

Math 243
Spring 2019
Practice Exam 2
Doomsday
Time Limit: Probably Enough

Name (Print): Solutions

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	35	
7	60	
Total:	145	

1. (10 points) For differentiable vector valued functions $u(t)$ and $v(t)$, prove that $\frac{d}{dt} \left(u(t) \cdot v(t) \right) = \frac{du}{dt} \cdot v(t) + u(t) \cdot \frac{dv}{dt}$.

Let $u = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$ and $v = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$.

$$\begin{aligned}\frac{d}{dt}(u \cdot v) &= \frac{d}{dt}(u_1 v_1 + u_2 v_2 + u_3 v_3) \\ &= u'_1 v_1 + u_1 v'_1 + u'_2 v_2 + u_2 v'_2 + u'_3 v_3 + u_3 v'_3 \\ &= u'_1 v_1 + u'_2 v_2 + u'_3 v_3 + u_1 v'_1 + u_2 v'_2 + u_3 v'_3 \\ &= u' \cdot v + u \cdot v'\end{aligned}$$

2. (10 points) Show that if $r(t)$ is a differentiable vector valued function and $|r(t)| = C$ for a constant C , then $r(t)$ and $\frac{dr}{dt}$ are orthogonal.

We have that $r(t) \cdot r(t) = |r(t)|^2 = C^2$, so

$$\begin{aligned}\frac{d}{dt}(C^2) &= \frac{d}{dt}(r(t) \cdot r(t)) \\ 0 &= r'(t) \cdot r(t) + r(t) \cdot r'(t) \\ &= 2r'(t) \cdot r(t).\end{aligned}$$

So, $0 = r'(t) \cdot r(t)$ and $\therefore r'(t)$ and $r(t)$ are orthogonal.

3. (10 points) Find $r(t)$ if

$$\frac{d^2r}{dt^2} = -32k, \quad r(0) = 100k, \quad \frac{dr}{dt} \Big|_{t=0} = 8i + 8j$$

$$\frac{dr}{dt} = \int \frac{d^2r}{dt^2} dt = -32tk + \vec{C}_1$$

$$8i + 8j = \frac{dr}{dt} \Big|_{t=0} = \vec{C}_1, \quad \therefore \frac{dr}{dt} = 8i + 8j - 32tk$$

$$r(t) = \int \frac{dr}{dt} dt = 8ti + 8tj - \frac{32}{2} t^2 k + \vec{C}_2$$

$$100k = r(0) = \vec{C}_2, \quad \text{so,}$$

$$r(t) = 8ti + 8tj + [100 - 16t^2]k$$

4. (10 points) Let $r(t) = t \sin(t^2) \mathbf{i} + \frac{1}{1+t^2} \mathbf{j} + t \sin(t) \mathbf{k}$. Find $\int r(t) dt$.

$$\int r(t) dt = -\frac{\cos(t^2)}{2} \mathbf{i} + \tan^{-1}(t) \mathbf{j} + \underbrace{(-t \cos t + \sin t)}_{\text{used parts}} \mathbf{k} + \vec{C}$$

5. (10 points) With $r(t)$ from the previous problem, find $\int_0^{\sqrt{\pi}} r(t) dt$.

$$\begin{aligned} & \int_0^{\sqrt{\pi}} r(t) dt \\ &= \left[-\frac{\cos((\sqrt{\pi})^2)}{2} + \frac{\cos(0^2)}{2} \right] \mathbf{i} \\ &+ \left[\tan^{-1}(\sqrt{\pi}) - \tan^{-1}(0) \right] \mathbf{j} \\ &+ \left[-\sqrt{\pi} \cos(\sqrt{\pi}) + \sin(\sqrt{\pi}) - (0 \cdot \cos(0) + \sin(0)) \right] \mathbf{k} \\ &= \mathbf{i} + \tan^{-1}(\sqrt{\pi}) \mathbf{j} + (-\sqrt{\pi} \cos(\sqrt{\pi}) + \sin(\sqrt{\pi})) \mathbf{k} \end{aligned}$$

6. Let $r(t) = t \cos(t) \mathbf{i} + t \sin(t) \mathbf{j} + \frac{2\sqrt{2}}{3} t^{3/2} \mathbf{k}$.

(a) (10 points) In a few words or a sketch, describe this curve for $t \geq 0$.

This is a friggin tornado.

