Math 307
Spring 2019
Exam 2
3/27/19
Time Limit: ∞/∞

Name (Print): Solutions

Problem	Points	Score
1	15	
2	20	
3	35	
4	30	
5	10	
6	20	
7	20	
8	20	
9	20	
10	20	
11	10	
12	10	
13	10	
14	10	
15	10	
16	10	
17	10	
Total:	280	

1. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be transformations defined by

$$S\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x+y\\x-y\end{bmatrix}, \qquad T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x-4y\\3x\end{bmatrix}$$

(a) (5 points) Show that S and T are both linear transformations.

S: $S\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}+\begin{bmatrix}x_2\\y_2\end{bmatrix}\right) = S\begin{bmatrix}x_1+x_2\\y_1+y_2\end{bmatrix} = \begin{bmatrix}2(x_1+x_2)+y_1+y_2\\x_1+x_2-(y_1+y_2)\end{bmatrix} = \begin{bmatrix}2x_1+y_1\\x_1-y_1\end{bmatrix}+\begin{bmatrix}2x_2+y_2\\\cancel{x_2}-y_2\end{bmatrix} = S\begin{bmatrix}x_1\\y_1\end{bmatrix}+S\begin{bmatrix}x_2\\y_2\end{bmatrix}$

$S\left(k\begin{bmatrix}x\\y\end{bmatrix}\right) = S\left(\begin{bmatrix}kx\\ky\end{bmatrix}\right) = \begin{bmatrix}2(kx)+ky\\kx-ky\end{bmatrix} = \begin{bmatrix}k(2x+y)\\k(x-y)\end{bmatrix} = k\begin{bmatrix}2x+y\\x-y\end{bmatrix} = kS\begin{bmatrix}x\\y\end{bmatrix}$

T: $T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}+\begin{bmatrix}x_2\\y_2\end{bmatrix}\right) = T\left(\begin{bmatrix}x_1+x_2\\y_1+y_2\end{bmatrix}\right) = \begin{bmatrix}x_1+x_2-4(y_1+y_2)\\3(x_1+x_2)\end{bmatrix} = \begin{bmatrix}x_1-4y_1\\3x_1\end{bmatrix}+\begin{bmatrix}x_2-4y_2\\3x_2\end{bmatrix}$
$= T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}\right)+T\left(\begin{bmatrix}\cancel{x_2}\\y_2\end{bmatrix}\right)$

$T\left(k\begin{bmatrix}x_1\\y_1\end{bmatrix}\right) = T\left(\begin{bmatrix}kx_1\\ky_1\end{bmatrix}\right) = \begin{bmatrix}kx_1-4ky_1\\3kx_1\end{bmatrix} = k\begin{bmatrix}x_1-4y_1\\3x_1\end{bmatrix}$
$= kT\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}\right).$

(b) (5 points) Find $ST\begin{bmatrix}x\\y\end{bmatrix}$ and $T^2\begin{bmatrix}x\\y\end{bmatrix}$.

$ST\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = S\left(\begin{bmatrix}x-4y\\3x\end{bmatrix}\right) = \begin{bmatrix}2(x-4y)+3x\\x-4y-3x\end{bmatrix}$

$T^2\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = T\left(T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)\right) = T\left(\begin{bmatrix}x-4y\\3x\end{bmatrix}\right) = \begin{bmatrix}x-4y-4(3x)\\3(x-4y)\end{bmatrix}$

(c) (5 points) Find the matrices of S and T with respect to the standard basis for $\mathbb{R}^2$.

$[S]_\alpha^\alpha = \begin{bmatrix}2&1\\1&-1\end{bmatrix}, \quad [T]_\alpha^\alpha = \begin{bmatrix}1&-4\\3&0\end{bmatrix}$

2. (a) (10 points) Prove or provide a counter example: If T is a linear transformation, then so is $T+T^2$.

