

MATH 244, Spring '11  
Exam 1

Name: **K E Y**

*INSTRUCTIONS:* Write legibly. Indicate your answer clearly. Show all work; explain your answers. Answers with work not shown might be worth zero points. No calculators, cell phones, or cheating.

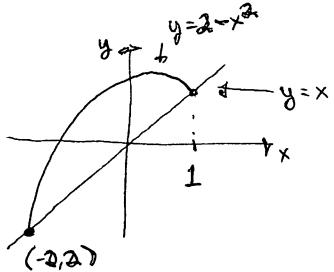
Problem	Worth	Score
1	10	
2	15	
3	15	
4	20	
5	20	
6	20	
Total	100	

(10) 1. Evaluate the following integral over  $0 \leq x \leq \ln 2$  and  $0 \leq y \leq \ln 2$ :

$$\begin{aligned} \iint e^{x-y} dA &= \int_0^{\ln(2)} \int_0^{\ln(2)} e^{x-y} dx dy \\ &= \int_0^{\ln(2)} e^{x-y} \Big|_0^{\ln(2)} dy \\ &= \int_0^{\ln(2)} e^{\ln(2)-y} - e^{-y} dy \\ &= \int_0^{\ln(2)} 2e^{-y} - e^{-y} dy \\ &= \int_0^{\ln(2)} e^{-y} dy \\ &= -e^{-y} \Big|_0^{\ln(2)} \\ &= -e^{\ln(2)} - (-e^0) \\ &= -e^{\ln(\frac{1}{2})} + 1 \\ &= -\frac{1}{2} + 1 \\ &= \frac{1}{2} \end{aligned}$$

- (15) 2. Find the volume of the solid that is bounded above by the cylinder  $z = x^2$  and below by the region enclosed by the parabola  $y = 2 - x^2$  and the line  $y = x$  in the  $xy$ -plane.

Note:



To find the intersection, solve:

$$x = 2 - x^2$$

$$\Leftrightarrow x^2 + x - 2 = 0$$

$$\Leftrightarrow (x+2)(x-1) = 0$$

$$\Rightarrow x = -2 \text{ or } x = 1$$

Volume:

$$\begin{aligned}
 & \int_{-2}^1 \int_x^{2-x^2} \int_0^{x^2} dz dy dx \\
 &= \int_{-2}^1 \int_x^{2-x^2} z \Big|_0^{x^2} dy dx \\
 &= \int_{-2}^1 \int_x^{2-x^2} x^2 dy dx \\
 &= \int_{-2}^1 (2-x^2)x^2 - x \cdot x^2 \cancel{dy} dx \\
 &= \int_{-2}^1 2x^2 - x^4 - x^3 dx \\
 &= \frac{2x^3}{3} - \frac{x^5}{5} - \frac{x^4}{4} \Big|_{-2}^1 \\
 &= \frac{2}{3} - \frac{1}{5} - \frac{1}{4} - \left( \frac{2(-2)^3}{3} - \frac{(-2)^5}{5} - \frac{(-2)^4}{4} \right)
 \end{aligned}$$

It's fine if you leave your answers like this on the test (and HW too!).

- (15) 3. Find the average value of the function  $f(x, y) = \sin(x-y)$  over the rectangle  $0 \leq x \leq \pi$  and  $0 \leq y \leq \pi/2$ .

$$\begin{aligned}
 \text{Ave Value} &= \frac{1}{\text{area of } R} \iint_R f(x, y) dA = \frac{1}{\pi^2/2} \int_0^{\pi/2} \int_0^\pi \sin(x-y) dx dy \\
 &= \frac{1}{\pi^2/2} \int_0^{\pi/2} -\cos(x-y) \Big|_0^\pi dy = \frac{1}{\pi^2/2} \int_0^{\pi/2} -\cos(\pi-y) + \cos(-y) dy \\
 &= \frac{1}{\pi^2/2} \left( \sin(\pi-y) - \sin(-y) \Big|_0^{\pi/2} \right) \\
 &= 4/\pi^2
 \end{aligned}$$

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- (20) 4. The region that lies inside the cardioid  $r = 1 + \cos\theta$  and outside the circle  $r = 1$  is the base of a solid right cylinder. The top of the cylinder lies in the plane  $\underbrace{z=x}_{z=r\cos(\theta)}$ .

- (a) Express the volume as an integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{1+\cos(\theta)} \int_{z=0}^{r\cos(\theta)} 1 dz r dr d\theta$$

- (b) Calculate the volume of the cylinder. (Skip the calculation if you ~~don't have enough time.~~)

*are not a robot.*

- (20) 5. Find the center of mass of a solid of constant density bounded below by the paraboloid  $z = x^2 + y^2$  and above by the plane  $z = 4$ .

Constant density:  $\delta = k$

$$\text{Mass: } \int_0^{2\pi} \int_0^2 \int_{r^2}^4 k r dz dr d\phi$$

$$= k \int_0^{2\pi} \int_0^2 4r - r^3 dr d\phi$$

$$= k \int_0^{2\pi} 2r^2 - \frac{r^4}{4} \Big|_0^2 d\phi$$

$$= k \int_0^{2\pi} 4 d\phi$$

$$= k 8\pi$$

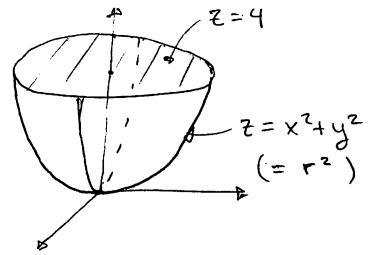
Let's assume that  $M_{zy} = M_{zx}$ .

$$\begin{aligned} M_{zx} &= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 k r \sin(\phi) \cdot r dz dr d\phi \\ &= \int_0^{2\pi} \int_0^2 k r^2 \sin(\phi) (4 - r^2) dr d\phi \\ &= \int_0^2 \int_0^{2\pi} k r^2 \sin(\phi) (4 - r^2) d\phi dr \\ &= 0 \end{aligned}$$

$$\begin{aligned} M_{zy} &= k \int_0^{2\pi} \int_0^2 \int_{r^2}^4 z r dz dr d\phi \\ &= \frac{k}{2} \int_0^{2\pi} \int_0^2 (kr^2 - r^5) dr d\phi \\ &= \frac{k}{2} \int_0^{2\pi} 32 - \frac{32}{3} d\phi \\ &= \frac{k 64\pi}{3} \end{aligned}$$

$$\text{Since } \frac{M_{zy}}{M} = \frac{\frac{k 64\pi}{3}}{k 8\pi} = \frac{8}{3},$$

C.O.M. is  $(0, 0, 8/3)$



- (20) 6. Find the average value of the function  $f(\rho, \phi, \theta) = \rho$  over the solid ball  $\rho \leq 1$ .

$$\begin{aligned} \text{Volume: } & \frac{4}{8}\pi , \quad \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho \cdot \rho^2 \sin(\phi) d\rho d\phi d\theta \\ &= \frac{1}{4} \int_0^{2\pi} \int_0^{\pi} \sin(\phi) d\phi d\theta \\ &= \frac{1}{2} \int_0^{2\pi} 1 d\theta = \pi \end{aligned}$$

Thus, ave. value is  $\frac{3}{4}$ .

Think of  $f(\rho, \phi, \theta) = \rho$  as density function. Express the moment of inertia about any central axis as an iterated integral.

For the z-axis;  $x^2 + y^2 = \rho^2 \sin^2(\phi)$  (after some algebra)

$$\text{Thus, } I_z = \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin^2(\phi) \cdot \rho^2 \sin^3(\phi) d\rho d\phi d\theta$$