Math 135 - Precalculus I
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Lines and Linear Equations

By a *linear equation* we mean an equation of the form

$$y = ax + b \, ,$$

where a and b are real numbers. The distingushing feature is the single power of the variable x.

Example 1. *The following are examples of linear equations.*

1. $y = 4x + 1$.

2. $y = x + 3$.

3. $y = x$.

4. $y = 5$.

A linear equation represents a line, that is the equation determines points in the plane which we can connect with a straight line. Moreover, given the graph of a line we can write down its (linear) equation. This requires two ingredients: the slope of the line and its y-intercept.

The slope represents the change along the line with respect to the y-axis versus the change with respect to the x-axis. Given two points in in the plane, (x_1, y_1) and (x_2, y_2), the slope of the line through them is found by computing

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} \, .$$

The letter m is commonly used for the slope of a line, thus equation (1) becomes

$$y = mx + b.$$

Whenever a linear equation has the above form we say that is is in the *slope-intercept form*.

Example 2. *Find the slope of the line passing through the points $(0, 1)$ and $(-1, 1)$.*
Let $(x_1, y_1) = (-1, 1)$ and let $(x_2, y_2) = (0, 1)$. Then,

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{1 - 1}{0 - (-1)} \\
&= \frac{0}{1} \\
&= 0.
\end{aligned}$$

Therefore, the slope of the line through the points $(0, 1)$ and $(-1, 1)$ is $m = 0$. See Figure 1.
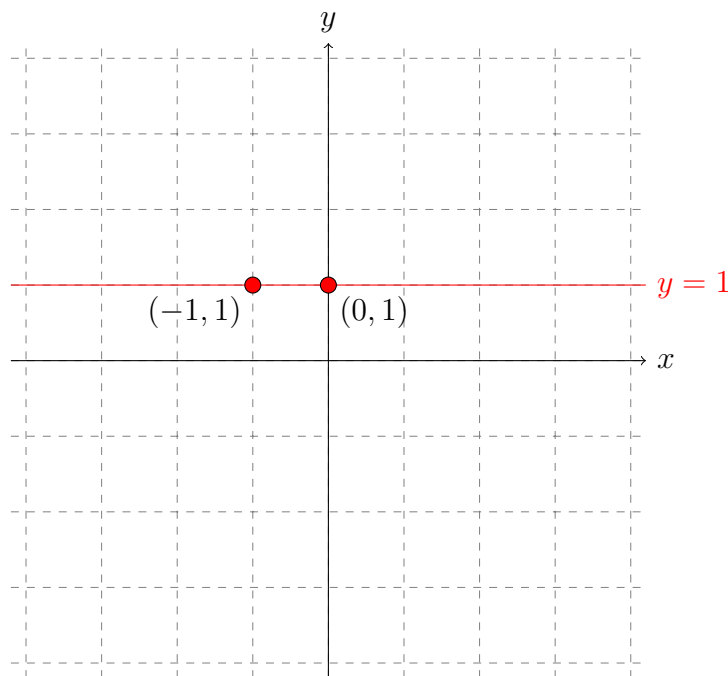
Figure 1: A line with slope zero.

It is important to designate a start- and an end-point in slope computations. In the above example the same answer is obtained when $(x_1, y_1) = (0, 1)$ and $(x_2, y_2) = (-1, 1)$. A common mistake is to take the points out of order.

Lines parallel to each other have the same slope. Observe that the line in Figure 1 is parallel to the x-axis. Another name for the x-axis is the line $y = 0$. It is clear that all horizontal lines have the same slope and are therefore all parallel to one another. Horizontal lines have a zero slope. These line have the form $y = c$, for some real number c. The line in the first example is a horizontal line with equation $y = 1$. Vertical lines are of the from $x = k$, where k is any real number.

Example 3. *Find the equation of a line perpendicular to the line in Figure 1.*
We already know that the equation of the line is $y = 1$. From this equation we read off the slope (the coefficient of x) to be 0. We cannot take the negative reciprocal without dividing by zero, but any vertical line will be perpendicular to $y = 1$. We can choose $x = -1$. The lines $y = 1$ and $x = -1$ are perpendicular vertical lines and intersect at the point $(-1, 1)$. See Figure 2.
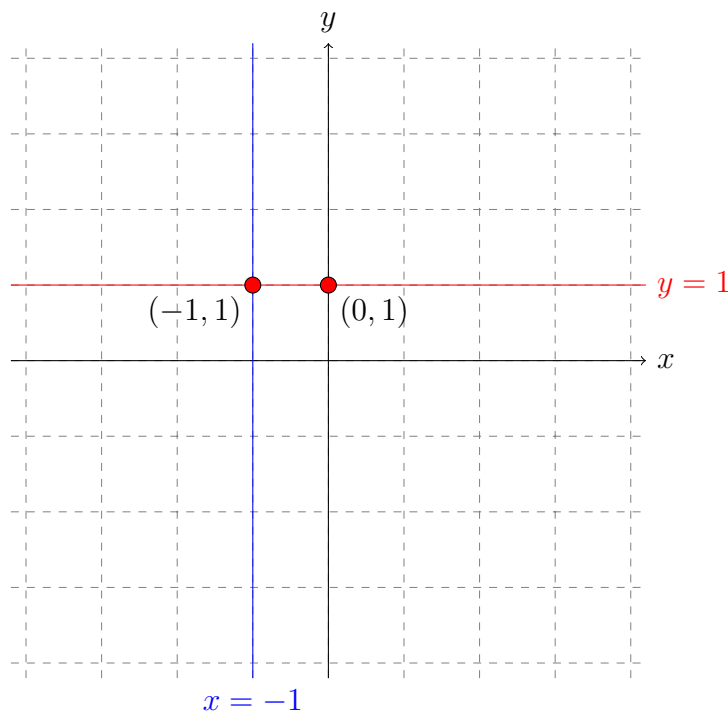
Figure 2: Vertical lines are perpendicular to horizontal lines.

The slopes of two non-vertical perpendicular lines multiply to -1. That is, if m_1 and m_2 are the slopes of two non-vertical perpendicular lines, then $m_1 \cdot m_2 = -1$. Thus, to find the slope of a line perpendicular to a given line one needs to find the *negative reciprocal* of the given slope. Vertical lines are perpendicular to horizontal lines, however, their slopes do not multiply to -1. This is because vertical lines have undefined slope: on a vertical line the change along the x-axis between consecutive points is 0, thus in the process of computing the slope of a vertical line we would be dividing by 0. We will use the superscript $\perp$ to denote perpendicular slopes.

Example 4. *Find the slope of a line perpendicular to the line given by $y = 3x + 1$. See Figure 3.* From the slope-intercept form of the line we read off the slope, the coefficient of x. We have $m = 3$ and $m^\perp = -\frac{1}{m} = -\frac{1}{3}$.

The final ingredient in determining the equation of a line is the y-intercept. This is the point where the graph of the line intersects the y-axis (the line $x = 0$) and is obtained by letting $x = 0$ in the line's equation. In the above example x is identically 0, thus the y-intercept is $b = 1$. In general, y-intercepts have the form $(0, b)$. Similarly, the x-intercept is the point on the graph of the line which intersects the x-axis (the line $y = 0$) and is obtained by letting $y = 0$ in the line's equation. Points that are x-intercepts have the form $(c, 0)$.
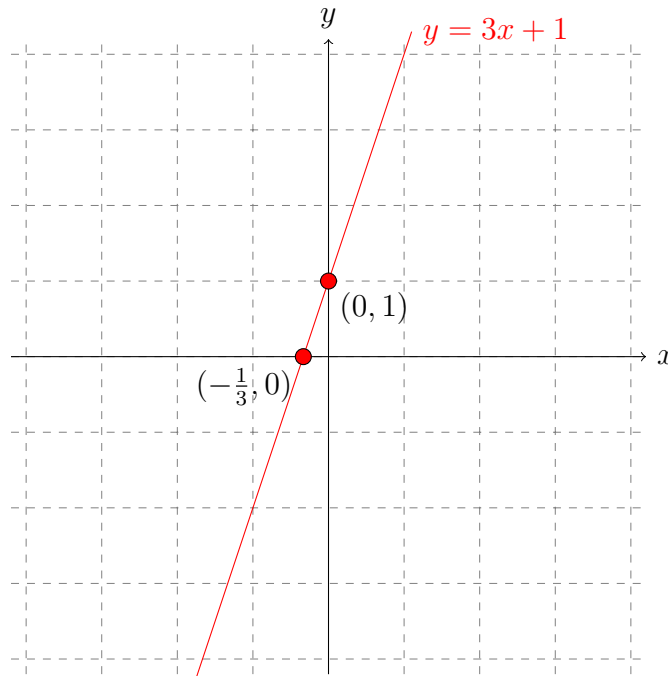
Figure 3: A line with positive slope.

Example 5. *Find the equation of a line parallel to the line $5x + y = 7$ and passing through the origin.*
Writing the equation in slope-intercept form we have $y = -5x + 7$. We read off the slope to be $m = -5$. A parallel line will have the same slope. Because the parallel line must pass through $(0,0)$, the y-intercept is $b = 0$. Therefore, the desired equation is $y = -5x$.

It is possible to write down the equation of a line without explicitly calculating its y-intercept (see Exercise 7). The *point-slope* form of a line through a point (x_1, y_1) is

$$y - y_1 = m(x - x_1) \ .$$

Here the y-intercept is disguised as $b = y_1 - mx_1$.

Figure 4: Two parallel lines.

Systems of Linear Equations

Given two or more linear equations we call the point (if one exists) where all the lines intersect the *solution* to this system of linear equations. In Figure 2 we saw that the lines $x = -1$ and $y = 1$ intersect at $(-1, 1)$. In other words, the solution to the system of equations

$$\begin{cases} y &= 1 \\ x &= \text{-1} \end{cases}$$

is the point $(-1, 1)$.
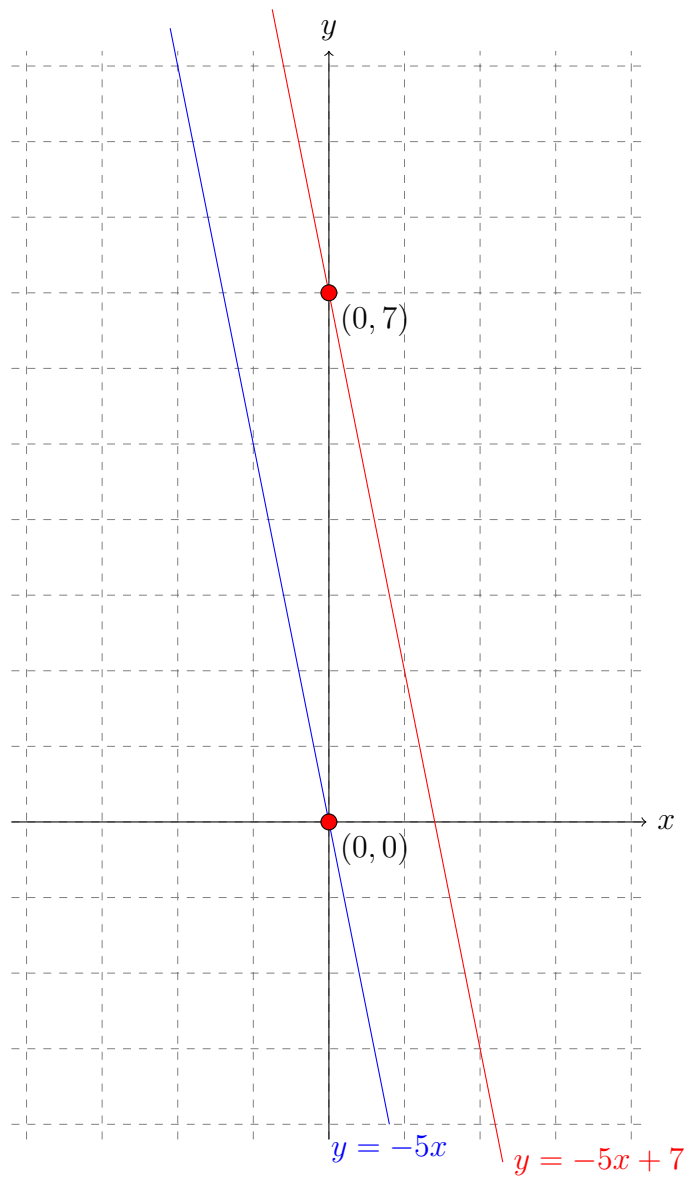
Example 6. The y-intercept of the line $y = mx + b$ is the solution to the system of equations given by

$$\begin{cases} y &= mx + b \\ x &= 0 \end{cases},$$

and the x-intercept is the solution to

$$\begin{cases} y &= mx + b \\ y &= 0 \end{cases}.$$

Example 7. The solution to the system of equations,

$$\begin{cases} 2x + 3y &= 3 \\ 8x + 12y &= 12 \\ x + 2y &= \text{-4} \end{cases},$$

is the point $(18, -11)$.

The main methods of solving a system of linear equations are elimination and substitution. When it is easy enough to solve for one variable, as in the example with the x-intercept, we do so and then make a substitution in the other equation, thereby obtaining an easy-to-solve linear equation in one variable. Remember that multiplying both sides of an equation by a number does not change the equation. To solve the system

$$\begin{cases} 8x + 12y &= 12 \\ x + 2y &= \text{-4} \end{cases}$$

by elimination we would first multiply the bottom equation by -6 and obtain the equivalent system,

$$\begin{cases} 8x + 12y &= 12 \\ -6x - 12y &= 24 \end{cases}.$$

Adding both equations *eliminates* the variable y and we have $2x = 36$, whence $x = 18$. It then follows that $y = -11$ and therefore the solution to the system is $(18, -11)$.

It is absolutely vital to check your answer by making sure that it is the solution to every equation.

1. Given is the line with equation $y = 3x - 2$.

   (a) Find five points on the line and arrange them in a table.

   (b) Graph the line.

   (c) Find the x-intercept and the y-intercept.

2. Find the slope-intercept form of the equation of the line through the points $(2, 7)$ and $(5, 2)$ and graph it.

3. Consider the line passing through the point $(2, 3)$ with slope $m = -1$.

   (a) Write down the point-slope equation of the line.

   (b) Write the equation in the slope-intercept form.

   (c) Find all intercepts.

4. Consider the line $y = 2x + 3$.

   (a) Find the equation in slope-intercept form of a parallel line through $(2, 5)$.

   (b) Find the equation of a perpendicular line through $(2, 7)$.

5. Consider the line L given by $2x + 3y = 6$.

   (a) Find the slope and intercepts of the line.

   (b) Find a point on the line and a point not on the line.

   (c) Write the equation of the line in point-slope form.

