Math 307
Spring 2019
Final - Practice

Name (Print): Solutions

Time Limit: 120 Minutes

Problem	Points	Score
1	80	
2	10	
3	10	
4	10	
5	15	
6	20	
7	50	
8	15	
9	20	
10	10	
11	10	
12	10	
13	10	
14	10	
15	20	
Total:	300	

1. (60 points) Answer true or false to the following statements:

If V is a vector space of dimension n, then V is an $n \times n$ matrix.

False

If 0 is an eigenvalue of A, then A is not invertible.

True

If V is a vector space of dimension n, then any set of n vectors must be linearly independent.

False

If $\det(A) \neq 0$ then A is invertible.

True

If A is an $n \times n$ invertible matrix, then the rows of A are linerarly independent

True

$\det(A + B) = \det(A) + \det(B)$

False

$\det(AB) = \det(A)\det(B)$

True

If T is a linear transformation, then so is $T + T^2$.

True

If A is invertible and B is invertible then $A + B$ is invertible.

False

If v_1 and v_2 are linearly independent then there exists nonzero constants, c_1 and c_2, such that $c_1v_1 + c_2v_2 = 0$.

False

If λ is an eigenvalue of A then $\det(\lambda I - A) = 0$.

True

If V is a vector space of dimension n, then any collection of n vectors forms a basis for V.

False

If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

True

If A is an $n \times n$ matrix with n distinct eigenvalues, then A is invertible.

False

If A is an $n \times n$ matrix that is invertible, then A is diagonalizable.

False

If A is an $n \times n$ matrix that is diagonalizable, then A is invertible.

False

(a) (10 points) Give a basis for the null space $NS(A)$ and the column space $CS(A)$ of the following matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix}$$

NS(A): The rref(A) is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$, so, any vector in NS(A) is of the form $t \begin{bmatrix} -4/3 \\ 1/3 \\ 1 \end{bmatrix}$, for some t, i.e. $\begin{bmatrix} -4/3 \\ 1/3 \\ 1 \end{bmatrix}$ forms a basis for NS(A).

CS(A): The rref(A^T) is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, and so $(\text{rref}(A^T))^T$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$.

Hence, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ form a basis for CS(A).

(b) (10 points) Solve the following linear system

$$\begin{aligned} x + y + z &= 0 \\ 2x - y + 3z &= 1 \\ -x + 2y - 2z &= -1 \end{aligned}$$

One way to do this using the first part of the problem is

1. Find one solution, say (x_0, y_0, z_0), to the system
2. All solutions will be of the form $(x_0, y_0, z_0) + NS(A)$.

To find one solution, set $z=0$ and see that $(1/3, -1/3, 0)$ gives a solution.

So, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4/3 \\ 1/3 \\ 1 \end{bmatrix}$ for $t \in \mathbb{R}$

gives all solutions to the system.

2. (10 points) Consider the matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Determine $\dim(E_1)$, $\dim(E_2)$ and $\dim(E_3)$.

$= 2$ $\quad = 1$ $\quad = 1$

3. (10 points) Determine which of the following matrices are similar to the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$.

Yes $\quad$ NO $\quad$ NOT in JNF, but it's easy to check that this is diagonalizable!

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Yes $\quad$ NO $\quad$ NO

4. (10 points) Suppose that the characteristic polynomial of A is $(\lambda-1)^4(\lambda-2)^3$. If $\dim(E_1) = 2$ and $\dim(E_2) = 1$, give the possible (up to permutation) Jordan Normal Forms of A.

The JNF of A is $\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$ where

$$B_1 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \underline{\text{OR}}$$

$$B_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

5. Consider the homogeneous first order linear differential system

$$x' = x - y + z$$
$$y' = 2y + z$$
$$z' = 2z$$

where x, y, z are all functions of the variable t.

(a) (5 points) Write the system in the matrix form $Y' = AY + G$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(b) (5 points) Is the matrix A diagonalizable? Explain your answer.

The eigenvalues of A are 1 and 2. Notice that

$$2I - A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So $\dim(E_2) = 1$. $\therefore$ A is not diagonalizable.

(c) (5 points) Find the general solution to the homogeneous system $Y' = AY$.

Firstly, $z = c_3 e^{2t}$. Next, $y' = 2y + e^{2t} \Rightarrow \frac{d}{dt}(ye^{-2t}) = c_3$.

