Math 307
Spring 2019
Exam 3 - Practice
4/17/19
Time Limit: 50 min.

Name (Print): Solutions

Problem	Points	Score
1	10	
2	15	
3	55	
4	35	
5	40	
Total:	155	

1. Consider the system of differential equations:

$$\begin{aligned} y_1' &= 2y_1 \\ y_2' &= -y_2 \\ y_3' &= 15y_3 \end{aligned}$$

(a) (5 points) Find the general solution, Y_H.

$$\varphi_H = \begin{bmatrix} c_1 e^{2x} \\ c_2 e^{-x} \\ c_3 e^{15x} \end{bmatrix}$$

(b) (5 points) Solve the initial value problem $Y(1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \varphi(1) = \begin{bmatrix} c_1 e^{2} \\ c_2 e^{-1} \\ c_3 e^{15} \end{bmatrix}$$

$$\begin{aligned} 1 = c_1 e^2 &\Rightarrow c_1 = e^{-2} \\ 2 = c_2 e^{-1} &\Rightarrow c_2 = 2e \\ 3 = c_3 e^{15} &\Rightarrow c_3 = 3e^{-15} \end{aligned}$$

So, the solution to the initial value problem is

$$\varphi = \begin{bmatrix} e^{-2} e^{2x} \\ 2e\, e^{-x} \\ 3e^{-15} e^{15x} \end{bmatrix} = \begin{bmatrix} e^{2x-2} \\ 2e^{1-x} \\ 3e^{15x-15} \end{bmatrix}$$

2. (15 points) Suppose that Z is a solution to $Y' = BY$, and that A and B are similar matrices. Prove that there exists an invertible matrix P such that PZ is a solution to $Y' = AY$.

If Z is a solution to $\varphi' = B\varphi$, then (*) $Z' = BZ$. If A and B are similar matrices, then there exists a P such that

(**) $P^{-1}AP = B$. Notice that this is the same as $AP = PB$.

From (*): $PZ' = PBZ$ (obtained by multiplying both sides by P).

Since the entries of P are numbers (meaning, the don't depend on any variables) $PZ' = (PZ)'$. Combining this with (**), we have $(PZ)' = A(PZ)$, which, says that PZ is a solution to $\varphi' = A\varphi$.

3. Consider the system of differential equations:

$$\begin{aligned} y_1' &= y_1 + 2y_2 + y_3 \\ y_2' &= 2y_1 + y_2 + y_3 \\ y_3' &= y_3 \end{aligned}$$

(a) (5 points) Write the system in the form $Y' = AY$.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}' = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

(b) (10 points) For the matrix A, find the eigenvalues.

$$\begin{aligned} p(\lambda) &= \det(\lambda I - A) \\ &= \det\begin{pmatrix} \lambda-1 & -2 & -1 \\ -2 & \lambda-1 & -1 \\ 0 & 0 & \lambda-1 \end{pmatrix} \\ &= (\lambda-1)\left((\lambda-1)^2\right) + 2(-2(\lambda-1)) \\ &= (\lambda-1)\left[(\lambda-1)^2 - 4\right] \\ &= (\lambda-1)(\lambda^2 - 2\lambda + 1 - 4) \\ &= (\lambda-1)(\lambda-3)(\lambda+1) \end{aligned}$$

$\therefore$ the eigenvalues are $1, 3$ and -1.

(c) (10 points) For each eigenvalue, find a corresponding eigenvector. In other words, find the eigenpairs.

$$\lambda = 1: \begin{pmatrix} 0 & -2 & -1 \\ -2 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \therefore \text{ eigenvector } \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$$\lambda = 3: \begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \therefore \text{ eigenvector } \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = -1: \begin{pmatrix} -2 & -2 & -1 \\ -2 & -2 & -1 \\ 0 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \therefore \text{ eigenvector } \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Note: I'm going to use the eigenpair

$$\left(1, \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}\right) \quad \left(\text{instead of } \left(1, \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}\right)\right)$$

(why? If $\begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$ is an eigenvector, then so is $k\begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$ for any k, and I hate fractions!)

