Math 307
Spring 2019
Exam 2
3/27/19
Time Limit: ∞/∞

Name (Print): ______________________________

Problem	Points	Score
1	15	
2	20	
3	35	
4	30	
5	10	
6	20	
7	20	
8	20	
9	20	
10	20	
11	10	
12	10	
13	10	
14	10	
15	10	
16	10	
17	10	
Total:	280	

1. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be transformations defined by

$$S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ x - y \end{bmatrix}, \qquad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - 4y \\ 3x \end{bmatrix}$$

(a) (5 points) Show that S and T are both linear transformations.

(b) (5 points) Find $ST\begin{bmatrix} x \\ y \end{bmatrix}$ and $T^2\begin{bmatrix} x \\ y \end{bmatrix}$.

(c) (5 points) Find the matrices of S and T with respect to the standard basis for $\mathbb{R}^2$.

2. (a) (10 points) Prove or provide a counter example: If T is a linear transformation, then so is $T + T^2$.

(b) (10 points) Let $D : C^\infty(-\infty, \infty) \to C^\infty(-\infty, \infty)$ be the usual derivative operator. Find a basis for the kernel of the operator

$$(D^2 - 4D + 4)^2(D^2 + 1)$$

3. Let α be the standard basis for $\mathbb{R}^3$ and β the basis consisting of the vectors

$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

(a) (5 points) Find the change of basis matrix from α to β.

(b) (10 points) Find the change of basis matrix from β to α.

(c) (10 points) Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x-y\\y-z\\2x+3y-3z\end{bmatrix}$, Find $[T]_\alpha^\alpha$.

(d) (10 points) Express $[T]_\beta^\beta$ as the product of the three matrices found above.

4. (a) (10 points) Let A be an $n \times n$ matrix. Prove that a number λ is an eigenvalue of A if and only if

$$\det(\lambda I - A) = 0.$$

(b) (10 points) Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$. Find the eigenvalue(s) and associated eigenvectors of A.

(c) (10 points) Define what it means for A to be similar to a matrix B, and what it means for A to be diagonalizable. Prove that if A is similar to B, and A is diagonalizable, then B is also diagonalizable.

5. (10 points) Find the eigenvalues and associated eigenvectors of the matrix

$$A = \begin{bmatrix} -4 & 5 \\ -4 & 4 \end{bmatrix}$$

6. Let $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   (a) (10 points) Find the eigenvalues and associated eigenvectors of A.

   (b) (10 points) Determine if A is diagonalizable. If it is, give the matrix P and the diagonal matrix D such that $P^{-1}AP = D$.

7. Let $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

(a) (10 points) Find the eigenvalues and associated eigenvectors of A.

(b) (10 points) Determine if A is diagonalizable. If it is, give the matrix P and the diagonal matrix D such that $P^{-1}AP = D$.

8. Let $A = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$

(a) (10 points) Find the eigenvalues and associated eigenvectors of A.

(b) (10 points) Determine if A is diagonalizable. If it is, give the matrix P and the diagonal matrix D such that $P^{-1}AP = D$.

9. Suppose that A is a matrix with characteristic polynomial $p(\lambda) = (\lambda - 3)^2(\lambda - 2)^2$.

   (a) (10 points) If $\dim(E_3) = 2$ and $\dim(E_2) = 2$ what is the Jordan Normal Form of A?

   (b) (10 points) If $\dim(E_3) = 1$ and $\dim(E_2) = 2$ what is the Jordan Normal Form of A?

10. (20 points) Suppose that A is a matrix with characteristic polynomial $p(\lambda) = (\lambda+1)^2(\lambda-5)^4$. If we decide on a Jordan Normal Form, J, of A as

$$J = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

where B_1 is a 2×2 matrix and B_2 is a 4×4 matrix, what are the possibilities (up to permutation of the Jordan blocks) of B_1 and B_2?

11. (10 points) Is the following $n \times n$ matrix diagonalizable?

$$\begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

12. (10 points) Prove that if A has one eigenvalue, r, then A is diagonalizable if and only if $A = rI$.

13. (10 points) Prove that a matrix A is invertible if and only if 0 is not an eigenvalue of A.

14. (10 points) Prove that if A is invertible and λ is an eigenvalue of A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

15. (10 points) Prove that if λ is an eigenvalue of A , then λ^k is an eigenvalue of A^k.

16. (10 points) Suppose that A and B are similar matrices. Show that they have the same eigenvalues.

17. (10 points) Prove or provide a counter example: Similar matrices have the same eigenvectors.