If $x, y \notin I$, then there is an $i \in I$ such that $i \vee x = i \vee y = 1$.
Then $(\mathrm{SD}_\vee)$ gives $i \vee (x \wedge y) = 1$, whence $x \wedge y \notin I$.

Corollary: There is no simple, finite, join semidistributive lattice except 2.

Where does semidistributivity come from? (not the stork)

- Skolem and Whitman solved the word problem for free lattices.
- Every $w \in \mathrm{FL}(X)$ is either join irreducible or has a **canonical join representation** $w = w_1 \vee \ldots \vee w_k$ such that $\{w_1, \ldots, w_k\}$ refines every other join representation.
- This implies $(\mathrm{SD}_\vee)$.



Jónsson and Kiefer

The following are equivalent for a finite lattice.

- L satisfies

$$(\mathrm{SD}_\vee) \qquad x \vee y \approx x \vee z \to x \vee y \approx x \vee (y \wedge z)$$

- L satisfies

$$(\mathrm{SD}_\vee)' \qquad w \approx \bigvee_i x_i \approx \bigvee_j y_j \to w \approx \bigvee_{i,j} (x_i \wedge y_j)$$

- Every $w \in L$ is either join irreducible or has a **canonical join representation**.

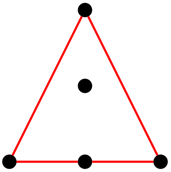
Moreover, the canonical joinands of 1 are join prime.

Convex geometries

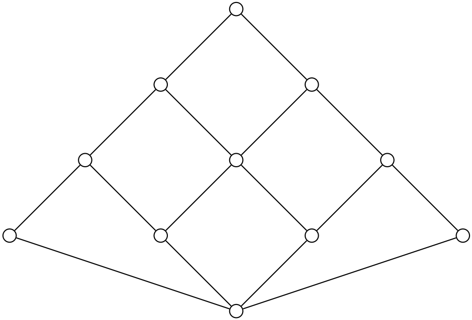
The following are equivalent for a finite lattice.

- L is join semidistributive and lower semimodular.
- Every element of L has a **unique** irredundant join decomposition.
- L is the closure lattice for an operator with the Anti-exchange Property.
- (many others)

The join prime elements of a finite convex geometry are its **extreme points**.



The lattice of convex subsets of a 4-element chain



Co(4)

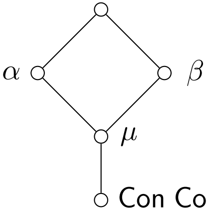
Day's Lemma

If L is a finite lattice and

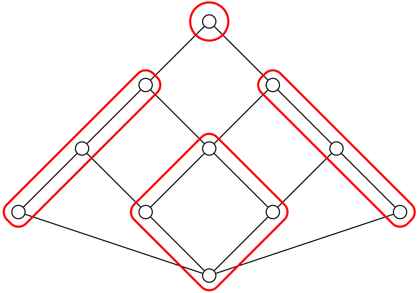
- $p \in \mathrm{J}(L)$,
- $p \leq \bigvee Q$ is a minimal nontrivial join cover,

then $\mathrm{con}(p, p_*) \leq \mathrm{con}(q, q_*)$ for each $q \in Q$.

Congruence relations on Co(n)



Con Co(n)



Co(4)

Wehrung's example

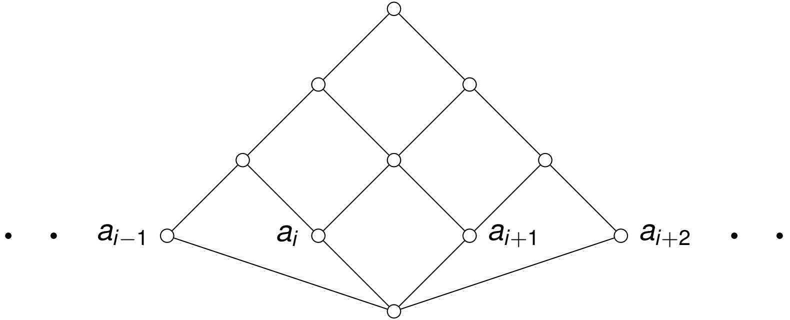
Let $\mathrm{FCo}(\mathbb{Z})$ denote the **finite** convex subsets of the integers.

- $\mathrm{FCo}(\mathbb{Z})$ is a join semidistributive lattice with no largest element.
- It is simple.

