Math 307
Spring 2019
Exam 1 - Practice
Due: ~~2/19/19~~
Time Limit: ~~75 Minutes~~

Name (Print): Solutions

Problem	Points	Score
1	20	
2	10	
3	10	
4	10	
5	40	
6	25	
7	20	
8	40	
9	20	
10	20	
11	20	
12	20	
13	20	
14	20	
Total:	295	

1. (20 points) Which of the following matrices are in reduced row-echelon form?

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ \begin{bmatrix}1&0&0&8\\0&1&0&5\\0&0&1&1\end{bmatrix},\ \begin{bmatrix}1&0&0&1\\0&0&0&0\\0&0&1&0\end{bmatrix},\ \begin{bmatrix}1&0&0&2\\0&0&1&0\\0&0&0&0\end{bmatrix},\ \begin{bmatrix}1&2&3&4&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},\ \begin{bmatrix}1&0&0&0\\0&1&1&0\\0&0&0&1\end{bmatrix},$$

$$\begin{bmatrix}1&0&0&1\\0&1&1&0\\0&0&0&1\end{bmatrix}.$$

2. (10 points) Prove that B^TB is always a symmetric matrix.

Proof: Let B be any matrix. Observe that

$$(B^TB)^T = B^T(B^T)^T = B^TB.$$

This shows that B^TB is symmetric.

3. (10 points) Let $A = \begin{bmatrix}2&0&0\\0&1&0\end{bmatrix}$ and $B = \begin{bmatrix}2&0\\0&1\\2&3\end{bmatrix}$. Find $(AB)^T$.

$AB = \begin{bmatrix}4&0\\0&1\end{bmatrix}$, so $(AB)^T = \begin{bmatrix}4&0\\0&1\end{bmatrix}$.

You can also do $(AB)^T = B^TA^T = \begin{bmatrix}2&0&2\\0&1&3\end{bmatrix}\begin{bmatrix}2&0\\0&1\\0&0\end{bmatrix}$

$$= \begin{bmatrix}4&0\\0&1\end{bmatrix}$$

4. (10 points) Let $A = \begin{bmatrix}1&2&0\\0&-1&1\\0&0&1\end{bmatrix}$, find A^{-1}.

$$\left[\begin{array}{ccc|ccc}1&2&0&1&0&0\\0&-1&1&0&1&0\\0&0&1&0&0&1\end{array}\right] R_2 - R_3 \rightarrow \left[\begin{array}{ccc|ccc}1&2&0&1&0&0\\0&-1&1&0&1&-1\\0&0&1&0&0&1\end{array}\right] R_1+2R_2$$

$$\rightarrow \left[\begin{array}{ccc|ccc}1&0&0&1&2&-2\\0&-1&0&0&+1&-1\\0&0&1&0&0&1\end{array}\right] -R_2$$

$$\rightarrow \left[\begin{array}{ccc|ccc}1&0&0&1&2&-2\\0&1&0&0&-1&1\\0&0&1&0&0&1\end{array}\right]$$

so, $A^{-1} = \begin{bmatrix}1&2&-2\\0&-1&1\\0&0&1\end{bmatrix}$

5. (a) (5 points) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

(b) (5 points) $\begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ -1 & -1 \end{bmatrix}$

(c) (5 points) If A is a 4×3 and B is a 3×4 matrix, what are the dimensions of AB and BA?

AB is a 4×4 and BA is a 3×3

(d) (5 points) True of false: $\det(AB) = \det(A)\det(B)$

True

(e) (5 points) True of false: $\det(A + B) = \det(A) + \det(B)$

False

(f) (5 points) Prove or provide a counter example to the following statement: If A is invertible and B is invertible then $A + B$ is invertible.

Counter Example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.
A and B are both invertible but $A+B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ which is not.

(g) (5 points) Prove or provide a counter example to the following statement: If A is invertible and B is invertible then AB is invertible.

Proof. $\det(AB) = \det(A) \cdot \det(B) \neq 0$ as $\det(A) \neq 0$ and $\det(B) \neq 0$ (they're both invertible). So, AB is invertible.

(h) (5 points) For the vectors $v_1, ..., v_n$, what is $\text{span}(v_1, ..., v_n)$?

$\text{span}(v_1, \dots, v_n)$ is the set of all linear combinations of $v_1, \dots, v_n$. AKA $\text{span}(v_1, \dots, v_n) = \{c_1 v_1 + \dots + c_n v_n \mid c_i \in \mathbb{R}\}$.

