Math 307
Spring 2019
Final - Practice

Name (Print): ____________________

Time Limit: 120 Minutes

Problem	Points	Score
1	80	
2	10	
3	10	
4	10	
5	15	
6	20	
7	50	
8	15	
9	20	
10	10	
11	10	
12	10	
13	10	
14	10	
15	20	
Total:	300	

1. (60 points) Answer true or false to the following statements:

If V is a vector space of dimension n, then V is an $n \times n$ matrix.

If 0 is an eigenvalue of A, then A is not invertible.

If V is a vector space of dimension n, then any set of n vectors must be linearly independent.

If $\det(A) \neq 0$ then A is invertible.

If A is an $n \times n$ invertible matrix, then the rows of A are linerarly independent

$\det(A + B) = \det(A) + \det(B)$

$\det(AB) = \det(A)\det(B)$

If T is a linear transformation, then so is $T + T^2$.

If A is invertible and B is invertible then $A + B$ is invertible.

If v_1 and v_2 are linearly independent then there exists nonzero constants, c_1 and c_2, such that $c_1 v_1 + c_2 v_2 = 0$.

If λ is an eigenvalue of A then $\det(\lambda I - A) = 0$.

If V is a vector space of dimension n, then any collection of n vectors forms a basis for V.

If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

If A is an $n \times n$ matrix with n distinct eigenvalues, then A is invertible.

If A is an $n \times n$ matrix that is invertible, then A is diagonalizable.

If A is an $n \times n$ matrix that is diagonalizable, then A is invertible.

(a) (10 points) Give a basis for the null space $NS(A)$ and the column space $CS(A)$ of the following matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix}$$

(b) (10 points) Solve the following linear system

$$\begin{aligned} x + y + z &= 0 \\ 2x - y + 3z &= 1 \\ -x + 2y - 2z &= -1 \end{aligned}$$

2. (10 points) Consider the matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Determine $\dim(E_1)$, $\dim(E_2)$ and $\dim(E_3)$.

3. (10 points) Determine which of the following matrices are similar to the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

4. (10 points) Suppose that the characteristic polynomial of A is $(\lambda-1)^4(\lambda-2)^3$. If $\dim(E_1) = 2$ and $\dim(E_2) = 1$, give the possible (up to permutation) Jordan Normal Forms of A.

5. Consider the homogeneous first order linear differential system

$$\begin{aligned} x' &= x - y + z \\ y' &= 2y + z \\ z' &= 2z \end{aligned}$$

where x, y, z are all functions of the variable t.

(a) (5 points) Write the system in the matrix form $Y' = AY + G$.

(b) (5 points) Is the matrix A diagonalizable? Explain your answer.

(c) (5 points) Find the general solution to the homogeneous system $Y' = AY$.

6. (20 points) Solve the initial value problem

$$Y' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} Y + \begin{bmatrix} e^x \\ 0 \end{bmatrix}, \qquad Y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

7. Let's consider the non-homogeneous first order linear differential system

$$x' = -4x - 3y + 3z$$
$$y' = 3x + 2y - 3z + e^t$$
$$z' = -3x - 3y + 2z$$

where x, y, z are all functions of the variable t.

(a) (5 points) Write the system in the matrix form $Y' = AY + G$.

(b) (15 points) Find a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$.

(c) (5 points) Find the general solution to the homogeneous system $Z' = DZ$.

(d) (5 points) Use the result of the previous question to find the general solution Y_H to the homogeneous system $Y' = AY$, and find a fundamental set, $Y_1,\ Y_2,\ Y_3$, of solutions to this system.

(e) (15 points) Using the matrix $M = [Y_1\ Y_2\ Y_3]$, compute a particular solution Y_p to the non-homogeneous system $Y' = AY + G$.

(f) (5 points) Use the result of the previous question to find the general solution to the non-homogeneous system $Y' = AY + G$.

8. Consider the following system:

$$y_1{}'' = y_1' + y_2' + e^x$$
$$y_2{}'' = y_1 + y_2 + \sin(x)$$

where y_1 and y_2 are functions of x.

(a) (10 points) Write this system as a system of first order liner differential equations in the form $Y' = AY + G$.

(b) (5 points) Give a condition on A so that we can solve this system with the techniques learned in this class.

9. (a) (5 points) Write the following differential equation as a system of first order linear equations:

$$y'' + 4y' + 3y = 0$$

(b) (10 points) Solve the system from part a).

(c) (5 points) Verify your solution by computing the kernel of the appropriate differential operator.

10. (10 points) Let be A an $n \times n$ matrix, prove that $\text{Rank}(A) = n$ if and only if A is invertible.

11. (10 points) Prove that if A and B are row equivalent, then $NS(A) = NS(B)$.

12. (10 points) Let A be an $n \times n$ matrix. Prove that a number λ is an eigenvalue of A if and only if

$$\det(\lambda I - A) = 0.$$

13. (10 points) Prove that if A is similar to B, and A is diagonalizable, then B is also diagonalizable.

14. (10 points) Prove that a matrix A is invertible if and only if 0 is not an eigenvalue of A.

15. (20 points) State 5 properties that are all equivalent to a matrix being invertible.