(d) (5 points) Find an invertible matrix P, and diagonal matrix D such that $P^{-1}AP = D$.

$$P = \begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(e) (5 points) Find the general solution to the system $Y' = DY$.

$$\varphi_{H,D} = \begin{bmatrix} c_1 e^{x} \\ c_2 e^{3x} \\ c_3 e^{-x} \end{bmatrix}$$

(f) (10 points) Find the general solution to the system $Y' = AY$.

$$\varphi_{H,A} = P\varphi_{H,D} = \begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 e^{x} \\ c_2 e^{3x} \\ c_3 e^{-x} \end{pmatrix}$$

$$= \begin{pmatrix} -c_1 e^{x} + c_2 e^{3x} - c_3 e^{-x} \\ -c_1 e^{x} + c_2 e^{3x} + c_3 e^{-x} \\ 2c_1 e^{x} \end{pmatrix}$$

(g) (10 points) Solve the initial value problem $Y(0) = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$ for $Y' = AY$.

$$\begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix} = \varphi_{H,A}(0) = \begin{bmatrix} -c_1 + c_2 - c_3 \\ -c_1 + c_2 + c_3 \\ 2c_1 \end{bmatrix}$$

Solving this system gives $c_1 = 3/2$, $c_2 = -1/2$, and $c_3 = 2$. So,

$$\varphi = \begin{bmatrix} -\frac{3}{2}e^{x} - \frac{1}{2}e^{3x} - 2e^{-x} \\ -\frac{3}{2}e^{x} - \frac{1}{2}e^{3x} + 2e^{-x} \\ 3e^{x} \end{bmatrix}$$

4. (35 points) Let $A = \begin{bmatrix} -2 & -4 \\ 5 & 2 \end{bmatrix}$. Give a real-valued general solution to the equation $Y' = AY$.

Our only hope is that A is diagonalizable:

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda+2 & 4 \\ -5 & \lambda-2 \end{pmatrix} = \lambda^2 - 4 + 20 = (\lambda+4i)(\lambda-4i)$$

So, eigenvalues are $\lambda = 4i$ and $\lambda = -4i$.

$$\lambda = 4i: \begin{pmatrix} 4i+2 & 4 \\ -5 & 4i-2 \end{pmatrix} \xrightarrow{(4i+2)R_2+5R_1} \begin{pmatrix} 4i+2 & 4 \\ 0 & 0 \end{pmatrix} \xrightarrow{(-4i+2)R_1}$$

$$\rightarrow \begin{pmatrix} 20 & 8-16i \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \frac{2}{5} - \frac{4}{5}i \\ 0 & 0 \end{pmatrix}$$

This gives the eigenpair $\left(4i, \begin{pmatrix} -\frac{2}{5} + \frac{4}{5}i \\ 1 \end{pmatrix}\right)$ and in turn, we can deduce that $\left(-4i, \begin{pmatrix} -\frac{2}{5} - \frac{4}{5}i \\ 1 \end{pmatrix}\right)$ is ~~the other~~ another eigenpair.

Now, if $P = \begin{pmatrix} \frac{-2}{5} + \frac{4}{5}i & \frac{-2}{5} - \frac{4}{5}i \\ 1 & 1 \end{pmatrix}$, then $P^{-1}AP = \underbrace{\begin{pmatrix} 4i & 0 \\ 0 & -4i \end{pmatrix}}_{D}$.

Further, $z = \begin{pmatrix} e^{4ix} \\ 0 \end{pmatrix}$ is a solution to $\varphi' = D\varphi$. As such, we obtain a solution, Pz, to $\varphi' = A\varphi$:

$$P\begin{pmatrix} e^{4ix} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{-2}{5} + \frac{4}{5}i & \frac{-2}{5} - \frac{4}{5}i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \cos(4x) + i\sin(4x) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-2}{5}\cos(4x) + \frac{4}{5}i\cos(4x) - \frac{2}{5}i\sin(4x) - \frac{4}{5}\sin(4x) \\ \cos(4x) + i\sin(4x) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-2}{5}\cos(4x) - \frac{4}{5}\sin(4x) \\ \cos(4x) \end{pmatrix} + i \begin{pmatrix} \frac{4}{5}\cos(4x) - \frac{2}{5}\sin(4x) \\ \sin(4x) \end{pmatrix}.$$