So, $y = c_2 e^{2t} + c_3 t e^{2t}$. Further,

~~$(e^{-t}x)' = x - (c_2e^{2t} + c_3te^{2t}) + c_3e^{t}$ which gives~~

$$x' = x - (c_2 e^{2t} + c_3 t e^{2t}) + c_3 e^{2t},$$ so,

$$\frac{d}{dt}(xe^{-t}) = -c_2 e^{t} - c_3 t e^{t} + c_3 e^{t}$$ and whence

$$x = c_1 e^{t} + (-c_2 + 2c_3 - c_3 t) e^{2t}.$$

6. (20 points) Solve the initial value problem

$$Y' = \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}}_{A} Y + \underbrace{\begin{bmatrix} e^x \\ 0 \end{bmatrix}}_{G}, \qquad Y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenpairs of A are $\left(2+i, \begin{bmatrix} -i \\ 1 \end{bmatrix}\right)$ and $\left(2-i, \begin{bmatrix} i \\ 1 \end{bmatrix}\right)$. As such, if $P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$, then $P^{-1}AP = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}$, and we see that $\begin{bmatrix} e^{(2+i)x} \\ 0 \end{bmatrix}$ is a solution to $\varphi' = D\varphi$. Since $e^{(2+i)x} = e^{2x}(\cos(x) + i\sin(x))$,

$$P\begin{bmatrix} e^{(2+i)x} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} e^{2x}\sin(x) \\ e^{2x}\cos(x) \end{bmatrix}}_{\varphi_1} + i \underbrace{\begin{bmatrix} -e^{2x}\cos(x) \\ e^{2x}\sin(x) \end{bmatrix}}_{\varphi_2}.$$

This gives our matrix of solutions to $\varphi' = A\varphi$ to be

$$M = \begin{bmatrix} e^{2x}\sin(x) & -e^{2x}\cos(x) \\ e^{2x}\cos(x) & e^{2x}\sin(x) \end{bmatrix} \text{ and, so, } M^{-1} = \begin{bmatrix} e^{-2x}\sin(x) & e^{-2x}\cos(x) \\ -e^{-2x}\cos(x) & e^{-2x}\sin(x) \end{bmatrix}.$$

We have $M^{-1}G = \begin{bmatrix} e^{-x}\sin(x) \\ -e^{-x}\cos(x) \end{bmatrix}$ and so $\int M^{-1}G = \begin{bmatrix} \frac{-e^{-x}}{2}\sin(x) + \frac{-e^{-x}}{2}\cos(x) \\ \frac{e^{-x}}{2}\cos(x) - \frac{e^{-x}}{2}\sin(x) \end{bmatrix}$

This gives $\underbrace{M\int M^{-1}G}_{\varphi_P} = \begin{bmatrix} -e^x/2 \\ -e^x/2 \end{bmatrix}$.

So, our solution to $\varphi' = A\varphi + G$ is $c_1\varphi_1 + c_2\varphi_2 + \varphi_P$.

<u>IVP:</u> $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \varphi(0) = \begin{bmatrix} -c_2 \\ c_1 \end{bmatrix} + \begin{bmatrix} -1/2 \\ -1/2 \end{bmatrix} \quad \therefore c_1 = 3/2, \ c_2 = -3/2.$

So the solution to the IVP is

$$\varphi = \frac{3}{2}\varphi_1 - \frac{3}{2}\varphi_2 + \varphi_P.$$

7. Let's consider the non-homogeneous first order linear differential system

$$x' = -4x - 3y + 3z$$
$$y' = 3x + 2y - 3z + e^t$$
$$z' = -3x - 3y + 2z$$

where x, y, z are all functions of the variable t.

(a) (5 points) Write the system in the matrix form $Y' = AY + G$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} -4 & -3 & 3 \\ 3 & 2 & -3 \\ -3 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ e^t \\ 0 \end{bmatrix}$$

(b) (15 points) Find a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$.

One finds $p(\lambda) = (\lambda+1)^2(\lambda-2)$ (make sure you can do this by hand!)