   (d) Find the equation of a line perpendicular to L, but passing through the same x-intercept as the line L.

6. Solve:
$$\begin{cases} y &= 2x - 1 \\ 2x - 5y &= 10 \end{cases}.$$

7. Derive the point-slope form of the equation for a line by following these steps.

   (a) Let L be the line passing through the fixed point (x_1, y_1) and an arbitrary point (x, y).

   (b) Find the general formula for the slope of L.

8. *Write down a system of 3 linear equations that has

   (a) Exactly one solution,

   (b) No solution,

   (c) Infinitely many solutions.

9. **Find a pair of points that together with the points $(-2, 1)$ and $(2, -2)$ are the vertices of a square.

10. ***Find *all* points such that together with the points $(-2, 1)$ and $(2, -2)$ they are the vertices of a right triangle.

Sample Midterm				Sample Final
1	A B C D			
4	A B C D			
36	A B C D			

1. Given is the line with equation $y = 3x - 2$.

   (a) Find five points on the line and arrange them in a table.

   Answer 1.

$y = 3x - 2$	
x	y
-2	-8
0	-2
1	1
3	7
10	28

   (b) Graph the line.

   Answer 2.
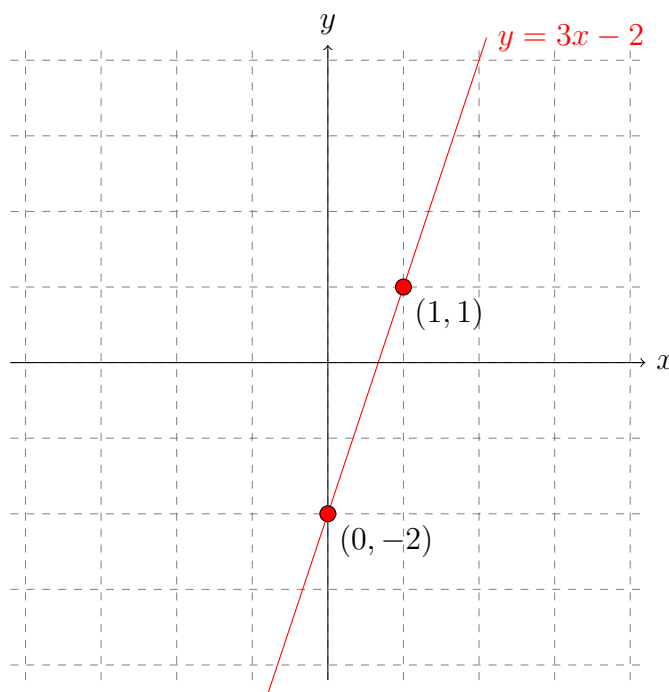
   (c) Find the x-intercept and the y-intercept.

   Answer 3. To find the x-intercept let $y = 0$ in $y = 3x - 2$ and solve for x.
   Thus, the x-intercept is the point $\left(\frac{2}{3}, 0\right)$.
   To find the y-intercept let $x = 0$ in $y = 3x - 2$ and solve for y.
   Thus, the y-intercept is the point $(0, -2)$.

2. Find the slope-intercept form of the equation of the line through the points $(2, 7)$ and $(5, 2)$ and graph it.

   Answer 4. First calculate the slope. $m = \frac{7-2}{2-5} = -\frac{5}{3}$. So far, we have

   $$y = -\frac{5}{3}x + b \,.$$

   To solve for b we substitute the coordinates of a point on the line, for example $(2, 7)$. Then at the point $(2, 7)$, we have

   $$7 = -\frac{5}{3} \cdot 2 + b \iff b = 7 + 2 \cdot \frac{5}{3} \iff b = \frac{31}{3} \,,$$

   and so the answer is

   $$y = -\frac{5}{3}x + \frac{31}{3} \,.$$

   We should check our work by verifying that the other point also lies on the line. In other words, substituting the point $(5, 2)$ we should obtain an identity. Indeed,

   $$2 = -\frac{5}{3} \cdot 5 + \frac{31}{3} = -\frac{25}{3} + \frac{31}{3} = \frac{6}{3} = 2 \,.$$

3. Consider the line passing through the point $(2, 3)$ with slope $m = -1$.

   (a) Write down the point-slope equation of the line.

   Answer 5.
   $$y - 3 = -(x - 2)$$

   (b) Write the equation in the slope-intercept form.

   Answer 6.
   $$y = -x + 5$$

   (c) Find all intercepts.

   Answer 7. The x-intercept is the point $(5, 0)$ and the y-intercept is the point $(0, 5)$.

4. Consider the line $y = 2x + 3$.

   (a) Find the equation in slope-intercept form of a parallel line through $(2, 5)$.

   Answer 8. The given line has slope $m = 2$, so we are looking for a line of the form $y = 2x + b$ and containing the point $(2, 5)$. Substituting $x = 2$, it follows that $b = 1$ in order for $y = 5$. Thus, we obtain

   $$y = 2x + 1 .$$

   (b) Find the equation of a perpendicular line through $(2, 7)$.

   Answer 9. The line has slope $m = 2$, so $m^\perp = -\frac{1}{m} = -\frac{1}{2}$, and a perpendicular line will have the form $y = -\frac{1}{2}x + b$. Substituting the point $(2, 7)$ and solving for b, we obtain

   $$y = -\frac{1}{2}x + 8 .$$

5. Consider the line L given by $2x + 3y = 6$.

   (a) Find the slope and intercepts of the line.

   Answer 10. In the slope-intercept form, we have $y = -\frac{2}{3}x + 2$ so the slope is $m = -\frac{2}{3}$. The x-intercept is the point $(3, 0)$ and the y-intercept is the point $(0, 2)$.

   (b) Find a point on the line and a point not on the line.

   Answer 11. The point $(0, 0)$ does not lie on the line, but $(3, 0)$ does.

   (c) Write the equation of the line in point-slope form.

   Answer 12. The slope we already know to be $m = -\frac{2}{3}$ and we can choose the point $(3, 0)$, so

   $$y = -\frac{2}{3}(x - 3) .$$

   (d) Find the equation of a line perpendicular to L, but passing through the same x-intercept as the line L.

   Answer 13. We have $m = -\frac{2}{3}$, so $m^\perp = \frac{3}{2}$. In slope-intercept form, we have

   $$y = \frac{3}{2}x + b ,$$

   and we need to have this line pass through the point $(3, 0)$. Substituting we find that $b = -\frac{9}{2}$, and so the answer is

   $$y = \frac{3}{2}x - \frac{9}{2} .$$

6. Solve:
   $$\begin{cases} y &= 2x - 1 \\ 2x - 5y &= 10 \end{cases}$$

Answer 14. Proceed by elimination: rewrite the system of equations and add them. We have,

$$\begin{cases} -2x + y & = & -1 \\ 2x - 5y & = & 10 \end{cases},$$

and whence $-4y = 9 \Rightarrow y = -\frac{9}{4}$. Then we substitute $y = -\frac{9}{4}$ into the first equation and solve for x and obtain $x = -\frac{5}{8}$. To make sure that $(-\frac{5}{8}, -\frac{9}{4})$ is the solution we check that it also solves the second equation,

$$2\left(-\frac{5}{8}\right) - 5\left(-\frac{9}{4}\right) = -\frac{5}{4} + \frac{45}{4} = \frac{40}{4} = 10 .$$

7. Derive the point-slope form of the equation for a line by following these steps.

   (a) Let L be the line passing through the fixed point (x_1, y_1) and an arbitrary point (x, y).

   (b) Find the general formula for the slope of L.

   Answer 15. The slope of L is given by $m = \frac{y - y_1}{x - x_1}$. Multiplying thru by $(x - x_1)$, we obtain the point-slope form.

8. *Write down a system of 3 linear equations that has

   (a) Exactly one solution.

   Answer 16. All the above problems have exactly one solution. Take, for example, Problem 6 and introduce a third line which passes through the solution $(-\frac{5}{8}, -\frac{9}{4})$. We use the slope-intercept form with an arbitrary slope, say $m = 2$, and obtain

   $$\begin{cases} y & = & 2x - 1 \\ y + \frac{9}{4} & = & 2(x + \frac{5}{8}) \\ 2x - 5y & = & 10 \end{cases}.$$

   (b) No solution.

   Answer 17. The only three lines in the plane that do not intersect are parallel lines. We can take for example the line $2x - 5y = 10$ and pick 3 different y-intercepts.

   $$\begin{cases} 2x - 5y & = & 0 \\ 2x - 5y & = & 5 \\ 2x - 5y & = & 10 \end{cases}.$$

   (c) Infinitely many solutions.

   Answer 18. Infinitely many solutions occur when the three lines are in fact the same line. That is, we have three parallel lines with the same y-intercept.

   $$\begin{cases} 2x - 5y & = & 0 \\ 4x - 10y & = & 0 \\ \frac{2}{3}x - \frac{5}{3}y & = & 0 \end{cases}$$

9. **Find a pair of points that together with the points $(-2, 1)$ and $(2, -2)$ are the vertices of a square.

Answer 19. Case 1: The segment $\overline{(-2, 1)(2, -2)}$ is a side of a square.
The line through the points $(-2, 1)$ and $(2, -2)$ has the equation

$$y' + 2 = -\frac{3}{4}(x' - 2) \,,$$

and the distance between these points is

$$d = \sqrt{(-2 - 1)^2 + (2 + 2)^2} = 5 \,.$$

Each of the two points we are looking for needs to lie on a line parallel to the one above and also on a line perpendicular to it and passing thru either $(-2, 1)$ or $(2, -2)$. That is, we need to solve the systems

$$\begin{cases} y' + 2 &=& \frac{4}{3}(x' - 2) \\ \sqrt{(y' + 2)^2 + (x' - 2)^2} &=& 5 \,, \end{cases}$$

and

$$\begin{cases} y' - 1 &=& \frac{4}{3}(x' + 2) \\ \sqrt{(y' - 1)^2 + (x' + 2)^2} &=& 5 \,. \end{cases}$$

In the first system, we substitute for $(y' + 2)$ in the second equation.

$$\sqrt{(y' + 2)^2 + (x' - 2)^2} = 5 \tag{1}$$

$$\sqrt{\left(\frac{4}{3}(x' - 2)\right)^2 + (x' - 2)^2} = 5 \tag{2}$$

$$\sqrt{\frac{16}{9}(x' - 2)^2 + (x' - 2)^2} = 5 \tag{3}$$

$$\sqrt{\frac{25}{9}(x' - 2)^2} = 5 \tag{4}$$

$$\frac{25}{9}(x' - 2)^2 = 25 \tag{5}$$

$$(x' - 2)^2 = 9 \tag{6}$$

$$x'^2 - 4x' - 5 = 0 \tag{7}$$

$$(x' - 5)(x' + 1) = 0 \tag{8}$$

Thus we obtain the solutions $(5, 2)$ and $(-1, -6)$. Following the same procedure for the second system, we obtain the solutions $(-5, -3)$ and $(1, 5)$. But our pair of solutions must lie on a line parallel to the one thru $(-2, 1)$ and $(2, -2)$; i.e., a line with slope $m = -\frac{3}{4}$. So, the possible solutions are the pairs of points $(1, 5), (5, 2)$, and $(-5, -3), (-1, -6)$.

Case 2: The segment $\overline{(-2,1)(2,-2)}$ lies on the diagonal of a square.
In this case we will obtain a smaller square of side length $\frac{5}{\sqrt{2}}$. We need to find the other diagonal; i.e., the line perpendicular to the segment $\overline{(-2,1)(2,-2)}$ and passing through its midpoint $(\frac{-2+2}{2}, \frac{1-2}{2}) = (0, -\frac{1}{2})$. This line has the equation

$$y + \frac{1}{2} = \frac{4}{3}x \ ,$$

and since half the diagonal is $\frac{5}{2}$, the two points we are looking for need to be distance $\frac{5}{2}$ away from the center of the square, the point $(0, -\frac{1}{2})$, as well as the endpoints of $\overline{(-2,1)(2,-2)}$. We need to solve the system

$$\begin{cases} y' + \frac{1}{2} &=& \frac{4}{3}x' \\ \sqrt{(y'+\frac{1}{2})^2 + (x')^2} &=& \left(\frac{5}{2}\right)^2 \end{cases} .$$

Omitting the algebra, we obtain the points $(\frac{3}{2}, \frac{3}{2})$ and $(-\frac{3}{2}, -\frac{5}{2})$, and this is the third possible solution.

10. ***Find *all* points such that together with the points $(-2, 1)$ and $(2, -2)$ they are the vertices of a right triangle.

Answer 20. We have two cases to consider. First, suppose that the line segment $\overline{(-2,1)(2,-2)}$ is the leg of a right triangle. Then, the third vertex lies on a line perpendicular to the line thru $(-2, 1)$ and $(2, -2)$, and passing thru either $(-2, 1)$ or $(2, -2)$. If (x', y') is the third vertex, then either

$$y' + 2 = \frac{4}{3}(x' - 2) \text{ or } y' - 1 = \frac{4}{3}(x' + 2) .$$

The second and more interesting case is that the line segment $\overline{(-2,1)(2,-2)}$ is the hypotenuse of a right triangle. From elementary geometry we recall the Theorem of Thales, which states that the triangle formed by the diameter of a circle and line segments joining an arbitrary point on the circle with the endpoints of the diameter is a right triangle. Thus, we need to find the equation of a circle whose diameter is the line segment $\overline{(-2,1)(2,-2)}$. Using the distance formula, we have the diameter

$$d = \sqrt{(-2 - 1)^2 + (2 + 2)^2} = 5 ,$$

and the midpoint of our circle,

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left(0, -\frac{1}{2} \right) .$$

This is the circle of radius $\frac{5}{2}$, centered at the point $(0, -\frac{1}{2})$.