6. (a) (15 points) Prove that A and B are row equivalent if and only if there is an invertible matrix C such that $CA = B$.

($\Rightarrow$) Suppose that A and B are ~~[illegible]~~ row equivalent, meaning that B can be obtained by a finite amount of row operations on A. This means that we can find elementary matrices $E_n, \dots, E_1$ such that

$$E_n \cdots E_1 A = B.$$

Since each E_i is invertible, $E_n \cdots E_1$ is invertible.

($\Leftarrow$) If there is an invertible matrix C such that

$$CA = B,$$

we can find $E_n, \dots, E_1$ such that $C = E_n \cdots E_1$ where each E_i is an elementary matrix. Now we observe that each E_i performs a row operation to A, (and we get B back) so we're done.

(b) (10 points) Compute

$$\det\left(\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 3 & 4 & 1 \\ -1 & -3 & -4 & -1 \end{bmatrix}\right) \leftarrow R_4 + R_3$$

$$= \det\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= 0$$

7. (a) (10 points) Compute

$$\det\left(\begin{bmatrix} -1 & 0 & 0 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 4 & 0 \end{bmatrix}\right)$$

$$= 1 \cdot \begin{vmatrix} -1 & 0 & 2 \\ 1 & 0 & 1 \\ 2 & 4 & 0 \end{vmatrix}$$ (Expanding about the 2nd Column)

$$= 1 \cdot (-4) \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix}$$ (Expanding about the 2nd column)

$$= -4(-1-2)$$

$$= 12$$

(b) (10 points) Suppose A is a 5×5 matrix such that $\det(A) = 2$. Let $E = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

What is $\det(EA) =$?

Solution 1

E does a row switch to A, so

$\det(EA) = -2$

Solution 2

$\det(E) = -1$, so

$$\det(EA) = \det(E)\det(A)$$
$$= -1 \cdot 2$$
$$= -2$$

8. (a) (10 points) Show that $x^2+1, x-1$ are linearly independent.

If $c_1(x^2+1)+c_2(x-1)=0$, then

$c_1x^2+c_2x+(c_1-c_2) = 0$. So,

$c_1=0$, and $c_2=0$, and thus x^2+1 and $x-1$ are linearly independent.

(b) (10 points) Show that the vectors $x^2+1, x-1$ do not span P_2.

Since $\dim(P_2)=3$, one needs, at a minimum, 3 vectors to span P_2.

(c) (10 points) Show that $x^2+x+1,\ x^2-1,\ x+5,\ 4$ are linearly **dependent**.

Any 4 vectors in P_2 are linearly dependent because $\dim(P_2)=3$.

(d) (10 points) Suppose that we have $v_1, ..., v_n \in \mathbb{R}^n$. Let A be the matrix whose i-th column is the vector v_i (recall that the notation for this is $A=[v_1...v_n]$). If $\det(A)\neq 0$, explain why $v_1, ..., v_n$ form a basis for $\mathbb{R}^n$.

For a basis, we need to span $\mathbb{R}^n$, (and to be L.I.). Solving for the c_i in the equation $c_1v_1+\dots+c_nv_n=\begin{bmatrix} b_1 \\ \vdots \\ b_n\end{bmatrix}$ is the same as ~~the~~ solving the matrix equation

$$(*)\quad A\begin{bmatrix} c_1 \\ \vdots \\ c_n\end{bmatrix}=\begin{bmatrix} b_1 \\ \vdots \\ b_n\end{bmatrix}.$$

Recall that $\det(A)\neq 0 \iff A$ is invertible.

If A is invertible, then $\begin{bmatrix} c_1 \\ \vdots \\ c_n\end{bmatrix}=A^{-1}\begin{bmatrix} b_1 \\ \vdots \\ b_n\end{bmatrix}$ (which gives us the c_i). So the v_i span. Linear independence is similar. One uses the same argument w/ $\begin{bmatrix} b_1 \\ \vdots \\ b_n\end{bmatrix}=0_{\mathbb{R}^n}$ and notes that $A^{-1}\begin{bmatrix} b_1 \\ \vdots \\ b_n\end{bmatrix}$ is the only solution to $(*)$.

9. (20 points) Suppose that

$$A = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{E_2} \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_3} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_4}$$

Determine if A is invertible by computing $\det(A)$. If A *is* invertible, express A^{-1} as a product of elementary matrices.