This is true. proof: $(T+T^2)(v) = T(v) + T(T(v))$, so, in order for $T+T^2$ to be defined, $T: V \to V$ (AKA, T is a linear operator). We have

$$\begin{aligned}(T+T^2)(v_1+v_2) &= T(v_1+v_2) + T(T(v_1+v_2)) \\ &= T(v_1) + T(v_2) + T(T(v_1) + T(v_2)) \\ &= T(v_1) + T(v_2) + T(T(v_1)) + T(T(v_2)) \\ &= T(v_1) + T(T(v_1)) + T(v_2) + T(T(v_2)) \\ &= (T+T^2)(v_1) + (T+T^2)(v_2),\end{aligned}$$

and $(T+T^2)(kv) = T(kv) + T(T(kv)) = kT(v) + T(kT(v))$
$= kT(v) + kT(T(v))$
$= k(T+T^2)(v).$

(b) (10 points) Let $D : C^\infty(-\infty,\infty) \to C^\infty(-\infty,\infty)$ be the usual derivative operator. Find a basis for the kernel of the operator

$$T = (D^2-4D+4)^2(D^2+1)$$

$(D^2-4D+4)^2(D^2+1) = (D-2)^4(D+i)(D-i)$, so, the vectors $y=e^{2x}$, $y=xe^{2x}$, $y=x^2e^{2x}$, $y=x^3e^{2x}$, $y=e^{ix}$, and $y=e^{-ix}$ would all be basis vectors for the $\ker(T)$.

3. Let α be the standard basis for $\mathbb{R}^3$ and β the basis consisting of the vectors

$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

(a) (5 points) Find the change of basis matrix from α to β.

$$[I]_\beta^\alpha = P = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\1 & 0 & 1\end{pmatrix}$$

(b) (10 points) Find the change of basis matrix from β to α.

Recall that $[I]_\alpha^\beta = \left([I]_\beta^\alpha\right)^{-1} = P^{-1}$. Now we simply find P^{-1}:

$$\left(\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 0 & 1 & 0\\1 & 0 & 1 & 0 & 0 & 1\end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & -1 & 0\\0 & 1 & 0 & 0 & 1 & 0\\0 & 0 & 1 & -1 & 1 & 1\end{array}\right),$$

and $\therefore$ $P^{-1} = \begin{pmatrix}1 & -1 & 0\\0 & 1 & 0\\-1 & 1 & 1\end{pmatrix}$

(c) (10 points) Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x-y\\y-z\\2x+3y-3z\end{bmatrix}$, Find $[T]_\alpha^\alpha$.

$$[T]_\alpha^\alpha = \begin{bmatrix}1 & -1 & 0\\0 & 1 & -1\\2 & 3 & -3\end{bmatrix}$$

(d) (10 points) Express $[T]_\beta^\beta$ as the product of the three matrices found above.

$$[T]_\beta^\beta = [I]_\alpha^\beta [T]_\alpha^\alpha [I]_\beta^\alpha = P^{-1}[T]_\alpha^\alpha P.$$

4a

4. (a) (10 points) Let A be an $n \times n$ matrix. Prove that a number λ is an eigenvalue of A if and only if

$$\det(\lambda I - A) = 0.$$

Proof: If λ is an eigenvalue of A, then there is a non-zero vector, v, such that $Av = \lambda v$. This says that $0 = \lambda v - Av \iff 0 = (\lambda I - A)(v)$. Since $v \neq 0$, $\lambda I - A$ has non-trivial solutions to the homogeneous equation, and therefore $\lambda I - A$ is not invertible, so $\det(\lambda I - A) = 0$. If $\det(\lambda I - A) = 0$, then $\lambda I - A$ is not invertible, so there is a non-zero v such that $(\lambda I - A)(v) = 0 \iff Av = \lambda v$, so λ is an eigenvalue.

(b) (10 points) Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$. Find the eigenvalue(s) and associated eigenvectors of A.

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda - 2 & -3 \\ -1 & \lambda - 3 \end{bmatrix}\right) = (\lambda-2)(\lambda-3) - 2 = \lambda^2 - 5\lambda + 4$$
$$= (\lambda - 1)(\lambda - 4)$$

$\lambda = 1$: $\begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ ∴ eigenvector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\lambda = 4$: $\begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ ∴ eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(c) (10 points) Define what it means for A to be similar to a matrix B, and what it means for A to be diagonalizable. Prove that if A is similar to B, and A is diagonalizable, then B is also diagonalizable.