$\lambda = -1$ $\quad -1 - A = \begin{bmatrix} +3 & +3 & -3 \\ -3 & -3 & +3 \\ 3 & 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore$ eigenvectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 2$ $\quad 2 - A = \begin{bmatrix} 6 & 3 & -3 \\ -3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore$ eigenvector $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(c) (5 points) Find the general solution to the homogeneous system $Z' = DZ$.

$$\varphi_{H,D} = \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{-t} \\ c_3 e^{2t} \end{bmatrix}$$

(d) (5 points) Use the result of the previous question to find the general solution Y_H to the homogeneous system $Y' = AY$, and find a fundamental set, Y_1, Y_2, Y_3, of solutions to this system.

$$\varphi_{H,A} = P\varphi_{H,D} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{-t} \\ c_3 e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{-t} + c_2 e^{-t} + c_3 e^{2t} \\ -c_1 e^{-t} - c_3 e^{2t} \\ c_2 e^{-t} + c_3 e^{2t} \end{bmatrix}$$

$$= c_1 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{-t}}_{\varphi_1} + c_2 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-t}}_{\varphi_2} + c_3 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{2t}}_{\varphi_3}$$

(e) (15 points) Using the matrix $M = [Y_1\ Y_2\ Y_3]$, compute a particular solution Y_p to the non-homogeneous system $Y' = AY + G$.

$$M = \begin{bmatrix} e^{-t} & e^{-t} & e^{2t} \\ -e^{-t} & 0 & -e^{2t} \\ 0 & e^{-t} & e^{2t} \end{bmatrix} \quad \text{one finds } M^{-1} = \begin{bmatrix} e^{t} & 0 & -e^{t} \\ e^{t} & e^{t} & 0 \\ -e^{-2t} & -e^{-2t} & e^{-2t} \end{bmatrix},$$

Now, $M^{-1}G = \begin{bmatrix} 0 \\ e^{2t} \\ -e^{-t} \end{bmatrix} \Rightarrow \int M^{-1}G = \begin{bmatrix} 0 \\ \frac{1}{2}e^{2t} \\ e^{-t} \end{bmatrix}$

and hence, $\underbrace{M\int M^{-1}G}_{Y_p} = \begin{bmatrix} \frac{3}{2}e^{t} \\ -e^{t} \\ \frac{3}{2}e^{t} \end{bmatrix}$

(f) (5 points) Use the result of the previous question to find the general solution to the non-homogeneous system $Y' = AY + G$.

The general solution to is

$Y_{H,A} + Y_p$!!!

8. Consider the following system:

$$y_1'' = y_1' + y_2' + e^x$$
$$y_2'' = y_1 + y_2 + \sin(x)$$

where y_1 and y_2 are functions of x.

(a) (10 points) Write this system as a system of first order liner differential equations in the form $Y' = AY + G$.

Let $v_1 = y_1$ Now, $v_1' = v_2$

$v_2 = y_1'$ $v_2' = v_2 + v_3 + e^x$

$v_3 = y_2$ $v_3' = v_3$

$v_4 = y_2'$ $v_4' = v_1 + v_3 + \sin(x)$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(b) (5 points) Give a condition on A so that we can solve this system with the techniques learned in this class.

If A is diagonalizable, then we can solve this system!

9. (a) (5 points) Write the following differential equation as a system of first order linear equations:

$$y'' + 4y' + 3y = 0$$

$V_1 = y$

$V_2 = y'$

$\therefore \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}}_{A} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$

$V_1' = V_2$

$V_2' = -3V_1 - 4V_2$

(b) (10 points) Solve the system from part a).

One finds the eigenvalues of A to be -3 and -1.

$-1 - A = \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ eigen vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$-3 - A = \begin{bmatrix} -3 & -1 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix}$ eigenvector $\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$.

$\varphi_H = \begin{bmatrix} -1 & -1/3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-x} \\ c_2 e^{-3x} \end{bmatrix} = \begin{bmatrix} -c_1 e^{-x} - \frac{1}{3} c_2 e^{-3x} \\ c_1 e^{-x} + c_2 e^{-3x} \end{bmatrix}$

(c) (5 points) Verify your solution by computing the kernel of the appropriate differential operator.

From φ_H we see that $y = v_1 = -c_1 e^{-x} - \frac{1}{3} c_2 e^{-3x}$

... But also, if $D^2 + 4D + 3$ is our operator, then we know that

$y = e^{-3x}$ and $y = e^{-x}$ form a basis for the kernel!

10. (10 points) Let be A an $n \times n$ matrix, prove that $\text{Rank}(A) = n$ if and only if A is invertible.

The proofs are all on old practice tests.