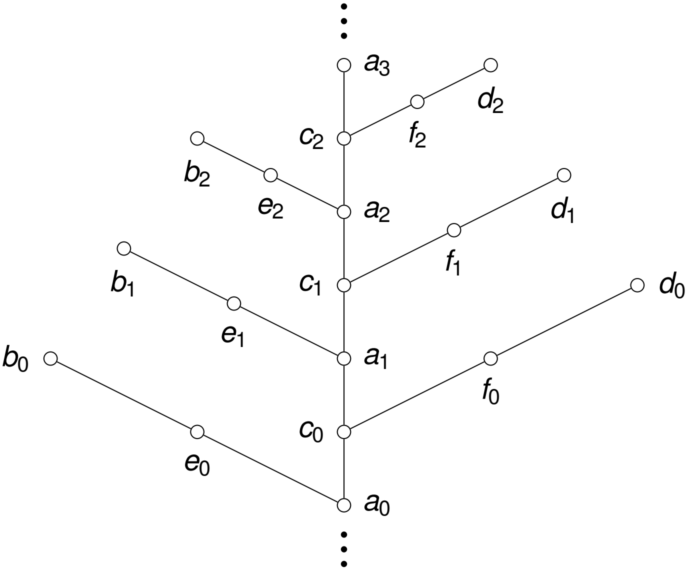
But it is not meet semidistributive.

Join semilattice presentation of $\mathrm{FCo}(\mathbb{Z})$

- Generated by an antichain of join irreducibles a_i $(i \in \mathbb{Z})$
- $a_i \leq a_{i-1} \vee a_{i+1}$ for all i
- $F = \bigcup_n F_n$ where $F_n = \mathrm{Co}[-n, n]$



Our example K: the join irreducibles



Our example K: the join relations

For all $i \in \mathbb{Z}$,

1. $a_i \leq b_{i-1} \vee c_{i-1}$
2. $b_i \leq e_i \vee a_{i+2}$
3. $c_i \leq a_i \vee d_{i-1}$
4. $d_i \leq f_i \vee c_{i+2}$
5. $e_i \leq a_i \vee e_{i-1}$
6. $f_i \leq c_i \vee f_{i-1}$
7. $b_i \leq e_i \vee b_{i+1}$
8. $d_i \leq f_i \vee d_{i+1}$

Example

K is a simple semidistributive lattice.

Example

K is a simple semidistributive lattice.

BUT WHY SHOULD YOU BELIEVE US?

From the free lattice book

In their early work on free lattices, Freese and Nation often had to put specific terms into canonical form. Whitman's algorithm for this, while efficient ... is tedious for humans. Freese used the following alternate procedure: he would stare at the term, looking for any obvious reason it was not in canonical form. If he could not find any such reason, he would take the term to Nation, whose office is one floor higher, and ask him to look at it. Nation used a dual procedure. The inadequacy of this procedure led us to develop computer programs to calculate in free lattices.

Checking the calculations

Let K be the lattice described above.

- $K = \bigcup_n K_n$ where K_n is the finite interval $[a_{-n}, b_{-n} \vee \cdots \vee b_{n-1} \vee d_{-n} \vee \cdots \vee d_{n-1}]$ in K.
- Ralph's programs for calculations in finitely presented lattices checked the claims of the proof in K_n for small n.

Minimal nontrivial join covers

First, we check that the rules (1)–(8) really are m.n.t.j.c.'s, i.e. for a rule $x \leq y \vee z$ that $x \nleq y_* \vee z$ and $x \nleq y \vee z_*$.
E.g., rule (1):

$$a_i \leq b_{i-1} \vee c_{i-1}$$

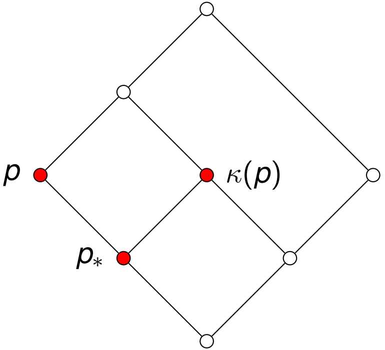
but

$$a_i \lneq e_{i-1} \vee c_{i-1}$$
$$a_i \nleq b_{i-1} \vee a_{i-1}$$

This ensures that K is simple.

$\kappa(p)$

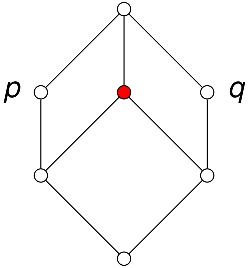
A finite lattice satisfies $(\mathrm{SD}_\wedge)$ iff for each $p \in \mathrm{J}(L)$ there is an element $\kappa(p)$ which is maximal w.r.t. $x \geq p_*$ but $x \ngeq p$.



If L is meet semidistributive, then $\kappa : \mathrm{J}(L) \to \mathrm{M}(L)$ is surjective.

Both ways semidistributivity

A finite lattice satisfies both semidistributive laws iff $\kappa : \mathrm{J}(L) \to \mathrm{M}(L)$ is a bijection.



In this example, the map κ is defined, so $(\mathrm{SD}_\wedge)$ holds, but not $(\mathrm{SD}_\vee)$ since $\kappa(p) = \kappa(q)$.

Calculations

Recall that our example is a union of intervals, $K = \bigcup_n K_n$. The program checks that our hand calculations of κ on K_n are correct, e.g.,

$$\kappa(a_i) = b_{-n} \vee \ldots \vee b_{i-2} \vee d_{-n} \vee \ldots \vee d_{i-1} \vee e_{i-1}$$

so that κ is a bijection.

Mahalo