From here we can conclude $\varphi_H = c_1 \begin{bmatrix} \frac{-2}{5}\cos(4x) - \frac{4}{5}\sin(4x) \\ \cos(4x) \end{bmatrix} + c_2 \begin{bmatrix} \frac{4}{5}\cos(4x) - \frac{2}{5}\sin(4x) \\ \sin(4x) \end{bmatrix}$

5. Suppose that the velocity of an object is given by the vector

$$v = \begin{bmatrix} 3x + 2y + z \\ 2y + 3z \\ 2z \end{bmatrix}$$

where x, y and z are the coordinates of the object's position (they are functions of time).

(a) (30 points) Find a general solution for the object's position. (part b) is on the next page)

$$x' = 3x + 2y + z$$
$$y' = 2y + 3z$$
$$z' = 2z$$

we have $z = C_3 e^{2t}$, and hence $y' = 2y + 3C_3 e^{2t}$, so

$$y' - 2y = 3C_3 e^{2t} \rightarrow \frac{d}{dt}(e^{-2t} y) = 3C_3$$
$$e^{-2t} y = 3C_3 t + C_2$$
$$y = 3C_3 t e^{2t} + C_2 e^{2t}.$$

Now, we have $x' = 3x + 2(3C_3 t e^{2t} + C_2 e^{2t}) + C_3 e^{2t}$

$$x' - 3x = 6C_3 t e^{2t} + 2C_2 e^{2t} + C_3 e^{2t}$$
$$\frac{d}{dt}(e^{-3t} x) = 6C_3 t e^{-t} + 2C_2 e^{-t} + C_3 e^{-t}$$
$$e^{-3t} x = 6C_3(-te^{-t} - e^{-t}) - 2C_2 e^{-t} - C_3 e^{-t} + C_1 \quad (*)$$
$$x = [-6C_3 t e^{-t} - 7C_3 e^{-t} - 2C_2 e^{-t} + C_1] e^{3t}$$
$$= -6C_3 t e^{2t} - 7C_3 e^{2t} - 2C_2 e^{2t} + C_1 e^{3t}$$

$(*)$ $\int te^{-t} dt = -te^{-t} + \int e^{-t} dt$
$= -te^{-t} - e^{-t} + C$

So, our position, P, has general solution

$$P_H = \begin{bmatrix} -6C_3 t e^{2t} - 7C_3 e^{2t} - 2C_2 e^{2t} + C_1 e^{3t} \\ 3C_3 t e^{2t} + C_2 e^{2t} \\ C_3 e^{2t} \end{bmatrix}$$

(b) (10 points) Give the object's position when $t = 1$ if it's position is $(-7, 2, 3)$ when $t = 0$.

$$-7 = x(0) = -7C_3 - 2C_2 + C_1$$

$$2 = y(0) = C_2$$

$$3 = z(0) = C_3$$

So,
$$-7 = -7(3) - 2(2) + C_1$$
$$18 = C_1.$$

So, the solution to the IVP is

$$P = \begin{bmatrix} -18te^{2t} - 21e^{2t} - 4e^{2t} + 18e^{3t} \\ 9te^{2t} + 2e^{2t} \\ 3e^{2t} \end{bmatrix} = \begin{bmatrix} -18te^{2t} - 25e^{2t} + 18e^{3t} \\ 9te^{2t} + 2e^{2t} \\ 3e^{2t} \end{bmatrix}$$

$$P(1) = \begin{bmatrix} -18e^2 - 25e^2 + 18e^3 \\ 9e^2 + 2e^2 \\ 3e^2 \end{bmatrix}$$

$$= \begin{bmatrix} -43e^2 + 18e^3 \\ 11e^2 \\ 3e^2 \end{bmatrix}$$