A similar to B means that there is an invertible P such that $P^{-1}AP = B$. A diagonalizable means that there is a Q such that $Q^{-1}AQ = D$, for a diagonal matrix D. Showing that B is diagonalizable amounts to finding ~~the~~ a matrix, M, so that $M^{-1}BM$ is diagonal. So, since $A = PBP^{-1}$,

$$(P^{-1}Q)^{-1}B(P^{-1}Q) = Q^{-1}PBP^{-1}Q$$
$$= Q^{-1}AQ = D. \square$$

5. (10 points) Find the eigenvalues and associated eigenvectors of the matrix

$$A = \begin{bmatrix} -4 & 5 \\ -4 & 4 \end{bmatrix}$$

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda+4 & -5 \\ 4 & \lambda-4 \end{pmatrix} = \lambda^2 - 16 + 20 = \lambda^2 + 4$$
$$= (\lambda - 2i)(\lambda + 2i)$$

So, $\lambda = 2i$ and $\lambda = -2i$ are eigenvalues.

$$2iI - A = \begin{pmatrix} 2i+4 & -5 \\ 4 & 2i-4 \end{pmatrix} \xrightarrow{(2i+4)R_2 - 4R_1} \begin{pmatrix} 2i+4 & -5 \\ 0 & 0 \end{pmatrix} (-2i+4)R_1$$

$$\rightarrow \begin{pmatrix} 20 & -20+10i \\ 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -1+\frac{1}{2}i \\ 0 & 0 \end{pmatrix}$$

So, $\begin{pmatrix} 1 - \frac{1}{2}i \\ 1 \end{pmatrix}$ is an eigenvector associated to $\lambda = 2i$.

$\therefore \begin{pmatrix} 1 + \frac{1}{2}i \\ 1 \end{pmatrix}$ is an eigenvector associated to $\lambda = -2i$.

(Note: Recall that if v is an eigenvector, then kv is too, for any scalar, so, for $\lambda = 2i$, you can also have the eigenvector,
$2\begin{pmatrix} 1 - \frac{1}{2}i \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - i \\ 2 \end{pmatrix}$.)

6. Let $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(a) (10 points) Find the eigenvalues and associated eigenvectors of A.

$\det(\lambda I - A) = (\lambda - 2)(\lambda - 3)(\lambda - 1)$;

$\lambda = 1$: $1I - A = \begin{pmatrix} -1 & -2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector.

$\lambda = 2$: $2I - A$ $\begin{pmatrix} 0 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector.

$\lambda = 3$: $3I - A = \begin{pmatrix} 1 & -2 & \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector.

(b) (10 points) Determine if A is diagonalizable. If it is, give ~~the~~ a matrix P and ~~the~~ a diagonal matrix D such that $P^{-1}AP = D$.

$\dim(E_1) + \dim(E_2) + \dim(E_3) = 3$, so diagonalizable

If $P = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, ~~then~~ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, then

$P^{-1}AP = D$.

7. Let $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

(a) (10 points) Find the eigenvalues and associated eigenvectors of A.

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda & -2 & -2 \\ -2 & \lambda & -2 \\ -2 & -2 & \lambda \end{pmatrix} = \lambda(\lambda^2-4) + 2\left(\overset{-2\lambda-4}{\cancel{4+2\lambda}}\right) - 2(4+2\lambda) = (\lambda+2)^2(\lambda-4)$$

$\lambda = 4$: $\begin{pmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $\therefore$ eigenvector is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$\lambda = -2$: $\begin{pmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\therefore$ $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are eigenvectors.

(b) (10 points) Determine if A is diagonalizable. If it is, give the matrix P and the diagonal matrix D such that $P^{-1}AP = D$.

$\dim(E_4) + \dim(E_{-2}) = 1 + 2 = 3$ $\therefore$ diagonalizable.

If $P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ then

$P^{-1}AP = D$.

8. Let $A = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$

(a) (10 points) Find the eigenvalues and associated eigenvectors of A.

Ch. Poly of A: $\lambda^3 - 2\lambda^2 - 15\lambda \underset{\text{important.}}{+36}$ Note: $\lambda = -4$ is a root of this polynomial.
as such, $(\lambda+4)$ is a factor... one can show that $\dfrac{\lambda^3 - 2\lambda^2 - 15\lambda + 36}{\lambda + 4} = (\lambda - 3)^2$
so that $\lambda^3 - 2\lambda^2 - 15\lambda + 36 = (\lambda-3)^2(\lambda+4)$. (Yes, I know this is a bit tricky).

$\lambda = -4$, $-4I - A = \begin{pmatrix} -9 & -6 & -2 \\ 0 & -3 & 8 \\ -1 & 0 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -8/3 \\ 0 & 0 & 0 \end{pmatrix}$ $\therefore$ eigenvector is $\begin{pmatrix} -2 \\ 8/3 \\ 1 \end{pmatrix}$

$\lambda = 3$, $3I - A = \begin{pmatrix} -2 & -6 & -2 \\ 0 & 4 & 8 \\ -1 & 0 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ $\therefore$ eigenvector is $\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$ (sorry)

(b) (10 points) Determine if A is diagonalizable. If it is, give the matrix P and the diagonal matrix D such that $P^{-1}AP = D$.