- (b) (10 points) Find the parametric equations of the tangent line to the curve when $t = \frac{\pi}{3}$.

$$\mathbf{r}'(t) = (\cos t - t \sin t) \mathbf{i} + (\sin t + t \cos t) \mathbf{j} + \sqrt{2t} \mathbf{k}$$

$$\mathbf{r}'(\pi/3) = \frac{3 - \sqrt{3}\pi}{6} \mathbf{i} + \frac{3\sqrt{3} + \pi}{6} \mathbf{j} + \sqrt{\frac{2\pi}{3}} \mathbf{k}$$

$$\mathbf{r}(\pi/3) = \frac{\pi}{6} \mathbf{i} + \frac{\sqrt{3}\pi}{6} \mathbf{j} + \frac{2\sqrt{2}}{3} \left(\frac{\pi}{3}\right)^{3/2} \mathbf{k}$$

$$x = \frac{\pi}{6} + \left(\frac{3 - \sqrt{3}\pi}{6}\right)t, \quad y = \frac{\sqrt{3}\pi}{6} + \left(\frac{3\sqrt{3} + \pi}{6}\right)t, \quad z = \frac{2\sqrt{2}}{3} \left(\frac{\pi}{3}\right)^{3/2} + \sqrt{\frac{2\pi}{3}}t$$

- (c) (15 points) Find the length of the curve from $t = 0$ to $t = \pi$.

The length of this curve is

$$\int_0^\pi \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + (\sqrt{2t})^2} dt$$

$$= \int_0^\pi \sqrt{t^2 + 2t + 1} dt$$

$$= \int_0^\pi \sqrt{(t+1)^2} dt$$

$$= \int_0^\pi t + 1 dt$$

$$= \frac{t^2}{2} + t \Big|_0^\pi = \frac{\pi^2}{2} + \pi$$

7. (60 points) For numbers $a, b \geq 0$, let

$$\mathbf{r}(t) = a \cos(t) \mathbf{i} + a \sin(t) \mathbf{j} + bt \mathbf{k}.$$

Find the unit tangent vector, T , the principle unit normal vector, N , the curvature, κ , the unit binormal, B , and the torsion, τ , of this curve. Give the equation of the osculating plane at $t = 0$.

$$\mathbf{v}(t) = \mathbf{r}'(t) = -a \sin t \mathbf{i} + a \cos t \mathbf{j} + b \mathbf{k}, \quad |\mathbf{v}| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{a^2 + b^2}} (-a \sin t \mathbf{i} + a \cos t \mathbf{j} + b \mathbf{k})$$

$$\frac{d\mathbf{T}}{dt} = \frac{1}{\sqrt{a^2 + b^2}} (-a \cos t \mathbf{i} - a \sin t \mathbf{j}), \quad \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{\sqrt{a^2 + b^2}} \sqrt{a^2 \cos^2 t + a^2 \sin^2 t} = \frac{a}{\sqrt{a^2 + b^2}}$$

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{\left| d\mathbf{T}/dt \right|} = \frac{\sqrt{a^2 + b^2}}{a} \left(\frac{1}{\sqrt{a^2 + b^2}} (-a \cos t \mathbf{i} - a \sin t \mathbf{j}) \right) = -\cos t \mathbf{i} - \sin t \mathbf{j}$$

$$\kappa = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{\sqrt{a^2 + b^2}} \cdot \frac{a}{\sqrt{a^2 + b^2}} = \frac{a}{a^2 + b^2}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin t & a \cos t & b \\ -a \cos t & -a \sin t & 0 \end{vmatrix} = \frac{1}{\sqrt{a^2 + b^2}} (b \sin t \mathbf{i} - b \cos t \mathbf{j} + a \mathbf{k})$$

$$\frac{d\mathbf{B}}{dt} = \frac{1}{\sqrt{a^2 + b^2}} (b \cos t \mathbf{i} + b \sin t \mathbf{j}), \quad \mathbf{T} = \frac{1}{|\mathbf{v}|} \left(\frac{d\mathbf{B}}{dt} \circ \mathbf{N} \right)$$

$$= \frac{-1}{\sqrt{a^2 + b^2}} \left(\frac{-b \cos^2 t}{\sqrt{a^2 + b^2}} - \frac{b \sin^2 t}{\sqrt{a^2 + b^2}} \right) \\ = \frac{b}{a^2 + b^2}$$

For the osculating plane, we can use the vector $\langle b \sin t, -b \cos t, a \rangle$ at $t = 0$, which gives $\langle 0, -b, a \rangle$. Since $\mathbf{r}(0) = a\mathbf{i}$, the equation is

$$-b(y - 0) + a(z - 0) = 0.$$