

Problem 1

~~x~~ (typo)

Integrate $G(x, y, z)$, over the parabolic cylinder $y = x^2$, $0 \leq x \leq 2$, $0 \leq z \leq 3$.

$$\mathbf{r}(x, z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$$

$$\mathbf{r}_x = 1\mathbf{i} + 2x\mathbf{j} + 0\mathbf{k}, \quad \mathbf{r}_z = 0\mathbf{i} + 0\mathbf{j} + 1\mathbf{k}$$

$$\mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} + \mathbf{j} + 0\mathbf{k}$$

$$|\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2 + 1}$$

$$\text{Thus, } \iint_S x \, d\sigma = \int_0^3 \int_0^2 x \sqrt{4x^2 + 1} \, dx \, dz = \frac{1}{4} ((57)^{3/2} - 1)$$

~~(this is a u-sub problem)~~

Problem 2

Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the sphere $x^2 + y^2 + z^2 = a^2$ (where a is a constant) in the direction away from the origin.

$$\text{From 15.5 (example 5)} \quad \mathbf{r}(\phi, \theta) = a\sin(\phi)\cos(\theta)\mathbf{i} + a\sin(\phi)\sin(\theta)\mathbf{j} + a\cos(\phi)\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin(\phi)$$

where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$

$$\text{and } \mathbf{r}_\phi \times \mathbf{r}_\theta = a^2 \sin^2(\phi) \cos(\theta)\mathbf{i} + a^2 \sin^2(\phi) \sin(\theta)\mathbf{j} + a^2 \sin(\phi) \cos(\phi)\mathbf{k}$$

Note: we should check to see if this vector points away from the origin. If not then reverse the order of the cross product.

$$\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|}, \quad \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^{2\pi} \int_0^\pi \mathbf{F} \cdot \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} \cdot |\mathbf{r}_\phi \times \mathbf{r}_\theta| \, d\phi \, d\theta$$

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = a\sin(\phi)\cos(\theta)\mathbf{i} + a\sin(\phi)\sin(\theta)\mathbf{j} + a\cos(\phi)\mathbf{k}$$

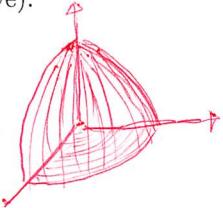
$$\mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = a^3 \sin(\phi) \quad (\text{after some simplification})$$

$$\text{Thus, Flux} = \int_0^{2\pi} \int_0^\pi a^3 \sin(\phi) \, d\phi \, d\theta$$

$$= \int_0^{2\pi} a^3 \cdot 2 \, d\theta = 4\pi a^3$$

Problem 3

Find the centroid of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ that lies in the first octant (x, y, z are all positive).



$$\text{Center of Mass} = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right)$$

From before, $r(\phi, \theta) = a \sin(\phi) \cos(\theta) \mathbf{i} + a \sin(\phi) \sin(\theta) \mathbf{j} + a \cos(\phi) \mathbf{k}$
 only now ω ~~not~~ $0 \leq \phi \leq \pi/2, 0 \leq \theta \leq \pi/2$

$$\begin{aligned} M &= \iint_S d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} |r(\phi, \theta)| d\phi d\theta \\ &= \int_0^{\pi/2} \int_0^{\pi/2} a^2 \sin(\phi) d\phi d\theta \\ &= \frac{a^3 \pi}{2} \end{aligned}$$

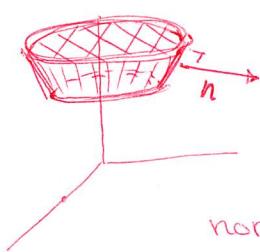
$$M_{yz} = \iint_S x d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} a^3 \sin^2(\phi) \cos(\theta) d\phi d\theta = \frac{a^3 \pi}{4}.$$

Geometry says that $M_{yz} = M_{xz} = M_{xy}$, so,

$$\text{Center of mass} = \left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2} \right)$$

Problem 4

Find the flux of $\mathbf{F} = -xi - yj + z^2k$ across the portion of the cone $z = \sqrt{x^2 + y^2}$ that lies between $z = 1$ and $z = 2$ in the direction away from the z -axis.



$$\begin{aligned} r(r, \theta) &= r \cos(\theta) \mathbf{i} + r \sin(\theta) \mathbf{j} + r \mathbf{k} \Rightarrow (|r_r \times r_\theta| = \sqrt{2}r, \\ &\theta \leq \theta \leq 2\pi, 1 \leq r \leq 2) \\ r_r \times r_\theta &= (r \cos(\theta)) \mathbf{i} + (-r \sin(\theta)) \mathbf{j} + r \mathbf{k} \end{aligned}$$

Note: we won't need this...

Note: This vector points toward the z -axis, so, the normal vector we want is $r_\theta \times r_r = r \cos(\theta) \mathbf{i} + r \sin(\theta) \mathbf{j} - r \mathbf{k}$
 (How did I know this? By looking at the picture, the normal vector needs to point down (negative k component))

$$\begin{aligned} \text{Now, Flux} &= \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_S \mathbf{F} \cdot \frac{(r_\theta \times r_r)}{|r_\theta \times r_r|} \cdot |r_\theta \times r_r| d\theta dr \\ &= \int_0^{2\pi} \int_1^2 -r^2 - r^3 dr d\theta \\ &= -\frac{73\pi}{6} \end{aligned}$$

Note: The negative flux is expected since vectors in \mathbf{F} point up and towards the z -axis (going into the cone).