$\dim(E_{-4}) + \dim(E_3) = 1 + 1 = 2 \neq 3$, so,

A is <u>not</u> diagonalizable.

9. Suppose that A is a matrix with characteristic polynomial $p(\lambda) = (\lambda - 3)^2(\lambda - 2)^2$.

(a) (10 points) If $\dim(E_3) = 2$ and $\dim(E_2) = 2$ what is the Jordan Normal Form of A?

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

(b) (10 points) If $\dim(E_3) = 1$ and $\dim(E_2) = 2$ what is the Jordan Normal Form of A?

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

10. (20 points) Suppose that A is a matrix with characteristic polynomial $p(\lambda) = (\lambda+1)^2(\lambda-5)^4$. If we decide on a Jordan Normal Form, J, of A as

$$J = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

where B_1 is a 2×2 matrix and B_2 is a 4×4 matrix, what are the possibilities (up to permutation of the Jordan blocks) of B_1 and B_2?

$B_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ and

B_2: $\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

$\begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

11. (10 points) Is the following $n \times n$ matrix diagonalizable?

$$A = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

NO. The only eigenvalue is 1, and by the below theorem, since it's no just 1's down the diagonal it's not diagonalizable.

(Also, you can do this: $1I - A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \\ 0 & 0 & 0 & & 0 \end{pmatrix}$, and see that there are $n-1$ free variables in $\ker(1I-A)$, so $\dim(E_1) = n-1 < n$ $\therefore$ not diagonalizable)

12. (10 points) Prove that if A has one eigenvalue, r, then A is diagonalizable if and only if $A = rI$.

($\Leftarrow$) If $A = rI$, then $(I)^{-1}AI = A = rI$, which is diagonal.

($\Rightarrow$) If A has only one eigenvalue and is diagonalizable, then $\dim(E_r) = n$ when A is an $n \times n$. Since E_r consists of vectors in $\mathbb{R}^n$, we see that $E_r = \mathbb{R}^n$. Therefore each stantard basis vector for $\mathbb{R}^n$ is an eigenvector for A w/ eigenvalue r. Now,

$$\begin{aligned} A = AI = A[e_1 e_2 \cdots e_n] &= [Ae_1 Ae_2 \cdots Ae_n] \\ &= [re_1\ re_2 \cdots re_n] \\ &= rI. \end{aligned}$$

$\square$.

13. (10 points) Prove that a matrix A is invertible if and only if 0 is not an eigenvalue of A.

($\Rightarrow$) If A is invertible, then $Av=0 \Rightarrow v=0$, so 0 is not an eigenvalue of A.

($\Leftarrow$) If 0 is not an eigenvalue of A, then if A is not inv. then $Av=0$ for $v\neq 0$, $\therefore$ 0 is an eigenvalue. $\therefore$ A must be invertible.

14. (10 points) Prove that if A is invertible and λ is an eigenvalue of A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

If $Av=\lambda v$, then $v=\lambda A^{-1}v$ and further that

$$\frac{1}{\lambda}v = A^{-1}v,$$

so $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. (Notice that $\lambda\neq 0$ as A is invertible).

15. (10 points) Prove that if λ is an eigenvalue of A , then λ^k is an eigenvalue of A^k.

This is easily seen as if $Av=\lambda v$, then $A^2v=A(\lambda v)=\lambda^2 v$

and (further) $A^k(v) = \overbrace{A(A(\cdots A(v)\cdots))}^{k\text{-times}} = \lambda^k v.$ $\square$

16. (10 points) Suppose that A and B are similar matrices. Show that they have the same eigenvalues.

If A and B are similar, then there is an inv. P such that $P^{-1}AP=B$. If $Bv=\lambda v$, then as $AP=PB$, $AP(v)=PB(v)=P(\lambda v)=\lambda Pv$, so λ is an eigenvalue of A (w/ eigenvector Pv). Since similarity is a reflexive property, we are done.

17. (10 points) Prove or provide a counter example: Similar matrices have the same eigenvectors.

The above proof should give you a big hint on how to construct a counter example (or even some of the previous problems...).