The answer is disappointing, because we do not explicitly give the coordinates of a point or points. In fact there are infinitely many possibilities, so listing them amounts to writing a formula which computes them for us. We have a formula for the coordinates of *every* point which solves the problem. If (x', y') is the third vertex of the right triangle with vertices $(-2, 1)$ and $(2, -2)$, then either

$$y' + 2 = \frac{4}{3}(x' - 2) ,$$

or

$$y' - 1 = \frac{4}{3}(x' + 2) ,$$

or

$$(x')^2 + \left(y' + \frac{1}{2} \right)^2 = \left(\frac{5}{2} \right)^2 .$$

Absolute Value

The *absolute value* of a number x, denoted $|x|$, is defined as a piece-wise linear function, meaning it is composed of lines. In particular, we are using the lines $y = x$ and $y = -x$,

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

We can graph these lines, but we are only using a portion of each graph to construct the graph of the absolute value function.
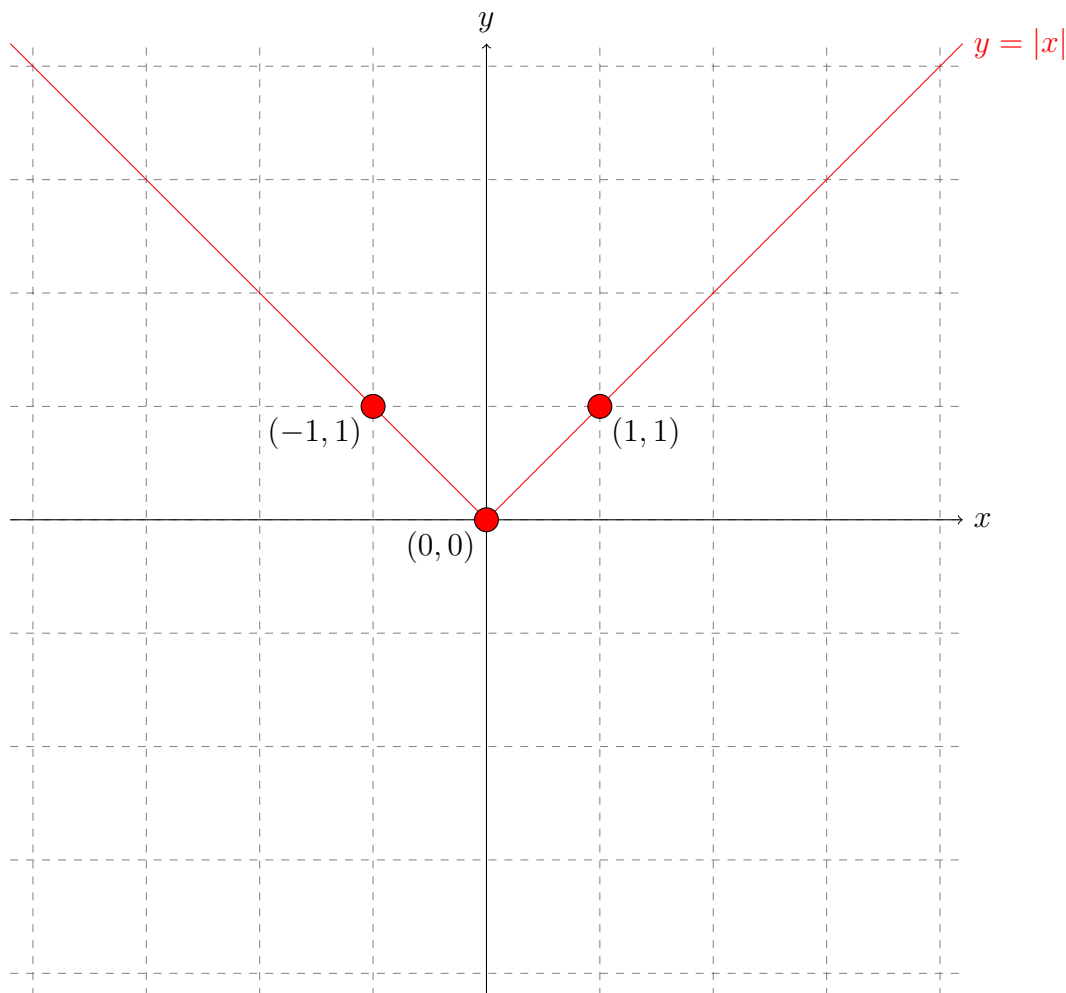

Figure 5: The absolute value function.

We interpret $|x|$ according to what x is. In case x is a number, the absolute value simply removes any and all minus signs.

Example 1.

1. $|-3| = 3$.

2. $|5| = 5$.

3. $|-\pi| = \pi$.

In the case that the expression inside the absolute value signs is not a number, we must follow the definition above and consider both possibilities, that the expression is either positive or negative.

Example 2. *Suppose $x \geq 3$. Rewrite $|x - 1|$ without using the absolute value sign.*
Without any additional information we do not know weather $x - 1$ is positive or negative. Thus it would not be correct to write $|x - 1| = x - 1$. However, because we are given the information that $x \geq 3$, we know that $x - 1$ is always at least 2 and is therefore a positive quantity. With this knowledge we apply the definition of absolute value, and obtain $|x - 1| = x - 1$.

Example 3. *Suppose $x \geq 3$. Rewrite $|2 - x|$ without using the absolute value sign.*
In this example, we know x is at least 3, hence $2 - x$ is a number that is less than -1. Consequently the expression inside the absolute value signs is always negative, and applying the definition of absolute value we obtain

$$|2 - x| = -(2 - x) = x - 2 \ .$$

The expression $x - 2$ is positive for all values of $x \geq 3$.

Another way to interpret the expression $|x - 1|$ is to think of the distance between x and 1. This becomes apparent if we draw a picture.
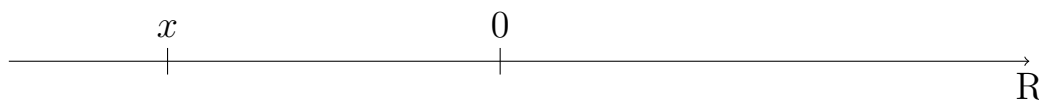
Now, as x is to the left of 0, it is a negative number, but the distance between x and 0 is a positive quantity by definition. Hence $|x|$ can be thought of as the distance between 0 and the point on the number line at position x. Shifting, we can generalize this situation,
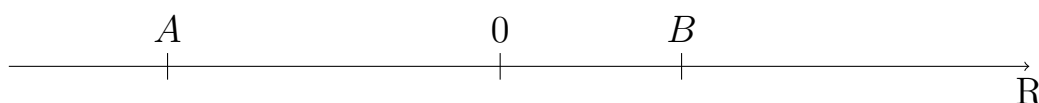
The distance between x and 0 is $|x|$, and we can also say the same thing by writing $|x - 0|$. Thus, the distance between A and B is expressed as $|A - B|$.
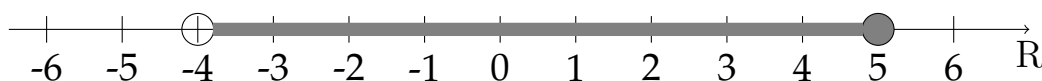
Interval Notation

We now introduce a notation scheme for denoting intervals on the number line. Say we want to denote the following shaded region,

The shaded region includes numbers between -4 and 5. In case we wish to say something about the endpoints -4 and 5, we have to be more careful. The expression,

$$-4 < x \le 5 \, ,$$

is pronounced "x is greater than -4 and less than or equal to 5". How do we indicate on the number line above that in addition to all numbers strictly between -4 and 5 we are also including the number 5? One common way to do this is to draw a solid dot at the point we wish to include and a hollow dot at each point we wish to omit, so our shaded region becomes,

Interval notation allows us to compactly express the shaded region. We list the endpoints, from lest to greatest, or from left to right on the number line. To indicate that we do not wish to include or shade in a particular endpoint, we use the symbols

$$\circ,), (, >, <,$$

and in case we do wish to include or shade in an endpoint, we use the following symbols,

$$\bullet,], [, \ge, \le \, ,$$

so the interval denoted above, consisting of all numbers greater than -4 and less than or equal to 5 can be expressed with the compound inequality $-4 < x \le 5$, and equivalently in interval notation as $(-4, 5]$.

Note that if we only wished to consider all numbers strictly between -4 and 5, that is $-4 < x < 5$, then in interval notation we would write $(-4, 5)$, which looks like and may be easily confused with the point with x-coordinate -4 and y-coordinate 5. It is therefore important, in order to avoid confusion, to use the word interval in conjunction with interval notation. We would say the interval $(-4, 5)$ is the set of all numbers x that satisfy the inequality $-4 < x < 5$, and now there is no confusion with the point $(-4, 5)$.

1. Draw the interval $(-2, 3]$ on the number line.

2. Arrange from least to greatest: $-2, |\pi|, |-2|, -|-1|, 1$. Use the symbols " $<$ " and " $\leq$ ".

3. Simplify to an integer: $|3(|4-7| \cdot |-1-2|) + 1|$.

4. Rewrite $|3-x| - |x+1|$ without using the absolute value sign where:

   (a) $x \geq 3$,

   (b) $x = 2$,

   (c) $x < -2$.

5. Write using the absolute value sign the expression representing the distance on the number line between 2 and -5.

6. Consider the intervals $(-3, 5]$ and $[0, 10]$.

   (a) Draw these intervals on the number line and mark the interval representing their intersection.

   (b) Express the intersection in interval notation.

   (c) Express the intersection in set notation without using the absolute value sign.

   (d) Express the intersection using the absolute value sign.

7. Write using the absolute value sign: "The distance between x and $-\frac{1}{3}$ is less than 2."

8. Write as an interval: $\{x : x \in \mathrm{R}\}$; i.e., "The set of all x, where x is a real number."

9. Plot on the number line $[0, 3) \cap \mathrm{N}$; i.e., "The intersection of $[0, 3)$ with the set of natural numbers."

10. Write as a union of two intervals: $\{x : |x - 5| \geq 1\}$.

11. Plot on the number line: $|x - 1| \leq 7$.

12. Plot on the number line: $|3 - x| \geq 3$.

13. Solve and write the answer in interval notation: $|x + 6| \geq 5$ and $x \leq -7$.

14. Solve and write the answer using absolute value: $3 \leq 2 - x \leq 10$.

15. Solve and write the answer in set notation: $-5 \leq x + 1 \leq -6$.

Sample Midterm				Sample Final			
2	A	B	C	D			
17	A	B	C	D			
22	A	B	C	D	2	A	B	C	D
23	A	B	C	D			
34	A	B	C	D			

1. Draw the interval $(-2, 3]$ on the number line.

   Answer 1.

2. Arrange from least to greatest: $-2, |\pi|, |-2|, -|-1|, 1$. Use the symbols " $<$ " and " $\leq$ ".

   Answer 2.
   $$-2 < -|-1| < 1 < |-2| < |\pi|$$

3. Simplify to an integer: $|3\left(|4 - 7| \cdot |-1 - 2|\right) + 1|$.

   Answer 3.
   $$
   \begin{aligned}
   |3\left(|4 - 7| \cdot |-1 - 2|\right) + 1| &= |3\left(|-3| \cdot |-3|\right) + 1| \\
   &= |3\left(3 \cdot 3\right) + 1| \\
   &= |3\left(9\right) + 1| \\
   &= |27 + 1| \\
   &= |28| \\
   &= 28.
   \end{aligned}
   $$

4. Rewrite $|3 - x| - |x + 1|$ without using the absolute value sign where:

   (a) $x \geq 3$.

   Answer 4. If $x \geq 3$, then $3 - x$ is negative and must be negated when the absolute value is removed. The expression $x + 1$ is positive so remains unchanged. So, for $x \geq 3$
   $$|3 - x| - |x + 1| = -(3 - x) - (x + 1) = -3 + x - x - 1 = -4 \ .$$

   (b) $x = 2$.

   Answer 5. Substitute and simplify,
   $$|3 - 2| - |2 + 1| = |1| - |3| = 1 - 3 = -2 \ .$$

   (c) $x < -2$.

   Answer 6. If $x < -2$, then $3 - x$ is positive and $x - 1$ negative. We must negate the second quantity if the absolute value sign is removed. So, for $x < -2$
   $$|3 - x| - |x + 1| = 3 - x + (x + 1) = 4 \ .$$

5. Write using the absolute value sign the expression representing the distance on the number line between 2 and -5.

Answer 7. $|2 - (-5)|$
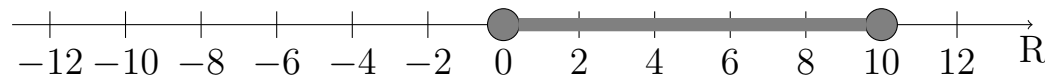
6. Consider the intervals $(-3, 5]$ and $[0, 10]$.

   (a) Draw these intervals on the number line and find their intersection.

   Answer 8. The interval $(-3, 5]$ is represented by the region
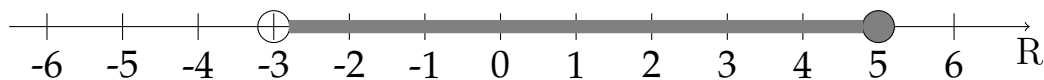
   The interval $[0, 10]$ is represented by the region

   The intersection of the intervals $(-3, 5]$ and $[0, 10]$ is given by the intersection symbol $\cap$ and we write $(-3, 5] \cap [0, 10]$.

   (b) Express the intersection in interval notation.

   Answer 9. $(-3, 5] \cap [0, 10] = [0, 5]$.

   (c) Express the intersection in set notation without using the absolute value sign.

   Answer 10. $\{x \ : \ 0 \le x \le 5\}$

   (d) Express the intersection using the absolute value sign.

   Answer 11. $\{x \ : \ |x - \frac{5}{2}| \le \frac{5}{2}\}$

7. Write using the absolute value sign: "The distance between x and $-\frac{1}{3}$ is less than 2."

Answer 12. $|x + \frac{1}{3}| < 2$

8. Write as an interval: $\{x \ : \ x \in \mathrm{R}\}$; i.e., "The set of all x, where x is a real number."

Answer 13. $(-\infty, \infty)$

9. Plot on the number line $[0, 3) \cap \mathrm{N}$; i.e., "The intersection of $[0, 3)$ with the set of natural numbers."

Answer 14. This is the set $\{1, 2\}$. Note that we do not consider 0 as a natural number.
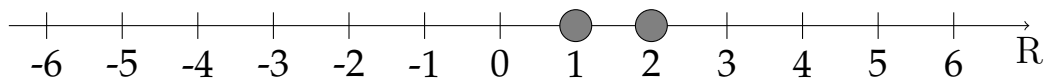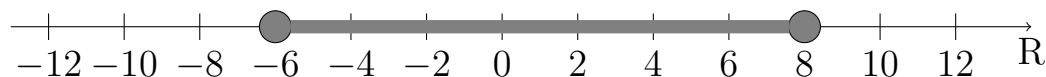

10. Write as a union of two intervals: $\{x \ : \ |x - 5| \geq 1\}$.

Answer 15. Either $x - 5 \geq 1$, in which case $x \geq 6$, or $x - 5 \leq -1$ and $x \leq 4$. In interval notation this is the union
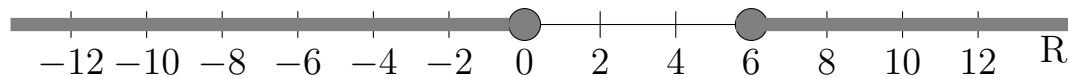
$$(-\infty, 4] \cup [6, \infty) \, .$$

11. Plot on the number line: $|x - 1| \leq 7$.

Answer 16.