Problem 1

Let $\mathbf{F} = x^2\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$ and C be the ellipse $4x^2 + y^2 = 4$ in the xy -plane traversed counterclockwise when viewed from above. Use a surface integral in Stokes' Theorem to find the circulation of \mathbf{F} around C .

For an ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, one parametrization is $x = a\cos(\phi)$, $y = b\sin(\phi)$, and since we'd like a "filled in" ellipse we can use

$$\mathbf{r}(r, \phi) = r\cos(\phi)\mathbf{i} + 2r\sin(\phi)\mathbf{j} + 0\mathbf{k} \quad 0 \leq r \leq 1, \quad 0 \leq \phi \leq 2\pi.$$

$$\nabla \times \mathbf{F} = 2\mathbf{k}, \text{ and } \mathbf{n} = \mathbf{k} \text{ so } \nabla \times \mathbf{F} \cdot \mathbf{n} = 2.$$

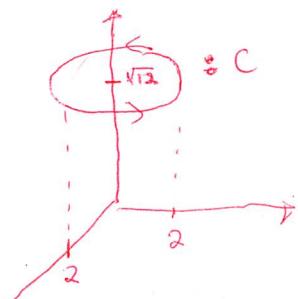
$$\mathbf{r}_r \times \mathbf{r}_\phi = 2r\mathbf{k}, \text{ thus, } |\mathbf{r}_r \times \mathbf{r}_\phi| = 2r. \text{ By Stokes' Theorem,}$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma = \int_0^{2\pi} \int_0^1 2 \cdot 2r \, dr \, d\phi$$

$$= 4\pi$$

Problem 2

Let $\mathbf{F} = x^2y^3\mathbf{i} + \mathbf{j} + z\mathbf{k}$ and C the intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$, counterclockwise when viewed from above. Use a surface integral in Stokes' Theorem to find the circulation of \mathbf{F} around C .



Recall that Any surface whose boundary is C can be used to find the circulation. So we could use (part of the sphere) OR just flat circle plate

For the flat circle plate: $\mathbf{r}(r, \phi) = r\cos(\phi)\mathbf{i} + r\sin(\phi)\mathbf{j} + \sqrt{12}\mathbf{k}$
for $0 \leq \phi \leq 2\pi$, $0 \leq r \leq 2$

$$|\mathbf{r}_r \times \mathbf{r}_\phi| = r\mathbf{k}, \mathbf{n} = \mathbf{k} \text{ and } \nabla \times \mathbf{F} = 3x^2y^2\mathbf{k}, \text{ Thus, } (\nabla \times \mathbf{F}) \cdot \mathbf{n} = 3x^2y^2$$

$$\text{So, } \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma = \iint_S -3x^2y^2 d\sigma = \int_0^{2\pi} \int_0^2 -3r^5 \sin^2(\phi) \cos^2(\phi) dr d\phi$$

$$= \cancel{-32} \int_0^{2\pi} \sin^2(\phi) \cos^2(\phi) d\phi$$

$$= \cancel{-32} \cdot \frac{\pi}{4} = -8\pi$$

Problem 3

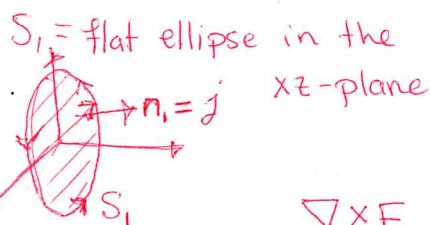
Let \mathbf{n} be the outer normal unit vector (away from the origin) of the parabolic shell

$$S : 4x^2 + y + z^2 = 4, \quad y \geq 0,$$

and let

$$\mathbf{F} = \left(-z + \frac{1}{2+x} \right) \mathbf{i} + \left(\tan^{-1}(y) \right) \mathbf{j} + \left(x + \frac{1}{4+z} \right) \mathbf{k}.$$

Find the value of



$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma$$

↑ S_1

By Stokes' Theorem



$$\nabla \times \mathbf{F} = -2 \mathbf{j}, \quad \text{so } (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1 = (\nabla \times \mathbf{F}) \cdot \mathbf{j} = -2$$

Short Cut: The area of S_1 is 2π (why?) and

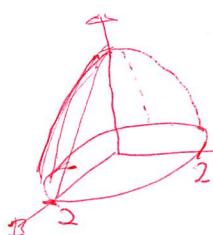
thus, $\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = -2 \iint_{S_1} d\sigma = -2(2\pi) = -4\pi$

since S_1 is flat, $\iint_{S_1} d\sigma$ is already known.

Also, if you pay close attention, we computed this

Problem 4 area already in problem #1.

Let $\mathbf{F} = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$, and S the surface parametrized by $\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + (4 - r^2) \mathbf{k}$ for $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 2$. Calculate the flux of $\nabla \times \mathbf{F}$ across S in the direction of the outward normal \mathbf{n} . Hint: Use Stokes' Theorem.



The flux of $\nabla \times \mathbf{F}$ across S is $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma$,

and yet again we can use Stokes' theorem...

The new surface can be a circle in the xy -plane

with radius 2, call it S_1 . Now $\mathbf{r}_1(r, \phi) = r \cos(\phi) \mathbf{i} + r \sin(\phi) \mathbf{j}$

where $0 \leq r \leq 2$ and $0 \leq \phi \leq 2\pi$. The new normal vector

is just the \mathbf{k} vector. So, $\nabla \times \mathbf{F} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $(\nabla \times \mathbf{F}) \cdot \mathbf{k} = 3$.

Thus, $\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, d\sigma = 3 \iint_{S_1} d\sigma = 3(4\pi) = 12\pi$

↑
area of S_1