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Problem 1.1. *Show that*

$$\lim_{x\to\infty} \sqrt{x^2+x+1} - x = \frac{1}{2}$$

Proof. First we do some algebraic manipulation:

$$\sqrt{x^2+x+1} - x = (\sqrt{x^2+x+1} - x)\frac{\sqrt{x^2+x+1}+x}{\sqrt{x^2+x+1}+x}$$

$$= \frac{x^2+x+1-x^2}{\sqrt{x^2+x+1}+x}$$

$$= \frac{x+1}{\sqrt{x^2(1+\frac{1}{x}+\frac{1}{x^2})}+x}$$

Remember that $\sqrt{x^2} = |x|$, so when we "pull out" that x^2 we get

$$= \frac{x+1}{|x|\sqrt{(1+\frac{1}{x}+\frac{1}{x^2})}+x}$$

Since x is positive ($x \to \infty$ so I would hope that this is the case), $|x| = x$

$$= \frac{x+1}{x\sqrt{(1+\frac{1}{x}+\frac{1}{x^2})}+x}$$

$$= \frac{x(1+\frac{1}{x})}{x(\sqrt{(1+\frac{1}{x}+\frac{1}{x^2})}+1)}$$

$$= \frac{1+\frac{1}{x}}{\sqrt{(1+\frac{1}{x}+\frac{1}{x^2})}+1}$$

Now we can say that

$$\lim_{x\to\infty} \sqrt{x^2+x+1} - x = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{\sqrt{(1+\frac{1}{x}+\frac{1}{x^2})}+1}$$

Remember that $\lim_{x\to\infty} \frac{1}{x} = 0$, so when we evaluate the limit we get

$$= \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

$\square$

Question: When we evaluated the limit, what limit laws did we use?

Now we work an almost identical problem but with a slight change. We will need another version of "curlee's theorem".

Theorem 1.2. ***(Curlee's Theorem 2)*** *Suppose that*

$$\lim_{x\to c} f(x) = a \in \mathbb{R} \neq 0 \quad \textbf{\textit{and}} \quad \lim_{x\to c} g(x) = 0$$

. Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \pm\infty \quad \textbf{\textit{or does not exist}}$$

Note: this works for any $c \in \mathbb{R}$ and ALSO for $c = \infty$ or $c = -\infty$.

Problem 1.3. *Show that*

$$\lim_{x\to -\infty} \sqrt{x^2+x+1} - x = -\infty$$

Proof. First we do some algebraic manipulation:

$$\sqrt{x^2+x+1} - x = (\sqrt{x^2+x+1} - x)\frac{\sqrt{x^2+x+1}+x}{\sqrt{x^2+x+1}+x}$$

$$= \frac{x^2+x+1-x^2}{\sqrt{x^2+x+1}+x}$$

$$= \frac{x+1}{\sqrt{x^2(1+\frac{1}{x}+\frac{1}{x^2})}+x}$$

Remember that $\sqrt{x^2} = |x|$, so when we "pull out" that x^2 we get

$$= \frac{x+1}{|x|\sqrt{(1+\frac{1}{x}+\frac{1}{x^2})}+x}$$

Since x is negative, $|x| = -x$

$$= \frac{x+1}{-x\sqrt{(1+\frac{1}{x}+\frac{1}{x^2})}+x}$$

$$= \frac{x(1+\frac{1}{x})}{x(-\sqrt{(1+\frac{1}{x}+\frac{1}{x^2})}+1)}$$

$$= \frac{1+\frac{1}{x}}{-\sqrt{(1+\frac{1}{x}+\frac{1}{x^2})}+1}$$

Now we can say that

$$\lim_{x\to -\infty} \sqrt{x^2+x+1} - x = \lim_{x\to -\infty} \frac{1+\frac{1}{x}}{-\sqrt{(1+\frac{1}{x}+\frac{1}{x^2})}+1}$$

Since the denominator goes to 0 and the numerator goes to 1 we may invoke Curlee's Theorem 2 to say that $\lim_{x\to-\infty} \frac{1+\frac{1}{x}}{-\sqrt{(1+\frac{1}{x}+\frac{1}{x^2})}-1} = \pm\infty$. To decide which infinity, we need to see that the denominator is positive (for any $x \leq -2$) and the numerator is always positive, hence,

$$\lim_{x\to-\infty} \frac{1+\frac{1}{x}}{-\sqrt{(1+\frac{1}{x}+\frac{1}{x^2})}+1} = \infty.$$

$\square$