12. Plot on the number line: $|3 - x| \geq 3$.

Answer 17.

13. Solve and write the answer in interval notation: $|x + 6| \geq 5$ and $x \leq -7$.

Answer 18. Either $x+6 \geq 5$, in which case $x \geq -1$, or $x+6 \leq -5$ and $x \leq -11$. Since x must be less than or equal than -7, the answer is $x \leq -11$ or, in interval notation, $(-\infty, -11]$.

14. Solve and write the answer using absolute value: $3 \le 2 - x \le 10$.

   Answer 19.

   $$3 \le 2 - x \le 10 \quad \Longleftrightarrow \quad 1 \le -x \le 8$$
   $$\Longleftrightarrow \quad -1 \ge x \ge -8 \ .$$

   The answer is the interval $[-8, -1]$. The center is -4.5 and the radius is 3.5, so using absolute value we write
   $$\left\{ x \ : \ \left| x + \frac{9}{2} \right| \le \frac{7}{2} \right\} \ .$$

15. Solve and write the answer in set notation: $-5 \le x + 1 \le -6$.

   Answer 20.

   $$-5 \le x + 1 \le -6 \quad \Longleftrightarrow \quad -6 \le x \le -7 \ .$$

   There is no number which is greater than or equal to -6 and less than or equal to -7. The answer is therefore the empty set, $\emptyset$.

Rational Expression

By a *rational expression* we mean an expression that has a numerator and denominator. We are, at this point, interested in solutions to equations involving rational expressions, that is all the possible values of x that make the equation a true statement.

Example 1. Solve:
$$\frac{x(x^2-4)}{(x-1)(x+2)} = 0 .$$

We begin by factoring into linear factors. Using the difference of squares formula, we obtain
$$\frac{x(x-2)(x+2)}{(x-1)(x+2)} = \frac{x(x-2)}{(x-1)} = 0 .$$

This rational expression is equal to zero provided its numerator is zero. Because, for any value of x, the numerator is a product of two numbers, we have to consider the possibility that either one of those numbers can be zero. Our factoring paid off, we have linear factors and setting each equal to zero amounts to finding the x-intercept of these lines,
$$x = 0 \text{ and } (x-2) = 0 \iff x = 2 .$$

Hence, the solutions are the numbers $x = 0$ and $x = 2$. In set notation, we would denote the set of solutions as $\{0, 2\}$.

For more complicated examples, we may have to do some algebra in order to transform our rational expression into a problem similar to the example above. Once we have a rational expression set equal to zero, where also the numerator and denominator are products of linear factors, then we proceed as in the example above. In all other cases, we must first

1. Introduce a zero by moving all rational expressions to one side of the equal sign.

2. Obtain a common denominator and rewrite the non-zero expression as one rational expression.

3. Factor the numerator and denominator into linear factors.

The following example illustrates a common error. The first step in the above procedure may not be omitted.

Example 2. Solve:
$$\frac{x(x-2)}{(x-1)} = 4 .$$

We set the linear factors from the numerator equal to 4, and obtain
$$x = 4 \text{ and } (x-2) = 4 \iff x = 6 .$$

However, neither $x = 4$, nor $x = 6$ are solutions, because substituting either into the equation does not result in a true statement. With $x = 4$, we have
$$\frac{4(4-2)}{(2-1)} = \frac{8}{1} = 8 \neq 0 ,$$

and using $x = 6$, we obtain
$$\frac{6(6-2)}{(6-1)} = \frac{24}{5} \neq 0 .$$

Factoring

The second step in the procedure of solving equations involving ration expressions depends on the ability to factor into linear factors. A linear factor is a polynomial of degree 1, or more concretely and expression of the form

$$(mx + b) \ .$$

We can check that a factor is a linear factor by deciding weather it has the above form, or weather graphing that expression would yield a line. The reason we are after a factorization using only linear factors is that these types of (linear) equations are the easiest of all to solve, as we have seen in Example 1 above.

     The process of factoring is a guess-and-check procedure, easily reversible so that we can check our work at each step.

Example 3. *Factor into linear factors the expression* $25x^2 + 30x + 9$.
We begin by writing some linear factors,

$$25x^2 + 30x + 9 = (ax + b)(cx + d) \ .$$

We know that $ac = 25$ and $bd = 9$, by applying the distributive law or by using the equivalent FOIL method. At this point we make some guesses, and check to see if they are correct. We have the factorizations $25 = 15 \cdot 1 = 5 \cdot 5$, and $9 = 9 \cdot 1 = 3 \cdot 3$, and setting $a = c = 5$ and $b = d = 3$, we have

$$(5x + 3)(5x + 3) = (5x + 3)^2 = 25x^2 + 30x + 9 \ .$$

     There are several useful factorization shortcuts, or formulas, that are used so often it is worth while to commit them to memory,

Difference of Squares $a^2 - b^2 = (a + b)(a - b)$,
Difference of Cubes $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$,
Sum of Cubes $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$,
                            $(a + b)^2 = (a^2 + 2ab + b^2)$,
                            $(a + b)^3 = (a^3 + 3a^2b + 3ab^2 + b^3)$.

As an exercise, you should be able to check that these are in fact true statements. For each formula, multiply and simplify on the right hand side to obtain the left hand side.

You must know the following identities.
(1) Difference of Squares: $a^2 - b^2 = (a+b)(a-b)$
(2) Difference of Cubes: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
(3) Sum of Cubes: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
(4) $(a+b)^2 = (a^2 + 2ab + b^2)$
(5) $(a+b)^3 = (a^3 + 3a^2b + 3ab^2 + b^3)$

1. Factor into linear factors whenever possible:

   (a) $x^2 + 5x + 6 = 0$

   (b) $x^3 - 3x^2 + 2x = 0$

   (c) $x(x-3) - 2x + 6 = 0$

   (d) $x^2 + 2x + 1 = 0$

   (e) $8x^3 - 27 = 0$

2. Solve and write the solutions in set notation:

   (a) $\frac{1}{x+1} = 3$

   (b) $\frac{-1}{x-1} + \frac{2x+1}{2} = 0$

   (c) $1 + \frac{1}{x+1} + \frac{2}{x-1} = 0$

Sample Midterm   Sample Final

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1. Factor into linear factors whenever possible:

   (a) $x^2 + 5x + 6 = 0$

   Answer 1.

   $$\begin{aligned} x^2 + 5x + 6 &= 0 \\ (x+3)(x+2) &= 0 \end{aligned}$$

   (b) $x^3 - 3x^2 + 2x = 0$

   Answer 2.

   $$\begin{aligned} x^3 - 3x^2 + 2x &= 0 \\ x(x^2 - 3x + 2) &= 0 \\ x(x-2)(x-1) &= 0 \end{aligned}$$

   (c) $x(x-3) - 2x + 6 = 0$

   Answer 3.

   $$\begin{aligned} x(x-3) - 2x + 6 &= 0 \\ x(x-3) - 2(x-3) &= 0 \\ (x-2)(x-3) &= 0 \end{aligned}$$

   (d) $x^2 + 2x + 1 = 0$

   Answer 4.

   $$\begin{aligned} x^2 + 2x + 1 &= 0 \\ (x+1)(x+1) &= 0 \\ (x+1)^2 &= 0 \end{aligned}$$

   (e) $8x^3 - 27 = 0$

   Answer 5.

   $$\begin{aligned} 8x^3 - 27 &= 0 \\ (2x)^3 - 3^3 &= 0 \\ (2x-3)((2x)^2 + 2x \cdot 3 + 3^2) &= 0 \\ (2x-3)(4x^2 + 6x + 9) &= 0 \end{aligned}$$

   The expression $(4x^2 + 6x + 9)$ does not factor at all. Later we will develop the tools that will allow us to see this easily.

2. Solve and write the solutions in set notation:

(a) $\frac{1}{x+1} = 3$

Answer 6.

$$\frac{1}{x+1} = 3$$
$$\frac{1}{x+1} - 3 = 0$$
$$\frac{1}{x+1} - 3 \cdot \frac{x+1}{x+1} = 0$$
$$\frac{1 - 3(x+1)}{x+1} = 0$$
$$\frac{-3x - 2}{x+1} = 0$$

Thus, $\frac{1}{x+1} = 3$ provided that $-3x - 2 = 0$ and $x \neq -1$. The solution set is $\{-\frac{2}{3}\}$.

(b) $\frac{-1}{x-1} + \frac{2x+1}{2} = 0$

Answer 7.

$$\frac{-1}{x-1} + \frac{2x+1}{2} = 0$$
$$\frac{-1}{x-1} \cdot \frac{2}{2} + \frac{2x+1}{2} \cdot \frac{x-1}{x-1} = 0$$
$$\frac{-2 + (2x+1)(x-1)}{2(x-1)} = 0$$
$$\frac{-2 + 2x^2 - x - 1}{2(x-1)} = 0$$
$$\frac{2x^2 - x - 3}{2(x-1)} = 0$$
$$\frac{(2x-3)(x+1)}{2(x-1)} = 0$$

Thus, $\frac{-1}{x-1} + \frac{2x+1}{2} = 0$ provided that $x = \frac{3}{2}$ or $x = -1$ and $x \neq 1$. The solution set is $\{-1, \frac{3}{2}\}$.

(c) $1 + \frac{1}{x+1} + \frac{2}{x-1} = 0$

Answer 8.

$$1 + \frac{1}{x+1} + \frac{2}{x-1} = 0$$

$$1 \cdot \frac{(x+1)(x-1)}{(x+1)(x-1)} + \frac{1}{x+1} \cdot \frac{x-1}{x-1} + \frac{2}{x-1} \cdot \frac{x+1}{x+1} = 0$$

$$\frac{(x+1)(x-1) + (x-1) + 2(x+1)}{(x+1)(x-1)} = 0$$

$$\frac{x^2 - 1 + x - 1 + 2x + 2}{(x+1)(x-1)} = 0$$

$$\frac{x^2 + 3x}{(x+1)(x-1)} = 0$$

$$\frac{x(x+3)}{(x+1)(x-1)} = 0$$

If $x = \pm 1$, then we are dividing by zero so these numbers, if they are solutions, are invalid. The numerator and hence the entire expression is 0, whenever $x = 0$ or $x = -3$. The solution set is $\{-3, 0\}$.

1. Solve using the key number method and write the solution in interval notation: $\frac{(2x-3)(7x-1)}{2-x} < 0$.

   Answer 1. The key numbers are the roots of the numerator and the roots of the denominator. The key numbers are

   $$x = \frac{3}{2}, \frac{1}{7}, 2$$

   We always exclude the roots of the denominator, because at these values of x the denominator is zero, and hence the whole expression is undefined. Depending on the type of inequality we may also need to exclude the zeros of the numerator. In this case we have a strict inequality and must exclude both $x = \frac{3}{2}$ and $x = \frac{1}{7}$.

   The key intervals are:

Key Interval	Test Value	$(2x-3)$	$(7x-1)$	$(2-x)$	$\frac{(2x-3)(7x-1)}{2-x}$
$(-\infty, \frac{1}{7})$	0	-	-	+	+
$(\frac{1}{7}, \frac{3}{2})$	1	-	+	+	-
$(\frac{3}{2}, 2)$	$\frac{7}{4}$	+	+	+	+
$(2, \infty)$	10	+	+	-	-

   The expression is negative on the intervals $(\frac{1}{7}, \frac{3}{2})$ and $(2, \infty)$, and the answer is the union

   $$\left(\frac{1}{7}, \frac{3}{2}\right) \cup (2, \infty) .$$

2. Solve using the key number method and write the solution in interval notation: $\frac{3-2x}{2-3x} \geq -\frac{1}{x}$.

   Answer 2. Before we do anything we need to introduce zero into the inequality. We know how to solve the previous type of problem so we reduce this problem into the form we had above.

   $$\frac{3-2x}{2-3x} \geq -\frac{1}{x} \quad \Longleftrightarrow \quad \frac{3-2x}{2-3x} + \frac{1}{x} \geq 0$$

   $$\Longleftrightarrow \quad \frac{3-2x}{2-3x} \cdot \frac{x}{x} + \frac{1}{x} \cdot \frac{2-3x}{2-3x} \geq 0$$

   $$\Longleftrightarrow \quad \frac{3x-2x^2 + (2-3x)}{x(2-3x)} \geq 0$$

   $$\Longleftrightarrow \quad \frac{3x-2x^2 + 2 - 3x}{x(2-3x)} \geq 0$$

   $$\Longleftrightarrow \quad \frac{-2x^2 + 2}{x(2-3x)} \geq 0$$

   $$\Longleftrightarrow \quad \frac{(-2x+2)(x+1)}{x(2-3x)} \geq 0$$

The equivalent problem is to solve

$$\frac{(-2x+2)(x+1)}{x(2-3x)} \geq 0 \ .$$

The solution set is $\{-1, 1\}$ and $x \neq 0, -\frac{2}{3}$. The key numbers are $x = -\frac{2}{3}, 1, -1, 0$. We exclude $x = 0$ and $x = -\frac{2}{3}$, because these are the roots of the denominator and keep $x = -1$ and $x = 1$, because we do not have strict inequality. The key intervals are

Interval	Test Value	(-2x+2)	(x+1)	(x-1)	(-2x-2)	x	(2x-3)	Eq 1	Eq 2
$(-\infty, -1]$	-10	+	-	-	+	-	+	+	+
$[-1, 0)$	$-\frac{1}{2}$	+	+	-	-	-	+	-	-
$(0, \frac{2}{3})$	$\frac{1}{2}$	+	+	-	-	+	+	+	+
$(\frac{2}{3}, 1]$	$\frac{3}{4}$	+	+	-	-	+	-	-	-
$[1, \infty)$	10	-	+	-	+	+	-	+	+

The expression in non-negative on the intervals $(-\infty, -1]$ and $(0, \frac{2}{3})$ and $[1, \infty)$. The answer is their union, namely

$$(-\infty, -1] \cup \left(0, \frac{2}{3}\right) \cup [1, \infty) \ .$$

You must know the following identities.
(1)   Difference of Squares: $a^2 - b^2 = (a+b)(a-b)$
(2)   Difference of Cubes: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
(3)   Sum of Cubes: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
(4) $(a+b)^2 = (a^2 + 2ab + b^2)$
(5) $(a+b)^3 = (a^3 + 3a^2b + 3ab^2 + b^3)$

Solve using the key number method and write the solution in interval notation.

