

Math 244
Summer 2015
Final
7/13/15
Time Limit: 80 Minutes

Name (Print): _____

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	20	
6	20	
7	15	
8	15	
Total:	130	

1. (15 points) Integrate $f(x, y, z) = 20x^5z^2$ over the path from $(0, 0, 0)$ to $(1, 1, 1)$ given by the curve

$$C : \quad r(t) = t\mathbf{i} + t^5\mathbf{j} + t\mathbf{k} \quad 0 \leq t \leq 1$$

$$\frac{dr}{dt} = 1\mathbf{i} + 5t^4\mathbf{j} + 1\mathbf{k}$$

$$\begin{aligned} \left| \frac{dr}{dt} \right| &= \sqrt{1^2 + (5t^4)^2 + 1^2} \\ &= \sqrt{2 + 25t^8} \end{aligned}$$

Along C , $f(x, y, z) = 20t^7$, so

$$\int_C f(x, y, z) ds = \int_0^1 20t^7 \sqrt{2 + 25t^8} dt$$

$$u = 2 + 25t^8$$

$$du = 200t^7 dt$$

$$\Rightarrow \frac{1}{10} \int_2^{27} \sqrt{u} du$$

$$= \frac{1}{10} \left(u^{\frac{3}{2}} \Big|_2^{27} \right)$$

$$= \frac{1}{15} \left((27)^{\frac{3}{2}} - (2)^{\frac{3}{2}} \right)$$

2. (15 points) Find the work done by $\mathbf{F} = 2y\mathbf{i} + 2x\mathbf{j} + 4z\mathbf{k}$ on the curve $C : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}$ for $0 \leq t \leq 1$.

$$\begin{aligned}\text{Work} &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2t^2\mathbf{i} + 2t\mathbf{j} + 4t^4\mathbf{k}) \cdot (1\mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k}) dt \\&= \int_0^1 2t^2 + 4t^2 + 16t^7 dt \\&= \int_0^1 6t^2 + 16t^7 dt \\&= \left[6\frac{t^3}{3} + \frac{16t^8}{8} \right]_0^1 \\&= 4\end{aligned}$$

3. (15 points) Show that $\mathbf{F} = e^{y+2z}(\mathbf{i} + x\mathbf{j} + 2x\mathbf{k})$ is conservative, then find the potential function, f , that satisfies $f(0) = 0$ (Hint: The $f(0) = 0$ part determines the constant term, C , in the potential function).

We check $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

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from the i^{th} component of \mathbf{F}

$$\Rightarrow f = e^{y+2z}x + g(y, z)$$

since $\frac{\partial f}{\partial y} = e^{y+2z}x + \frac{\partial g}{\partial y} \Rightarrow g(y, z) = h(z)$.

and $\frac{\partial f}{\partial z} = e^{y+2z} \cdot 2x + h'(z) \Rightarrow h'(z) = 0$
 $\Rightarrow h(z) = C$.

Thus, $f = e^{y+2z}x + C$

and $e^{0+0} \cdot 0 + C = f(0) = 0 \Rightarrow C = 0$.

Thus, $f = x e^{y+2z}$

4. (15 points) Compute the circulation $\oint_C (6y + x) dx + (y + 2x) dy$, where C is the circle $(x - 2)^2 + (y - 3)^2 = a^2$ (a is a constant) traversed once in the counterclockwise direction. Feel free to use facts about the area of a circle in your computation.

$$\text{Green's Theorem for circulation: } \oint_C M dx + N dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} dA$$

$$\text{Thus, } \oint_C (6y + x) dx + (y + 2x) dy$$

$$= \iint_R 2 - 6 dA$$

$$= -4 \iint_R 1 dA$$

$$= -4 (\pi a^2)$$

5. (a) (10 points) Parametrize the portion of the plane $x+y+z = 2$ inside the cylinder $x^2+y^2 = 4$

