**INSTRUCTIONS:** Write legibly. Indicate your answer clearly. Show all work; explain your answers. Answers with work not shown might be worth **zero** points. No calculators, cell phones, or cheating.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Worth</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
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<td><strong>Total</strong></td>
<td><strong>100</strong></td>
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1. Evaluate the following integral over $0 \leq x \leq \ln 2$ and $0 \leq y \leq \ln 2$:

\[
\iint e^{x-y} \, dA = \int_0^{\ln(2)} \int_0^{\ln(2)} e^{x-y} \, dx \, dy
\]

\[
= \int_0^{\ln(2)} e^{-y} \bigg|_0^{\ln(2)} \, dy
\]

\[
= \int_0^{\ln(2)} e^{-y} - e^{-\ln(2)} \, dy
\]

\[
= \int_0^{\ln(2)} 2e^{-y} - e^{-y} \, dy
\]

\[
= \int_0^{\ln(2)} e^{-y} \, dy
\]

\[
= -e^{-y} \bigg|_0^{\ln(2)}
\]

\[
= -e^{-\ln(2)} - (-e^0)
\]

\[
= -e^{-\frac{\ln(2)}{\ln(e)}} + 1
\]

\[
= -\frac{1}{2} + 1
\]

\[
= \frac{1}{2}
\]
2. Find the volume of the solid that is bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the $xy$-plane.

Note:

To find the intersection, solve:

$$x = 2 - x^2$$

$$\Rightarrow x^2 + x - 2 = 0$$

$$\Rightarrow (x+2)(x-1) = 0$$

$$\Rightarrow x = -2 \text{ or } x = 1$$

Volume:

$$\int_{-2}^{1} \int_{x}^{2-x^2} \int_{0}^{x^2} dz \, dy \, dx$$

$$= \int_{-2}^{1} \int_{x}^{2-x^2} x^2 \, dy \, dx$$

$$= \int_{-2}^{1} \left( \int_{x}^{2-x^2} x^2 \, dy \right) \, dx$$

$$= \int_{-2}^{1} \left. (2-x^2)x^2 - x \cdot x^2 \right|_{x}^{2-x^2} \, dx$$

$$= \int_{-2}^{1} 2x^2 - x^4 - x^3 \, dx$$

$$= \left. \frac{2x^3}{3} - \frac{x^5}{5} - \frac{x^4}{4} \right|_{-2}^{1}$$

$$= \frac{2}{3} - \frac{1}{5} - \frac{1}{4} - \left( \frac{2(-2)^3}{3} - \frac{(-2)^5}{5} - \frac{(-2)^4}{4} \right)$$

It's fine if you leave your answers like this on the test (and HW too!).
(15) 3. Find the average value of the function \( f(x, y) = \sin(x-y) \) over the rectangle \( 0 \leq x \leq \pi \) and \( 0 \leq y \leq \pi/2 \).

\[
\text{Ave} = \frac{1}{A} \iint_R f(x,y) \, dA = \frac{1}{\pi/2} \int_0^\pi \int_0^{\pi/2} \sin(x-y) \, dx \, dy
\]

\[
= \frac{1}{\pi/2} \int_0^{\pi/2} \left[ -\cos(x-y) \right]_0^\pi \, dy = \frac{1}{\pi/2} \int_0^{\pi/2} \left[ -\cos(\pi-y) + \cos(-y) \right] \, dy
\]

\[
= \frac{1}{\pi/2} \left( \sin(y) - \sin(y) \right) \bigg|_0^{\pi/2} = \frac{1}{\pi/2} = \frac{2}{\pi}
\]

(20) 4. The region that lies inside the cardioid \( r = 1 + \cos \theta \) and outside the circle \( r = 1 \) is the base of a solid right cylinder. The top of the cylinder lies in the plane \( z = x \).

(a) Express the volume as an integral

\[
\int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos(\theta)} \int_{0}^{\cos(\theta)} 1 \, dz \, r \, dr \, d\theta
\]

(b) Calculate the volume of the cylinder. (Skip the calculation if you don't have enough time.)

are not a robot.
5. Find the center of mass of a solid of constant density bounded below by the paraboloid \( z = x^2 + y^2 \) and above by the plane \( z = 4 \).

\[ \text{Constant density: } \delta = k \]

\[ \text{Mass: } \int_0^{2\pi} \int_0^2 \int_0^4 k \cdot r \ dz \ dr \ d\phi \]

\[ = k \int_0^{2\pi} \int_0^2 \left( 4r - r^3 \right) \ dr \ d\phi \]

\[ = k \int_0^{2\pi} 2r^2 - \frac{r^4}{4} \bigg|_0^2 \ d\phi \]

\[ = k \int_0^{2\pi} 4 \ d\phi \]

\[ = k \ 8\pi \]

Let's assume that \( M_{z'z} = M_{xz} \).

\[ M_{xz} = \int_0^{2\pi} \int_0^2 \int_0^4 k r \sin(\phi) \cdot r \ dz \ dr \ d\phi \]

\[ = \int_0^{2\pi} \int_0^2 k r^2 \sin(\phi) \left( 4r - r^3 \right) \ dr \ d\phi \]

\[ = \int_0^2 \int_0^{2\pi} k r^2 \sin(\phi) \left( 4r - r^3 \right) \ d\phi \ dr \]

\[ = 0 \]

\[ M_{yz} = k \int_0^{2\pi} \int_0^2 \int_0^4 z \ r \ dz \ dr \ d\phi \]

\[ = \frac{k}{2} \int_0^{2\pi} \int_0^2 kr \left( 4r - r^3 \right) \ dr \ d\phi \]

\[ = \frac{k}{2} \int_0^{2\pi} 32 - \frac{32}{3} \ d\phi \]

\[ = \frac{k \ 64\pi}{3} \]

Since \( \frac{M_{yz}}{M} = \frac{k \ 64\pi/3}{k \ 8\pi} = \frac{8}{3} \),

\[ \text{COM. is } (0, 0, 8/3) \]
6. Find the average value of the function \( f(\rho, \phi, \theta) = \rho \) over the solid ball \( \rho \leq 1 \).

\[
\text{Volume: } \frac{4}{3} \pi \rho^3, \quad \int_0^{2\pi} \int_0^\pi \int_0^\rho \rho^2 \sin^2(\phi) \, d\rho d\phi d\omega \\
= \frac{1}{4} \int_0^{2\pi} \int_0^\pi \sin(\phi) \, d\phi d\omega \\
= \frac{1}{2} \int_0^{2\pi} 1 \, d\omega = \pi
\]

Thus, ave. value is \( \frac{3}{4} \).

Think of \( f(\rho, \phi, \theta) = \rho \) as density function. Express the moment of inertia about any central axis as an iterated integral.

For the \( z \)-axis: \( x^2 + y^2 = \rho^2 \sin^2(\phi) \) (after some algebra)

Thus, \( I_z = \int_0^{2\pi} \int_0^\pi \int_0^\rho \rho^2 \sin^2(\phi) \cdot \rho^2 \sin^3(\phi) \, d\rho d\phi d\omega \)