1. $x^2 + 4x + 4 < 0$

2. $(x-3)(x+1)(3x-1) \geq 0$

3. $-x(x-2) \leq 1$

4. $\dfrac{x^2 - 1}{x^2 + 7x + 12} \geq 0$

5. $\dfrac{x}{x-1} \leq 1$

6. $x + \dfrac{4}{x-2} > 6$

Sample Midterm   Sample Final
26   A   B   C   D   25   A   B   C   D

Solve using the key number method and write the solution in interval notation:

1. $x^2 + 4x + 4 < 0$

Answer 1. Factoring we have

$$x^2 + 4x + 4 < 0 \iff (x+2)(x+2) < 0 .$$

There is no denominator and hence no values at which the expression is undefined. The key value is the one root of the numerator, $x = -2$. Note that the strictly less than inequality does not allow us to include $x = -2$. The key intervals are

Key Interval	Test Value	(x+2)	(x+2)	(x+2)(x+2)
$(-\infty, -2)$	-5	-	-	+
$(-2, \infty)$	0	+	+	+

Since the expression is nowhere negative the solution is the emptyset, $\emptyset$.

2. $(x-3)(x+1)(3x-1) \geq 0$

Answer 2. The expression is already factored and there is no denominator to worry about. The three key values are $x = 3, -1, \frac{1}{3}$, and produce these four key intervals.

Key Interval	Test Value	(x-3)	(x+1)	(3x-1)	(x-3)(x+1)(3x-1)
$(-\infty, -1]$	-5	-	-	-	-
$[-1, \frac{1}{3}]$	0	-	+	-	+
$[\frac{1}{3}, 3]$	1	-	+	+	-
$[3, \infty)$	10	+	+	+	+

We have two intervals for which the expression in non-negative and the answer is the union $[-1, \frac{1}{3}] \cup [3, \infty)$.

3. $-x(x-2) \leq 1$

Answer 3. We first need to introduce zero into the inequality.

$$\begin{aligned}
-x(x-2) &\leq 1 \\
x(x-2) &\geq -1 \\
x(x-2) + 1 &\geq 0 \\
x^2 - 2x + 1 &\geq 0 \\
(x-1)(x-1) &\geq 0
\end{aligned}$$

The key number is $x = 1$ and as before we have two key intervals

Key Interval	Test Value	(x-1)	(x-1)	(x-1)(x-1)
$(-\infty, 1]$	0	-	-	+
$[1, \infty)$	10	+	+	+

Since the expression is always non-negative the solution is the entire real line. In interval notation, we would write $(-\infty, \infty)$.

4. $\frac{x^2-1}{x^2+7x+12} \geq 0$

Answer 4. To find the key values we factor the expression

$$\frac{x^2-1}{x^2+7x+12} = \frac{(x+1)(x-1)}{(x+4)(x+3)} \geq 0 \ .$$

The key numbers are $x = \pm 1, -4, -3$. We have a denominator to worry about now and this produces additional intervals. Note the use of $(\ )'s$ around the key values which come from the denominator: the roots of the denominator cannot be included in the key intervals.

Key Interval	Test Value	(x-1)	(x+1)	(x+3)	(x+4)	$\frac{(x+1)(x-1)}{(x+4)(x+3)}$
$(-\infty, -4)$	-5	-	-	-	-	+
$(-4, -3)$	$-\frac{7}{2}$	-	-	-	+	-
$(-3, -1]$	-2	-	-	+	+	+
$[-1, 1]$	0	-	+	+	+	-
$[1, \infty)$	10	+	+	+	+	+

The intevals corresponding to non-negative values of the expression are $(-\infty, -4]$ and $(-3, -1]$ and $[1, \infty)$. Note that the zeros of the numerator are included since our expression only needs to be non-negative. The answer is the union

$$(-\infty, -4) \cup (-3, -1] \cup [1, \infty)$$

5. $\frac{x}{x-1} \leq 1$

Answer 5. Introduce a zero into the inequality and factor:

$$\begin{aligned}
\frac{x}{x-1} &\leq 1 \\
\frac{x}{x-1} - 1 &\leq 0 \\
\frac{x}{x-1} - 1 \cdot \frac{x-1}{x-1} &\leq 0 \\
\frac{1}{x-1} &\leq 0
\end{aligned}$$

The key value is $x = 1$ and we have

Key Interval	Test Value	(x-1)	$\frac{1}{(x-1)}$
$(-\infty, 1)$	-1	-	-
$(1, \infty)$	10	+	+

The expression is negative on the interval $(-\infty, 1)$, and that is the answer.

6. $x + \frac{4}{x-2} \geq 6$

Answer 6.

$$
\begin{aligned}
x + \frac{4}{x-2} &\geq 6 \\
x + \frac{4}{x-2} - 6 &\geq 0 \\
x \cdot \frac{x-2}{x-2} + \frac{4}{x-2} - 6 \cdot \frac{x-2}{x-2} &\geq 0 \\
\frac{x(x-2) + 4 - 6(x-2)}{x-2} &\geq 0 \\
\frac{x^2 - 2x + 4 - 6x + 12}{x-2} &\geq 0 \\
\frac{x^2 - 8x + 16}{x-2} &\geq 0 \\
\frac{(x-4)(x-4)}{x-2} &\geq 0
\end{aligned}
$$

The key number is $x = 2, 4$. The key intervals are

Key Interval	Test Value	(x-2)	(x-4)	(x-4)	$\frac{(x-2)(x-2)}{(x-2)}$
$(-\infty, 2)$	0	-	-	-	-
$(2, 4]$	3	+	-	-	+
$[4, \infty)$	5	+	+	+	+

The expression is non-negative on the intervals $(2, 4]$ and $[4, \infty)$, and the union is the answer, namely

$$(2, 4] \cup [4, \infty) = (2, \infty).$$

Rules for working with exponents. Let x, a, b be any real numbers. Then:

1. $x^0 = 1, x \neq 0$

   Example 1. $\frac{1}{b^0} = 1, (x^2)^0 = 1$

2. $x^a \cdot x^b = x^{a+b}$

   Example 2. $3^5 \cdot 3 = 3^{5+1} = 3^6, b^3 \cdot b^{-3} = b^0 = 1, a^{-5} \cdot a^2 = a^{-5+2} = a^{-3}$

   Common mistake: $a^4 \cdot b^4 \neq (a+b)^4, 2^3 \cdot 3^2 \neq 1$

3. $\frac{x^a}{x^b} = x^{a-b}$

   Example 3. $\frac{3}{3^2} = 3^{1-2} = 3^{-1}, \frac{6^2}{6^{-2}} = 6^{2+2} = 6^4$

   Common mistake: $\frac{3^2}{2^3} \neq 1, \frac{b^x}{a^{-x}} \neq \frac{b}{a}$

4. $(x^a)^b = x^{a \cdot b}$

   Example 4. $(5^2)^2 = 5^{2 \cdot 2} = 5^4, \left(\frac{1}{a^2}\right)^3 = \frac{1}{a^{2 \cdot 3}} = \frac{1}{a^6}, (7^{-1})^{-3} = 7^3$

   Common mistake: $(2^3)^2 \neq 2^5$

5. $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}, y \neq 0$

   Example 5. $\left(\frac{3}{2}\right)^2 = \frac{3^2}{2^2} = \frac{9}{4}, \left(\frac{1}{b}\right)^4 = \frac{1^4}{b^4} = \frac{1}{b^4}$

   Common mistake: $\left(\frac{1}{5}\right)^5 \neq \frac{5}{5}$

6. $x^{-1} = \frac{1}{x}, x \neq 0$

   Example 6. $\frac{1}{b^{-1}} = \left(\frac{1}{b}\right)^{-1} = b, \left(\frac{3}{5}\right)^{-1} = \frac{5}{3}, -3^{-1} = -\frac{1}{3}$

   Common mistake: $(-5)^{-1} \neq 5, \frac{-1}{4^{-1}} \neq \frac{1}{4}$

7. $(xy)^b = x^b \cdot y^b$

   Example 7. $(2 \cdot 3)^2 = 2^2 \cdot 3^2 = 4 \cdot 9 = 36 = 6^2$

   Common mistake: $(3 \cdot 4)^{-1} \neq (3 - 4)$

Whenever you are asked to simplify an expression with integer exponents the final form should contain no negative exponents. As always, reduce fractions to lowest terms.

Example 8.
$$\frac{3zx^3y^4}{6zy^5} = \frac{3z^{1-1}x^3y^{4-5}}{6} = \frac{3z^0x^3y^{-1}}{6} = \frac{x^3}{2y} .$$

Example 9.
$$\left(\frac{2h^2 + f}{2^3 fh}\right)^{-3} = \left(\frac{2^3 fh}{2h^2 + f}\right)^3 = \frac{(2^3 fh)^3}{(2h^2 + f)^3} = \frac{2^9 f^3 h^3}{(2h^2 + f)^3} .$$

Recall the rules for working with exponents. Let x, a, b be any real numbers. Then:

1. $x^0 = 1, x \neq 0$

2. $x^a \cdot x^b = x^{a+b}$

3. $\frac{x^a}{x^b} = x^{a-b}, x \neq 0$

4. $(x^a)^b = x^{a \cdot b}$

5. $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}, y \neq 0$

6. $x^{-1} = \frac{1}{x}, x \neq 0$

7. $(xy)^b = x^b \cdot y^b$

Simplify:

1. $\frac{3zx^3y^4}{6z^2y}$

2. $\left(\frac{2^2 fh^2}{2^3 fh}\right)^3$

3. $\frac{3zx^3y^4}{(6z^2y)^{-2}}$

4. $\frac{(3zx^3y^4)^y}{(6z^2y)^x}$

5. $\left(\frac{2^2 fh^2 + f}{2^3 fh + 2}\right)^{-3}$

6. $(xyz + zyx + yxz)^4$

7. $\frac{3zx^3y^4}{6z^2y} \cdot \frac{3zx^3y^4 - 3zx^3y^4}{(6z^2y)^{-2}} \cdot \left(\frac{2^2 fh^2 + f}{2^3 fh + 2}\right)^3$ [Hint: Single digit]

8. $\left(\frac{1}{d^{-1}x}\right)^{-d}$

9. $\frac{3zx^3y^4}{6z^2y} \cdot \frac{3zx^3y^4}{(6z^2y)^{-2}} \cdot \left(\frac{2^2 fh^2 + f}{2^3 fh + 2}\right)^3$

10. $\frac{4^{2^3} - 4^7}{4^7}$ [Hint: Single digit]

11. $(5^3) \cdot (25^2)^{-1}$

Explain why the expression 0^0 is undefined.

Simplify:

1. $\frac{3zx^3y^4}{6z^2y}$

   Answer 1.
   $$\frac{3zx^3y^4}{6z^2y} = \frac{3}{6}z^{1-2}x^3y^{4-1} = \frac{1}{2}z^{-1}x^3y^3 = \frac{x^3y^3}{2z}$$

2. $\left(\frac{2^2fh^2}{2^3fh}\right)^3$

   Answer 2.
   $$\left(\frac{2^2fh^2}{2^3fh}\right)^3 = \frac{(2^2fh^2)^3}{(2^3fh)^3} = \frac{2^6f^3h^6}{2^9f^3h^3} = 2^{6-9}f^{3-3}h^{6-3} = 2^{-3}f^0h^3 = \frac{h^3}{8}$$

3. $\frac{3zx^3y^4}{(6z^2y)^{-2}}$

   Answer 3.
   $$\frac{3zx^3y^4}{(6z^2y)^{-2}} = \left(3zx^3y^4\right)\left(6z^2y\right)^2 = \left(3zx^3y^4\right)\left(6^2z^4y^2\right) = 3\cdot36z^{1+4}x^3y^{4+2} = 108z^5x^3y^6$$

4. $\frac{\left(3zx^3y^4\right)^y}{(6z^2y)^x}$

   Answer 4.
   $$\frac{\left(3zx^3y^4\right)^y}{(6z^2y)^x} = \frac{3^yz^yx^{3y}y^{4y}}{6^xz^{2x}y^x} = \frac{3^yz^{y-2x}x^{3y}y^{4y-x}}{(2\cdot3)^x} = \frac{3^{y-x}z^{y-2x}x^{3y}y^{4y-x}}{2^x}$$

5. $\left(\frac{2^2fh^2+f}{2^3fh+2}\right)^{-3}$

   Answer 5.
   $$\left(\frac{2^2fh^2+f}{2^3fh+2}\right)^{-3} = \left(\frac{2^3fh+2}{2^2fh^2+f}\right)^3$$

6. $(xyz + zyx + yxz)^4$

   Answer 6.
   $$(xyz + zyx + yxz)^4 = (3xyz)^4 = 3^4x^4y^4z^4 = 81x^4y^4z^4$$

7. $\frac{3zx^3y^4}{6z^2y} \cdot \frac{3zx^3y^4 - 3zx^3y^4}{(6z^2y)^{-2}} \cdot \left(\frac{2^2fh^2+f}{2^3fh+2}\right)^3$ [Hint: Single digit]

Answer 7.

$$\frac{3zx^3y^4}{6z^2y} \cdot \frac{3zx^3y^4 - 3zx^3y^4}{(6z^2y)^{-2}} \cdot \left(\frac{2^2fh^2+f}{2^3fh+2}\right)^3 = \frac{3zx^3y^4}{6z^2y} \cdot \frac{0}{(6z^2y)^{-2}} \cdot \left(\frac{2^2fh^2+f}{2^3fh+2}\right)^3 = 0$$

8. $\left(\frac{1}{d^{-1}x}\right)^{-d}$

Answer 8.

$$\left(\frac{1}{d^{-1}x}\right)^{-d} = \left(d^{-1}x\right)^d = d^{-d}x^d = \frac{x^d}{d^d}$$

9. $\frac{3zx^3y^4}{6z^2y} \cdot \frac{3zx^3y^4}{(6z^2y)^{-2}} \cdot \left(\frac{2^2fh^2+f}{2^3fh+2}\right)^3$

Answer 9.

$$\begin{aligned}
\frac{3zx^3y^4}{6z^2y} \cdot \frac{3zx^3y^4}{(6z^2y)^{-2}} \cdot \left(\frac{2^2fh^2+f}{2^3fh+2}\right)^3 &= \frac{(3zx^3y^4)^2}{(6z^2y)^{-1}} \cdot \left(\frac{2^2fh^2+f}{2^3fh+2}\right)^3 \\
&= (3zx^3y^4)^2 (6z^2y) \cdot \left(\frac{2^2fh^2+f}{2^3fh+2}\right)^3 \\
&= 9 \cdot 6z^{2+2}x^6y^{8+1} \cdot \left(\frac{2^2fh^2+f}{2^3fh+2}\right)^3 \\
&= 54z^4x^6y^9 \cdot \left(\frac{2^2fh^2+f}{2^3fh+2}\right)^3
\end{aligned}$$

10. $\frac{4^{2^3}-4^7}{4^7}$ [Hint: Single digit]

Answer 10.

$$\frac{4^{2^3}-4^7}{4^7} = \frac{4^8-4^7}{4^7} = \frac{4^7(4-1)}{4^7} = 3$$

11. $\left(5^3\right) \cdot \left(25^2\right)^{-1}$

Answer 11.

$$\left(5^3\right) \cdot \left(25^2\right)^{-1} = \left(5^3\right) \cdot \left(5^4\right)^{-1} = \left(5^3\right) \cdot \left(5^{-4}\right) = 5^{3-4} = 5^{-1} = \frac{1}{5}$$

Explain why the expression 0^0 is undefined.