$$x = r \cos(\phi), \quad y = r \sin(\phi) \dots \quad z = 2 - x - y$$

$$\mathbf{r}(r, \phi) = r \cos(\phi) \mathbf{i} + r \sin(\phi) \mathbf{j} + (2 - r \cos(\phi) - r \sin(\phi)) \mathbf{k}$$

$$0 \leq r \leq 2, \quad 0 \leq \phi \leq 2\pi$$

- (b) (10 points) Parametrize the portion of the cone $z = \sqrt{x^2 + y^2}$ that lies above the circle $(x-2)^2 + y^2 = 4$. inside the cylinder

$$\mathbf{r}(r, \phi) = (r \cos(\phi) + 2) \mathbf{i} + r \sin(\phi) \mathbf{j} + \sqrt{(r \cos(\phi) + 2)^2 + (r \sin(\phi))^2} \mathbf{k}$$

$$0 \leq r \leq 2, \quad 0 \leq \phi \leq 2\pi$$

6. (20 points) Set up the integral that gives the flux of $\mathbf{F} = -x\mathbf{i} - y\mathbf{j} + z^2\mathbf{k}$ across the portion of the cone $z = \sqrt{x^2 + y^2}$ that lies between $z = 1$ and $z = 2$ in the direction away from the z -axis.

See HW15 Solutions

7. (15 points) Let \mathbf{n} be the outer normal unit vector (away from the origin) of the sphere

$$S : \quad x^2 + y^2 + z^2 = 1, \quad z \geq 0,$$

and let

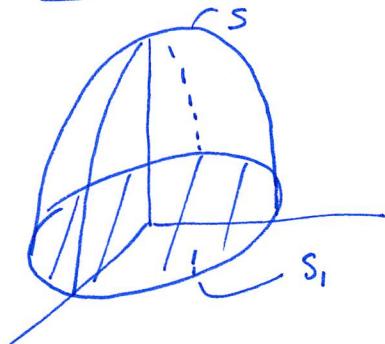
$$\mathbf{F} = \left(-y + \frac{1}{2+x} \right) \mathbf{i} + \left(\tan^{-1}(y) \right) \mathbf{j} + \left(x + \frac{1}{4+z} \right) \mathbf{k}.$$

Find the value of

$$\iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n} \, d\sigma$$

Hint: Use Stokes' Theorem to rewrite this as a surface integral over the unit circle in the xy -plane, then use a formula for area.

S_1



$$\text{Then } \mathbf{n} = \hat{\mathbf{k}}$$

$$\begin{aligned} \text{and } (\nabla \times \mathbf{F}) \cdot \mathbf{n} &= (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{k}} \\ &= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma \\ &= \iint_{S_1} 1 \, d\sigma \\ &= \pi \end{aligned}$$

8. (15 points) Find the outward (away from the origin) flux of $\mathbf{F} = \frac{x^2}{2}\mathbf{i} + 2\mathbf{j} + x^2y^2\mathbf{k}$ across a sphere centered at the origin with radius 2 cut by the first octant. The surface includes the sides and is closed.

Divergence Theorem:

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \mathbf{F} dV$$

$$\nabla \cdot \mathbf{F} = x + 0 + 0$$

$$\begin{aligned} \iiint_D x dV &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^2 r \sin(\phi) \cos(\theta) r^2 \sin(\phi) dr d\phi d\theta \\ &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{r^4}{4} \int_0^2 \sin^2(\phi) \cos(\theta) dr d\phi d\theta \\ &= 4 \int_0^{\pi/2} \int_0^{\pi/2} \sin^2(\theta) \cos(\theta) d\theta d\phi \\ &= \pi \int_0^{\pi/2} \cos(\theta) d\theta \\ &= \pi \left(\sin(\theta) \Big|_0^{\pi/2} \right) \\ &= \pi \end{aligned}$$

Note:

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \sin^2(\phi) d\phi \\ &= \int_0^{\pi/2} \frac{1 - \cos(2\phi)}{2} d\phi \\ &= \left(\frac{1}{2}\phi - \frac{\sin(2\phi)}{4} \right) \Big|_0^{\pi/2} \\ &= \frac{\pi}{4} \end{aligned}$$