Notice that $E_1, \dots, E_4$ are all elementary matrices, where, $\det(E_1) = 2$, $\det(E_2) = 1$, $\det(E_3) = 1$, $\det(E_4) = -1$. This tells us that $\det(A) = -2$.

$$A^{-1} = E_4^{-1} E_3^{-1} E_2^{-1} E_1^{-1}$$
$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

10. (20 points) State 5 properties that are all equivalent to a matrix being invertible. (lets say A is $n \times n$)

1. $AX = 0$ has only one solution $(x = 0)$. (x is an $n \times 1$ vector)
2. A is row equivalent to I.
3. $\det(A) \neq 0$.
4. A is a product of elementary matrices.
5. $\operatorname{Rank}(A) = n$. (why? this says that $\dim(\ker(A)) = 0$ and $\therefore AX = 0$ has only one solution)

Extras:

6. $AX = B$ has a solution for all B (and its unique.)
7. There is a matrix B such that $AB = I$. (okay, so that's the definition of A^{-1})

11. (20 points) Let

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

Find a basis for the null space and row space of A, the dimension of the null space of A, and the rank of A. Explain why the standard basis for $\mathbb{R}^4$ is a basis for $CS(A)$.

If $AX=0$ then
$$x_1 = x_3 - x_5 - 3x_7$$
$$x_2 = -x_5 - x_7$$
$$x_4 = -4x_5 - 2x_7$$
$$x_6 = -3x_7$$

and so
$$X = \begin{bmatrix} x_3 - x_5 - 3x_7 \\ -x_5 - x_7 \\ x_3 \\ -4x_5 - 2x_7 \\ x_5 \\ -3x_7 \\ x_7 \end{bmatrix}$$

$$= x_3 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + x_5 \underbrace{\begin{bmatrix} -1 \\ -1 \\ 0 \\ -4 \\ 1 \\ 8 \end{bmatrix}}_{v_2} + x_7 \underbrace{\begin{bmatrix} -3 \\ -1 \\ 0 \\ -2 \\ 0 \\ -3 \\ 1 \end{bmatrix}}_{v_3}$$

v_1, v_2 and v_3 form a basis for $NS(A)$, so $\dim(NS(A)) = 3$.

Since ~~3 + dim(~~ $3 + \text{Rank}(A) = 7$,
$\text{Rank}(A) = 4$.

Since A has column vectors in $\mathbb{R}^4$ and $\dim(CS(A)) = 4$, any basis for $\mathbb{R}^4$ will also be a basis for for $CS(A)$.
(Also, A has the standard basis as 4 of its columns so that's another way to argue this fact)

12. (20 points) Prove that if A is an $n \times n$ matrix and $\text{Rank}(A) = n$, then A is invertible.

We have $\dim(\ker(A)) + \text{Rank}(A) = n$, and so $\dim(\ker(A)) = 0$, and so the only vector that A sends to zero is $0_{\mathbb{R}^n}$, and this implies that A is invertible.

13. (20 points) For a matrix A, prove that the set of vectors $\overbrace{\{X \in \mathbb{R}^n : AX = 0\}}^{=V}$ form a vector space.

This is a subset of $\mathbb{R}^n$. If X_1 and X_2 are in V, then $A(X_1 + X_2) = AX_1 + AX_2 = 0$ shows that ~~X_1 and X_2 are both~~ $X_1 + X_2$ is in V.

Also, $A(kX_1) = kAX_1 = k0 = 0$, so kX_1 is in V (for any scalar, k) (hence V is a subspace of $\mathbb{R}^n$)

14. (20 points) Prove that if A and B are row equivalent, then $NS(A) = NS(B)$.

If A and B are r.e., then we can find $E_n, \dots, E_1$ such that $E_n \cdots E_1 A = B$. If $X \in NS(A)$, then $AX = 0$ says that $E_n \cdots E_1 A X = E_n \cdots E_1 0 = 0$, so $BX = 0$. Hence $NS(A) \subseteq NS(B)$. Further, If $BX = 0$, then ~~then~~ $AX = E_1^{-1} \cdots E_n^{-1} B X = E_1^{-1} \cdots E_n^{-1} 0 = 0$, So, $NS(B) \subseteq NS(A)$. This gives us that $NS(A) = NS(B)$.

EXTRA CREDIT:

(10 points) Define non-standard operations on $\mathbb{R}$ that satisfy all the properties of a vector space EXCEPT property 7.

(20 points) Define non-standard operations on $\mathbb{R}$ that satisfy all the properties of a vector space EXCEPT property 8.