Answer 12. Let $a \neq 0$ and write:

$$0^0 = 0^{(a-a)} = \frac{0^a}{0^a} = \frac{0}{0}$$

We are secretly dividing by zero, which is never allowed and is the reason why 0^0 is undefined.

Rules for working with rational exponents. Let x, a, b be any real numbers and let m, n be any integers but not zero, i.e., m and n are of the set $\{\ldots, -3, -2, -1, 1, 2, 3 \ldots\}$.

1. **Rule:**
$$x^{\frac{m}{n}} = (\sqrt[n]{x})^m = \sqrt[n]{x^m}$$

   Example 1.
$$64^{\frac{3}{2}} = \sqrt{64^3} = \sqrt{262144} = 512$$

   However, this is much easier to do using the other expression as follows (especially if you don't have a calculator around):

$$64^{\frac{3}{2}} = (\sqrt{64})^3 = 8^3 = 512$$

   Note also that there is another way to handle a situation like this and avoid "big" numbers using the methods of the last lecture:

$$64^{\frac{3}{2}} = (2^6)^{\frac{3}{2}} = 2^{6 \cdot \frac{3}{2}} = 2^9 = 512$$

2. **Rule:** Recall that $x^{-n} = \frac{1}{x^n} = (\frac{1}{x})^n$ so
$$x^{-\frac{m}{n}} = \frac{1}{x^{\frac{m}{n}}} = \frac{1}{(\sqrt[n]{x})^m} = \frac{1}{\sqrt[n]{x^m}}$$

   Example 2.
$$\left(\frac{25}{4}\right)^{-\frac{5}{2}} = \left(\frac{4}{25}\right)^{\frac{5}{2}} = \left(\sqrt{\frac{4}{25}}\right)^5 = \left(\frac{\sqrt{4}}{\sqrt{25}}\right)^5 = \left(\frac{2}{5}\right)^5 = \frac{2^5}{5^5} = \frac{32}{3125}$$

   Note however that this is a lot of steps. Students would be expected to answer more like this:

$$\left(\frac{25}{4}\right)^{-\frac{5}{2}} = \left(\frac{4}{25}\right)^{\frac{5}{2}} = \left(\sqrt{\frac{4}{25}}\right)^5 = \left(\frac{2}{5}\right)^5 = \frac{32}{3125}$$

3. Recall that $(ab)^n = a^n b^n$ and $(x^n)^m = x^{nm}$. We use these when simplifying expressions like

$$(64x^9)^{\frac{2}{3}} = 64^{\frac{2}{3}}(x^9)^{\frac{2}{3}} = (\sqrt[3]{64})^2 x^{9 \cdot \frac{2}{3}} = 4^2 x^6 = 16x^6$$

4. Simplifying rational expressions works just as before. Recall also from last time that whenever you are asked to simplify an expression with integer exponents the final form should contain no negative exponents. As always reduce fractions to lowest terms. In addition, square roots should contain no perfect squares, cube roots no perfect cubes, etc.

$$\frac{2zx^{\frac{2}{3}}y^{\frac{3}{2}}}{6z^{\frac{1}{2}}y^{-\frac{7}{2}}} = \frac{z^{1-\frac{1}{2}}x^{\frac{2}{3}}y^{\frac{3}{2}-(-\frac{7}{2})}}{3} = \frac{z^{\frac{1}{2}}x^{\frac{2}{3}}y^5}{3}$$

$$\sqrt[3]{32x^5y^9} = \sqrt[3]{2^3 \cdot 2^2x^3 \cdot x^2(y^3)^3} = 2xy^3\sqrt[3]{2^2x^2} = 2xy^3\sqrt[3]{4x^2}$$

Evaluate or simplify each expression. You should have no negative exponents in any answers.

1. $625^{\frac{1}{4}}$

2. $64^{\frac{2}{3}}$

3. $64^{-\frac{2}{3}}$

4. $49^{\frac{3}{2}}$

5. $49^{-\frac{3}{2}}$

6. $0.001^{\frac{1}{3}}$

7. $0.001^{-\frac{4}{3}}$

8. $(-8)^{\frac{5}{3}}$

9. $(-8)^{-\frac{5}{3}}$

10. $(x^2+1)^{\frac{2}{3}}(x^2+1)^{\frac{4}{3}}$

11. $\dfrac{2z^{\frac{3}{2}}x^{\frac{1}{3}}y^{\frac{5}{3}}}{6z^{\frac{1}{2}}y^{-\frac{4}{3}}x^{\frac{1}{3}}}$

12. $\dfrac{(2x^2+1)^{-\frac{6}{5}}(2x^2+1)^{\frac{6}{5}}(x^2+1)^{-\frac{1}{5}}}{(x^2+1)^{\frac{9}{5}}}$

Sample Midterm │ Sample Final
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Evaluate or simplify each expression. You should have no negative exponents in any answers.

1. $625^{\frac{1}{4}}$

   Answer 1.
   $$(5^4)^{\frac{1}{4}} = 5$$

2. $64^{\frac{2}{3}}$

   Answer 2.
   $$64^{\frac{2}{3}} = (2^6)^{\frac{2}{3}} = 2^4 = 16$$

3. $64^{-\frac{2}{3}}$

   Answer 3.
   $$64^{-\frac{2}{3}} = \left((64)^{\frac{2}{3}}\right)^{-1} = (16)^{-1} = \frac{1}{16}$$

4. $49^{\frac{3}{2}}$

   Answer 4.
   $$49^{\frac{3}{2}} = (7^2)^{\frac{3}{2}} = 7^3 = 343$$

5. $49^{-\frac{3}{2}}$

   Answer 5.
   $$49^{-\frac{3}{2}} = \left(49^{\frac{3}{2}}\right)^{-1} = 343^{-1} = \frac{1}{343}$$

6. $0.001^{\frac{1}{3}}$

   Answer 6.
   $$0.001^{\frac{1}{3}} = \left(10^{-3}\right)^{\frac{1}{3}} = 10^{-1} = \frac{1}{10}$$

7. $0.001^{-\frac{4}{3}}$

   Answer 7.
   $$0.001^{-\frac{4}{3}} = \left(10^{-3}\right)^{-\frac{4}{3}} = 10^4 = 10000$$

8. $(-8)^{\frac{5}{3}}$

   Answer 8.
   $$(-8)^{\frac{5}{3}} = \left((-2)^3\right)^{\frac{5}{3}} = (-2)^5 = -32$$

9. $(-8)^{-\frac{5}{3}}$

   Answer 9.
   $$(-8)^{-\frac{5}{3}} = \left((-8)^{\frac{5}{3}}\right)^{-1} = (-32)^{-1} = -\frac{1}{32}$$

10. $(x^2+1)^{\frac{2}{3}}(x^2+1)^{\frac{4}{3}}$

    Answer 10.
    $$(x^2+1)^{\frac{2}{3}}(x^2+1)^{\frac{4}{3}} = (x^2+1)^{\frac{2}{3}+\frac{4}{3}} = (x^2+1)^{\frac{6}{3}} = (x^2+1)^2 = x^4 + 2x^2 + 1$$

11. $\dfrac{2z^{\frac{3}{2}}x^{\frac{1}{3}}y^{\frac{5}{3}}}{6z^{\frac{1}{2}}y^{-\frac{4}{3}}x^{\frac{1}{3}}}$

    Answer 11.
    $$\frac{2z^{\frac{3}{2}}x^{\frac{1}{3}}y^{\frac{5}{3}}}{6z^{\frac{1}{2}}y^{-\frac{4}{3}}x^{\frac{1}{3}}} = \frac{1}{3}z^{\frac{3}{2}-\frac{1}{2}}x^{\frac{1}{3}-\frac{1}{3}}y^{\frac{5}{3}+\frac{4}{3}} = \frac{1}{3}z^1 x^0 y^3 = \frac{1}{3}zy^3$$

12. $\dfrac{(2x^2+1)^{-\frac{6}{5}}(2x^2+1)^{\frac{6}{5}}(x^2+1)^{-\frac{1}{5}}}{(x^2+1)^{\frac{9}{5}}}$

    Answer 12.
    $$
    \begin{aligned}
    \frac{(2x^2+1)^{-\frac{6}{5}}(2x^2+1)^{\frac{6}{5}}(x^2+1)^{-\frac{1}{5}}}{(x^2+1)^{\frac{9}{5}}} &= (2x^2+1)^{-\frac{6}{5}+\frac{6}{5}}(x^2+1)^{-\frac{1}{5}-\frac{9}{5}} \\
    &= (2x^2+1)^0(x^2+1)^{-\frac{10}{5}} \\
    &= (x^2+1)^{-2} \\
    &= \frac{1}{(x^2+1)^2} \\
    &= \frac{1}{x^4+2x^2+1}
    \end{aligned}
    $$

Remember these important identities! For all real numbers a, b:

Difference of Squares: $a^2 - b^2 = (a+b)(a-b)$

Difference of Cubes: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Sum of Cubes: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$(a+b)^2 = (a^2 + 2ab + b^2)$

$(a+b)^3 = (a^3 + 3a^2 b + 3ab^2 + b^3)$

WARNING! Avoid the "Freshman's Dream"

Remember to multiply all the factors in expressions such as (4) and (5) above. In general:

$$(a+b)^n \neq a^n + b^n$$

We will be using the difference of squares extensively. For example, how would you simplify the following expression so that the denominator no longer contains a square root?

$$\frac{5}{7 + \sqrt{x}}$$

The key idea is to realize that $(7 + \sqrt{x})(7 - \sqrt{x}) = 49 - x$, i.e. we have a difference of squares. Then if we multiply our equation by $1 = \frac{7 - \sqrt{x}}{7 - \sqrt{x}}$ we are not changing the equation, but the denominator will no longer have a radical.

$$\frac{5}{7 + \sqrt{x}} \cdot \frac{7 - \sqrt{x}}{7 - \sqrt{x}} = \frac{5(7 - \sqrt{x})}{49 - x} = \frac{35 - 5\sqrt{x}}{49 - x}$$

Remember that a and b in the difference of squares expression are parameters. This means that we can replace a and b with pretty much any expression we wish. Consider the following example. We are still using the difference of squares.

$$\frac{-(x-2)^2}{2\sqrt{x-1} - x} = \frac{-(x-2)^2}{2\sqrt{x-1} - x} \cdot \left(\frac{2\sqrt{x-1} + x}{2\sqrt{x-1} + x} \right) \tag{9}$$

$$= \frac{-(x-2)^2 \cdot (2\sqrt{x-1} + x)}{4(x-1) - x^2} \tag{10}$$

$$= \frac{(x-2)^2 \cdot (2\sqrt{x-1} + x)}{x^2 - 4x + 4} \tag{11}$$

$$= \frac{(x-2)^2 \cdot (2\sqrt{x-1} + x)}{(x-2)^2} \tag{12}$$

$$= 2\sqrt{x-1} + x \tag{13}$$

In the above examples both the pairs $7 - \sqrt{x}$ and $7 + \sqrt{x}$ and the pairs $2\sqrt{x-1} - x$ and $2\sqrt{x-1} + x$ are called *conjugates*. In the difference of squares formula $(a+b)$ is the conjugate of $(a-b)$. So conjugation amounts to switching the sign in the given expression. The process of multiplying an expression whose denominator contains a radical by 1 in the form of a fraction with the numerator and denominator both being conjugates of the expression's denominator is called *rationalizing the denominator*.

Example 1. Rationalize the Denominator.

$$\frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{5}}{\sqrt{3}} \cdot \left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{\sqrt{5} \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{15}}{3}$$

Example 2. Rationalize the denominator.

$$\frac{\sqrt{2}}{\sqrt[5]{4}} = \frac{\sqrt{2}}{\sqrt[5]{4}} \cdot \left(\frac{\sqrt[5]{4}}{\sqrt[5]{4}}\right)^4 = \frac{\sqrt{2} \cdot \sqrt[5]{4^4}}{4} = \frac{\sqrt{2} \cdot \sqrt[5]{2^8}}{4} = \frac{2 \cdot \sqrt{2} \cdot \sqrt[5]{2^3}}{4} = \frac{\sqrt{2} \cdot \sqrt[5]{2^3}}{2}$$

Example 3. Find the conjugate of $\sqrt{x^2 - \frac{x}{2} - 2} + x$.

$$\sqrt{x^2 - \frac{x}{2} - 2} - x.$$

Example 4. Rationalize the denominator.

$$\frac{3x + 4}{2\sqrt{x^2 - \frac{x}{2} - 2} + x} = \frac{3x + 4}{2\sqrt{x^2 - \frac{x}{2} - 2} + x} \cdot \left(\frac{2\sqrt{x^2 - \frac{x}{2} + 2} - x}{2\sqrt{x^2 - \frac{x}{2} + 2} - x}\right) \tag{14}$$

$$= \frac{(3x + 4)(2\sqrt{x^2 - \frac{x}{2} - 2} - x)}{4(x^2 - \frac{x}{2} + 2) - x^2} \tag{15}$$

$$= \frac{(3x + 4)(2\sqrt{x^2 - \frac{x}{2} - 2} - x)}{3x^2 - 2x + 8} \tag{16}$$

$$= \frac{(3x + 4)(2\sqrt{x^2 - \frac{x}{2} - 2} - x)}{(3x + 4)(x - 2)} \tag{17}$$

$$= \frac{2\sqrt{x^2 - \frac{x}{2} - 2} - x}{x - 2} \tag{18}$$

$$\tag{19}$$

Rationalize the denominator:

1. $\frac{\sqrt{2}}{\sqrt{3}}$

2. $\frac{\sqrt{2}}{1-\sqrt{2}}$

3. $\frac{1+\sqrt{3}}{1-\sqrt{3}}$

4. $\frac{a+\sqrt{b}}{a-\sqrt{b}}$

5. $\frac{1}{\sqrt{5}} + 4\sqrt{45}$

6. $\frac{4}{\sqrt[3]{16}}$

7. $\frac{3}{\sqrt[4]{3}}$

8. $\frac{3}{\sqrt[4]{27a^5b^{11}}}$

9. $\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}-\sqrt{2}}$

10. $\frac{\sqrt{x}-2\sqrt{y}}{\sqrt{x}+2\sqrt{y}}$

Rationalize the denominator:

1. $\frac{\sqrt{2}}{\sqrt{3}}$

   Answer 1.
   $$\frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}} \cdot \left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{\sqrt{2}\sqrt{3}}{3} = \frac{\sqrt{6}}{3}$$

2. $\frac{\sqrt{2}}{1-\sqrt{2}}$

   Answer 2.
   $$\frac{\sqrt{2}}{1-\sqrt{2}} = \frac{\sqrt{2}}{1-\sqrt{2}} \cdot \left(\frac{1+\sqrt{2}}{1+\sqrt{2}}\right) = \frac{\sqrt{2}+2}{1-2} = -\sqrt{2}-2$$

3. $\frac{1+\sqrt{3}}{1-\sqrt{3}}$

   Answer 3.
   $$\frac{1+\sqrt{3}}{1-\sqrt{3}} = \frac{1+\sqrt{3}}{1-\sqrt{3}} \cdot \left(\frac{1+\sqrt{3}}{1+\sqrt{3}}\right) = \frac{4+2\sqrt{3}}{1-3} = -2-\sqrt{3}$$

4. $\frac{a+\sqrt{b}}{a-\sqrt{b}}$

   Answer 4.
   $$\frac{a+\sqrt{b}}{a-\sqrt{b}} = \frac{a+\sqrt{b}}{a-\sqrt{b}} \cdot \left(\frac{a+\sqrt{b}}{a+\sqrt{b}}\right) = \frac{a^2+2a\sqrt{b}+b}{a^2-b}$$

5. $\frac{1}{\sqrt{5}} + 4\sqrt{45}$

   Answer 5.
   $$\frac{1}{\sqrt{5}} + 4\sqrt{45} = \frac{1}{\sqrt{5}} + 4 \cdot 3\sqrt{5} = \frac{1}{\sqrt{5}} + \frac{12 \cdot 5}{\sqrt{5}} = \frac{61}{\sqrt{5}} = \frac{61\sqrt{5}}{5}$$

6. $\frac{4}{\sqrt[3]{16}}$

   Answer 6.
   $$\frac{4}{\sqrt[3]{16}} \cdot \left(\frac{\sqrt[3]{16}}{\sqrt[3]{16}}\right)^2 = \frac{4\sqrt[3]{16^2}}{16} = \frac{\sqrt[3]{2^8}}{4} = \frac{2^2\sqrt[3]{2^2}}{4} = \sqrt[3]{4}$$

7. $\frac{3}{\sqrt[4]{3}}$

Answer 7.

$$\frac{3}{\sqrt[4]{3}} = \frac{3}{\sqrt[4]{3}} \cdot \left(\frac{\sqrt[4]{3}}{\sqrt[4]{3}}\right)^3 = \frac{3\sqrt[4]{3^3}}{3} = \sqrt[4]{3^3}$$

8. $\frac{3}{\sqrt[4]{27a^5b^{11}}}$

Answer 8.

$$\frac{3}{\sqrt[4]{27a^5b^{11}}} = \frac{3}{\sqrt[4]{27a^5b^{11}}} \cdot \left(\frac{\sqrt[4]{27a^5b^{11}}}{\sqrt[4]{27a^5b^{11}}}\right)^3 = \frac{3\sqrt[4]{(27a^5b^{11})^3}}{27a^5b^{11}}$$

$$= \frac{3\sqrt[4]{3^9a^{15}b^{33}}}{3^3a^5b^{11}}$$

$$= \frac{3^3a^3b^8\sqrt[4]{3a^3b}}{3^3a^5b^{11}}$$

$$= \frac{\sqrt[4]{3a^3b}}{a^2b^3}$$

9. $\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}-\sqrt{2}}$

Answer 9.

$$\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}-\sqrt{2}} = \frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}-\sqrt{2}} \cdot \left(\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}\right) = \frac{x+2\sqrt{2x}+2}{x-2}$$

10. $\frac{\sqrt{x}-2\sqrt{y}}{\sqrt{x}+2\sqrt{y}}$

Answer 10.

$$\frac{\sqrt{x}-2\sqrt{y}}{\sqrt{x}+2\sqrt{y}} = \frac{\sqrt{x}-2\sqrt{y}}{\sqrt{x}+2\sqrt{y}} \cdot \left(\frac{\sqrt{x}-2\sqrt{y}}{\sqrt{x}-2\sqrt{y}}\right) = \frac{x-4\sqrt{xy}+4y}{x-4y}$$

A *perfect square* is a number a such that $a = b^2$ for some real number b. Some examples of perfect squares are $4 = 2^2, 16 = 4^2, 169 = 13^2$. We wish to have a method for finding b when a is an expression. For instance, you should remember that $a^2 + 2ab + b^2$ is a perfect square, because it is exactly $(a + b)^2$. How would you turn the expression $x^2 + ax$ into a perfect square?

A moment of thought should convince you that if we add $(\frac{a}{2})^2$ to $x^2 + ax$ we obtain a perfect square, because $(x + \frac{a}{2})^2 = x^2 + ax + (\frac{a}{2})^2$. The addition of $(\frac{a}{2})^2$ is called *completing the square*, because the new expression can now be written as a square of some other expression.

Example 1. Complete the square: $x^2 + 4x = 0$

$$x^2 + 4x = 0 \iff (x^2 + 4x + 4) = 4 \iff (x + 2)^2 = 4$$

We have added *the square of half the coefficient of x* to the original equation, and therefore to maintain equality it was necessary to add the same amount to the other side of the equation.

Warning 2. The coefficient of x^2 must be equal to 1 in order to complete the square.

Example 3. Complete the square: $2x^2 + 8x = 0$

$$2x^2 + 8x = 0 \iff 2(x^2 + 4x) = 0 \iff 2(x^2 + 4x + 4) = 8 \iff 2(x + 2)^2 = 8$$

We added 4, the square of half the coefficient of x, inside the parentheses. Note that this amounts to adding 8 to the left side of the equation, because everything inside the parentheses is multiplied by 2. Therefore, to maintain equality we add 8 to the right side of the equation. In case we cannot set our expression equal to 0, we must subtract whatever number we add to the expression:

Example 4. Complete the square: $2x^2 + 8x$

$$2x^2 + 8x = 2(x^2 + 4x) = 2(x^2 + 4x + 4) - 8 = 2(x + 2)^2 - 8$$

Example 5. $(x - h)^2 + (y - k)^2 = r^2$ is the equation of a circle of radius r centered at the point (h, k). Using the method of completing the square (twice) find the radius and center of the circle given by the equation $x^2 + y^2 + 8x - 6y + 21 = 0$.

$$
\begin{align}
x^2 + y^2 + 8x - 6y + 21 &= 0 \tag{1} \\
(x^2 + 8x) + (y^2 - 6y) &= -21 \tag{2} \\
(x^2 + 8x + 16) + (y^2 - 6y + 9) &= -21 + 16 + 9 \tag{3} \\
(x + 4)^2 + (y - 3)^2 &= 4 \tag{4}
\end{align}
$$

We have now the form $(x - (-4))^2 + (y - 3)^2 = 2^2$ which is a circle of radius $r = 2$ centered at the point $(h, k) = (-4, 3)$.

Deriving the Quadratic Formula   Given a quadratic equation, i.e. an equation of this form:

$$ax^2 + bx + c = 0, a \neq 0 \tag{5}$$

where a, b, and c are real numbers, we wish to have a formula that will give us the explicit values of x for which the quadratic equation is zero. That is, we need a formula that produces x_1 and x_2 such that

$$ax_1^2 + bx_2 + c = 0 \text{ and } ax_2^2 + bx_2 + c = 0 \tag{6}$$

The quadratic formula tells us exactly how to find our set of solutions $\{x_1, x_2\}$, but it also tells how large this set is. We can have two distinct solutions and this happens whenever the discriminant is a positive number. We can have just one solution if the discriminant is zero. In this case we say that the root $x_1(= x_2)$ has multiplicity 2, because it occurs twice. Finally, when the discriminant is a negative number, we have a square root of a negative number and hence no (real) solutions. Recall the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ where the discriminant is equal to } b^2 - 4ac \tag{7}$$

How do we know that this is indeed correct? We can apply the method of completing the square to our quadratic equation (1) and verify that equation (2) is correct. Here are the details:

$$ax^2 + bx + c = 0 \tag{8}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \tag{9}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \tag{10}$$

$$x^2 + \frac{b}{a}x + \left(\frac{\frac{b}{a}}{2}\right)^2 = \left(\frac{\frac{b}{a}}{2}\right)^2 - \frac{c}{a} \tag{11}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \tag{12}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \tag{13}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \cdot \frac{4a}{4a} \tag{14}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} \tag{15}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \tag{16}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \tag{17}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} \tag{18}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \tag{19}$$

$$x = -\frac{b}{2a} + \frac{\pm\sqrt{b^2 - 4ac}}{2a} \tag{20}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{21}$$

So which of the above steps do we call "completing the square"? The answer is (4) to (7); the rest deal with writing the equation in the form $x = \ something$. Let's review:

Suppose you are given your favorite quadratic $ax^2 + bx + c$ and need to solve for x. You are no longer amused by factoring and decide to complete the square instead.

Step 1: Check the coefficients. If $a = 0$ you don't need to complete the square. If $a \neq 1$ then you need to factor out a. So suppose that $a \neq 1$ and $a \neq 0$.

$$ax^2 + bx + c = a \left[x^2 + \frac{b}{a}x + \frac{c}{a} \right] \tag{22}$$

Step 2: Group the x terms together. You complete the square only on the terms containing the variable x. Notice that inside the brackets [ ] we now have a new quadratic equation with coefficients $a = 1, b = \frac{b}{a}$ and $c = \frac{c}{a}$.

$$ax^2 + bx + c = a \left[\left(x^2 + \frac{b}{a}x \right) + \frac{c}{a} \right] \tag{23}$$

Step 3: Complete the square: add the square of half of the coefficient of x to the terms in side the parentheses ( ).

$$ax^2 + bx + c = a \left[\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a} \right)^2 \right) + \frac{c}{a} \right] \tag{24}$$

Step 4: Up until now we have not altered the equation, but adding something to the right side requires subtracting the same number. We have added $\frac{b^2}{4a^2}$ inside the brackets [ ] and everything inside [ ] is multiplies by a. Therefore, to keep the equation unchanged, we now subtract from the right side the number $a \cdot \frac{b^2}{4a^2}$ and obtain

$$ax^2 + bx + c = a \left[\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a} \right)^2 \right) + \frac{c}{a} \right] - \frac{b^2}{4a} \tag{25}$$

Step 5: Simplify. The term in the parentheses ( ) is a perfect square and so

$$ax^2 + bx + c = a \left[\left(x + \frac{b}{2a} \right)^2 + \frac{c}{a} \right] - \frac{b^2}{4a} = a \left(x + \frac{b}{2a} \right)^2 - \frac{b^2 - 4ac}{4a} \tag{26}$$

This form should look familiar. If we were to set line (22) equal to zero we would have the standard quadratic equation. Then dividing by a (legal since $a \neq 0$) and moving terms around returns us to equation (12).

Viete's Equations, or how to pick out the correct pair of solutions to a quadratic equation ...

Proposition 6. *Given a quadratic equation with real coefficients a, b, c*

$$ax^2 + bx + c = 0, a \neq 0$$

If the solutions exist, then they have the following form

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

and they obey the following algebraic equations:

$$x_1 + x_2 = \frac{-b}{a}$$

$$x_1 \cdot x_2 = \frac{c}{a}$$

If you are given a quadratic equation to solve *and* are allowed to use the quadratic formula, then you may follow these steps and save yourself some work.

Step 1: Make sure that the solutions exist, i.e. $b^2 - 4ac \geq 0$

Step 2: Look at the quadratic equation you have to solve and determine the values of a, b, c and compute $\frac{-b}{a}$ and $\frac{c}{a}$.

Step 3: Compute $x_1 + x_2$ and $x_1 \cdot x_2$ for each set of solutions your are given as a choice.

Step 4: Compare the results of steps 2 and 3. If you find a match, you have found the solution. If there is no match, then none of the possible choices is a solution. (Is it possible to have more than one set of matching solutions?)

Using the method of completing the square, put each circle into the form

$$(x - h)^2 + (y - k)^2 = r^2$$

.

Then determine the center and radius of each circle.

1. $x^2 + y^2 - 10x + 2y + 17 = 0$.

2. $x^2 + y^2 + 8x - 6y + 16 = 0$.

3. $9x^2 + 54x + 9y^2 - 18y + 64 = 0$.

4. $4x^2 - 4x + 4y^2 - 59 = 0$.
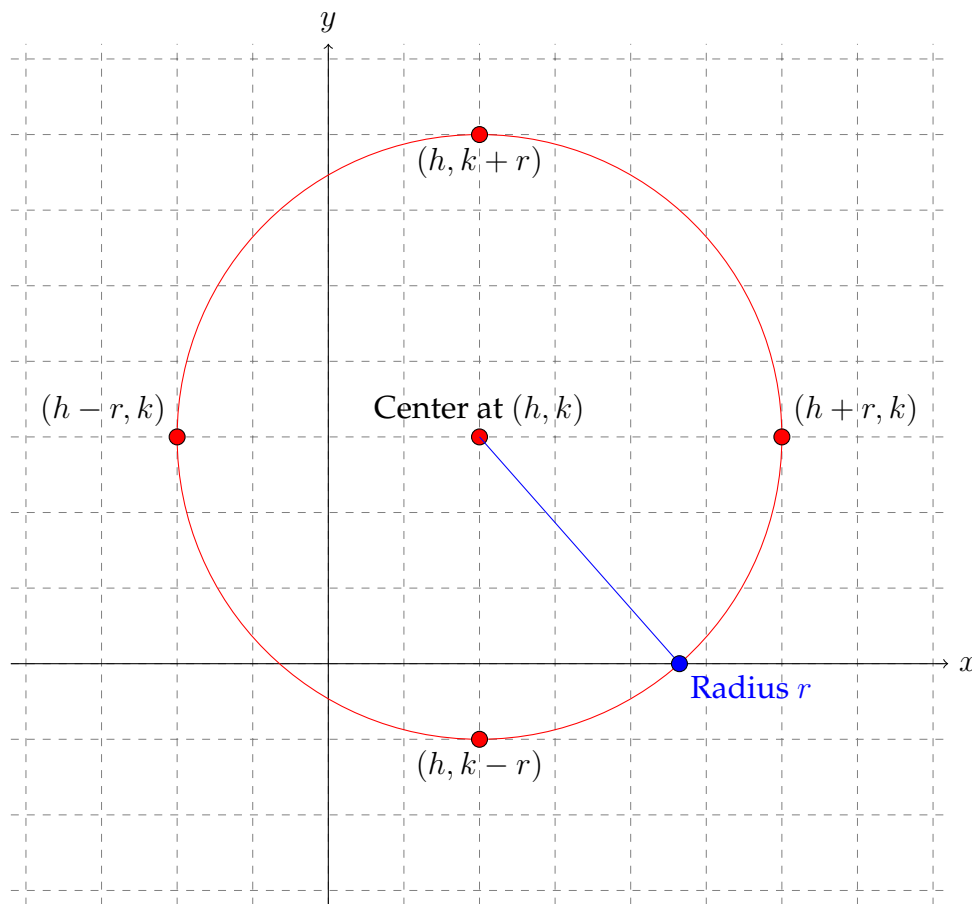
Sample Midterm				Sample Final
3	A	B	C	D
8	A	B	C	D
11	A	B	C	D
29	A	B	C	D
31	A	B	C	D
32	A	B	C	D
35	A	B	C	D

Using the method of completing the square, put each circle into the form

$$(x - h)^2 + (y - k)^2 = r^2$$

.

Then determine the center and radius of each circle.

1. $x^2 + y^2 - 10x + 2y + 17 = 0$.

    Answer 1.

$$
\begin{aligned}
x^2 + y^2 - 10x + 2y + 17 &= 0 \\
(x^2 - 10x) + (y^2 + 2y) &= -17 \\
(x^2 - 10x + 25) + (y^2 + 2y + 1) &= -17 + 25 + 1 \\
(x - 5)^2 + (y + 1)^2 &= 3^2
\end{aligned}
$$

    Circle of radius $r = 3$ centered at $(5, -1)$.

2. $x^2 + y^2 + 8x - 6y + 16 = 0$.

    Answer 2.

$$
\begin{aligned}
x^2 + y^2 + 8x - 6y + 16 &= 0 \\
(x^2 + 8x) + (y^2 - 6y) &= -16 \\
(x^2 + 8x + 16) + (y^2 - 6y + 9) &= -16 + 16 + 9 \\
(x + 4)^2 + (y - 3)^2 &= 3^2
\end{aligned}
$$

    Circle of radius $r = 3$ centered at $(-4, 3)$.

3. $9x^2 + 54x + 9y^2 - 18y + 64 = 0$.

    Answer 3.

$$
\begin{aligned}
9x^2 + 54x + 9y^2 - 18y + 64 &= 0 \\
(9x^2 + 54x) + (9y^2 - 18y) &= -64 \\
9(x^2 + 6x) + 9(y^2 - 2y) &= -64 \\
9(x^2 + 6x + 9) + 9(y^2 - 2y + 1) &= -64 + 81 + 9 \\
9(x + 3)^2 + 9(y - 1)^2 &= 26 \\
(x + 3)^2 + (y - 1)^2 &= \left(\frac{\sqrt{26}}{3}\right)^2
\end{aligned}
$$

    Circle of radius $r = \frac{\sqrt{26}}{3}$ centered at $(-3, 1)$.

4. $4x^2 - 4x + 4y^2 - 59 = 0$.

Answer 4.

$$
\begin{aligned}
4x^2 - 4x + 4y^2 - 59 &= 0 \\
(4x^2 - 4x) + (4y^2) &= 59 \\
4(x^2 - x) + 4(y^2) &= 59 \\
4(x^2 - x + \frac{1}{4}) + 4(y^2) &= 59 + 1 \\
4(x - \frac{1}{2})^2 + 4(y^2) &= 60 \\
(x - \frac{1}{2})^2 + (y^2) &= 15
\end{aligned}
$$

Circle of radius $\sqrt{15}$ centered at $(\frac{1}{2}, 0)$.

1. Compute:

    (a) $f(x) = 2x + 1$, $f(2x) =$

    (b) $f(x) = x^2 - 1$, $f(x + 1) =$

    (c) $f(x) = 7x + 11$, $f(f(x) - 9) =$

    (d) $f(x) = \frac{1}{x-1}$, $f(\frac{1}{x}) =$

    (e) $f(x) = 2x + 4$, $\frac{f(x+h) - f(x)}{h} =$

2. Determine if the following are functions.

    (a) $y = 4$

    (b) $\{(-2, 0), (3, 5), (-2, 4), (1, 5)\}$

    (c) The relation which assigns to each person the month and day of their birthday.

    (d)
    $$y = \begin{cases} x^2, & -3 \leq x \leq 0; \\ x, & 0 < x \leq 2; \\ x^3, & 2 \leq x < 5. \end{cases}$$

3. Find the domain and range of the following functions. Write the answer in interval notation.

    (a) $y = 2$

    (b) $\{(-1, 0), (3, 1), (0, -1), (1, -1), (2, \sqrt{2}), (4, 1)\}$

    (c) The relation which assign to each person the first letter of their last name.

    (d)
    $$g(x) = \begin{cases} 0, & -3 \leq x \leq 0; \\ x, & 0 < x < 2; \\ x^2, & 2 \leq x < 5. \end{cases}$$

Sample Midterm				Sample Final
5	A B C D			
7	A B C D			
12	A B C D			
13	A B C D			
16	A B C D			
37	A B C D			

1. Compute:

   (a) $f(x) = 2x + 1$, $f(2x) =$

   Answer 1.
   $$f(2x) = 2(2x) + 1 = 4x + 1$$

   (b) $f(x) = x^2 - 1$, $f(x + 1) =$

   Answer 2.
   $$f(x + 1) = (x + 1)^2 - 1 = x^2 + 2x + 1 - 1 = x^2 + 2x = x(x + 2)$$

   (c) $f(x) = 7x + 11$, $f(f(x) - 9) =$

   Answer 3.
   $$\begin{aligned}
   f(f(x) - 9) &= 7(f(x) - 9) + 11 \\
   &= 7(7x + 11 - 9) + 11 \\
   &= 7(7x + 2) + 11 \\
   &= 49x + 14 + 11 \\
   &= 49x + 25
   \end{aligned}$$

   (d) $f(x) = \frac{1}{x-1}$, $f(\frac{1}{x}) =$

   Answer 4.
   $$f\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x} - 1} = \frac{1}{\frac{1}{x} - \frac{x}{x}} = \frac{1}{\frac{1-x}{x}} = \frac{x}{1-x}$$

   (e) $f(x) = 2x + 4$, $\frac{f(x+h) - f(x)}{h} =$

   Answer 5.
   $$\frac{f(x + h) - f(x)}{h} = \frac{2(x + h) + 4 - (2x + 4)}{h} = \frac{2x + 2h + 4 - 2x - 4}{h} = \frac{2h}{h} = 2$$

2. Determine if the following are functions.

   (a) $y = 4$

   Answer 6. Yes, this is a horizontal line and passes the vertical line test.

   (b) $\{(-2, 0), (3, 5), (-2, 4), (1, 5)\}$

   Answer 7. No, because for $x = -2$ we have two different values.

   (c) The relation which assigns to each person the month and day of their birthday.

   Answer 8. Yes, each person has one birthday.

   (d)
   $$y = \begin{cases} x^2, & -3 \le x \le 0; \\ x, & 0 < x \le 2; \\ x^3, & 2 \le x < 5. \end{cases}$$

   Answer 9. No, at $x = 2$ there are two different possible values: 2 and $2^3 = 8$.

3. Find the domain and range of the following functions.

    (a) $y = 2$

    Answer 10.
    Domain: $(-\infty, \infty)$
    Range: $\{2\}$

    (b) $\{(-1, 0), (3, 1), (0, -1), (1, -1), (2, \sqrt{2}), (4, 1)\}$

    Answer 11.
    Domain: $\{-1, 0, 1, 2, 3, 4\}$
    Range: $\{-1, 0, 1, \sqrt{2}\}$

    (c) The relation which assign to each person the first letter of their last name.

    Answer 12.
    Domain: The set of all people.
    Range: The alphabet.

    (d)

$$g(x) = \begin{cases} 0, & -3 \leq x \leq 0; \\ x, & 0 < x < 2; \\ x^2, & 2 \leq x < 5. \end{cases}$$

    Answer 13.
    Domain: $[-3, 0] \cup (0, 2) \cup [2, 5) = [-3, 5]$
    Range: $\{0\} \cup (0, 2) \cup [4, 25) = [0, 2) \cup [4, 25)$

1. The constant function: $f(x) = c, c \in \mathrm{R}$.

Domain	$(-\infty, \infty)$
Range	$\{c\}$
Intervals of Increase	
Intervals of Decrease	none
Turning Points	none
Local Maxima	
Local Minima	
Global Maxima	$x = c$
Global Minima	$x = c$
Symmetry	

2. The linear function: $f(x) = ax + b; a, b \in \mathrm{R}$.

Domain	$(-\infty, \infty)$
Range	$\begin{cases} \{b\}, & a = 0; \\ (-\infty, \infty), & a \neq 0. \end{cases}$
Intervals of Increase	$\begin{cases} \text{none}, & a \leq 0; \\ (-\infty, \infty), & a > 0. \end{cases}$
Intervals of Decrease	$\begin{cases} \end{cases}$
Turning Points	none
Local Maxima	$\begin{cases} \text{none}, & a \neq 0; \\ b, & a = 0. \end{cases}$
Local Minima	$\begin{cases} \text{none}, & a \neq 0; \\ b, & a = 0. \end{cases}$
Global Maxima	$\begin{cases} \end{cases}$
Global Minima	$\begin{cases} \end{cases}$
Symmetry	

3. The square function: $f(x) = x^2$.

Domain	$(-\infty, \infty)$
Range	
Intervals of Increase	$[0, \infty)$
Intervals of Decrease	$(-\infty, 0]$
Turning Points	x=0
Local Maxima	none
Local Minima	$x = 0$
Global Maxima	none
Global Minima	$x = 0$
Symmetry	

4. The cube function: $f(x) = x^3$.

Domain	
Range	
Intervals of Increase	$(-\infty, \infty)$
Intervals of Decrease	none
Turning Points	none
Local Maxima	
Local Minima	
Global Maxima	
Global Minima	
Symmetry	

5. The inverse function: $f(x) = \frac{1}{x}$.

Domain	
Range	
Intervals of Increase	none
Intervals of Decrease	$(-\infty, 0) \cup (0, \infty)$
Turning Points	none
Local Maxima	none
Local Minima	none
Global Maxima	none
Global Minima	none
Symmetry	

6. The inverse square function: $f(x) = \frac{1}{x^2}$.

Domain	$(-\infty, 0) \cup (0, \infty)$
Range	
Intervals of Increase	
Intervals of Decrease	
Turning Points	none
Local Maxima	none
Local Minima	none
Global Maxima	none
Global Minima	none
Symmetry	

7. The square root function: $f(x) = \sqrt{x}$.

Domain	$[0, \infty)$
Range	$[0, \infty)$
Intervals of Increase	
Intervals of Decrease	
Turning Points	none
Local Maxima	
Local Minima	
Global Maxima	
Global Minima	
Symmetry	

8. The cube root function: $f(x) = \sqrt[3]{x}$.

Domain	
Range	
Intervals of Increase	
Intervals of Decrease	none
Turning Points	none
Local Maxima	none
Local Minima	none
Global Maxima	none
Global Minima	none
Symmetry	

9. The absolute value function: $f(x) = |x|$.

Domain	
Range	
Intervals of Increase	$[0, \infty)$
Intervals of Decrease	$(-\infty, 0]$
Turning Points	$x = 0$
Local Maxima	none
Local Minima	
Global Maxima	none
Global Minima	
Symmetry	

Sample Midterm				Sample Final
15	A	B	C	D
20	A	B	C	D
24	A	B	C	D
25	A	B	C	D

1. The constant function: $f(x) = c, c \in \mathrm{R}$.

Domain	$(-\infty, \infty)$
Range	$\{c\}$
Intervals of Increase	none
Intervals of Decrease	none
Turning Points	none
Local Maxima	$c, x = c$
Local Minima	$c, x = c$
Global Maxima	$c, x = c$
Global Minima	$c, x = c$
Symmetry	y-axis

2. The linear function: $f(x) = ax + b; a, b \in \mathbb{R}$.

Domain	$(-\infty, \infty)$
Range	$\begin{cases} \{b\}, & a = 0; \\ (-\infty, \infty), & a \neq 0. \end{cases}$
Intervals of Increase	$\begin{cases} \text{none}, & a \leq 0; \\ (-\infty, \infty), & a > 0. \end{cases}$
Intervals of Decrease	$\begin{cases} \text{none}, & a \geq 0; \\ (-\infty, \infty), & a < 0. \end{cases}$
Turning Points	none
Local Maxima	$\begin{cases} \text{none}, & a \neq 0; \\ b, & a = 0. \end{cases}$
Local Minima	$\begin{cases} \text{none}, & a \neq 0; \\ b, & a = 0. \end{cases}$
Global Maxima	$\begin{cases} \text{none}, & a \neq 0; \\ b, & a = 0. \end{cases}$
Global Minima	$\begin{cases} \text{none}, & a \neq 0; \\ b, & a = 0. \end{cases}$
Symmetry	$\begin{cases} \text{none}, & a \neq 0; \\ y\text{-axis}, & a = 0. \end{cases}$

3. The square function: $f(x) = x^2$.

Domain	$(-\infty, \infty)$
Range	$[0, \infty)$
Intervals of Increase	$[0, \infty)$
Intervals of Decrease	$(-\infty, 0]$
Turning Points	$x = 0$
Local Maxima	none
Local Minima	$x = 0$
Global Maxima	none
Global Minima	$x = 0$
Symmetry	y-axis

4. The cube function: $f(x) = x^3$.

Domain	$(-\infty, \infty)$
Range	$(-\infty, \infty)$
Intervals of Increase	$(-\infty, \infty)$
Intervals of Decrease	none
Turning Points	none
Local Maxima	none
Local Minima	none
Global Maxima	none
Global Minima	none
Symmetry	origin

5. The inverse function: $f(x) = \frac{1}{x}$.

Domain	$(-\infty, 0) \cup (0, \infty)$
Range	$(-\infty, 0) \cup (0, \infty)$
Intervals of Increase	none
Intervals of Decrease	$(-\infty, 0) \cup (0, \infty)$
Turning Points	none
Local Maxima	none
Local Minima	none
Global Maxima	none
Global Minima	none
Symmetry	origin

6. The inverse square function: $f(x) = \frac{1}{x^2}$.

Domain	$(-\infty, 0) \cup (0, \infty)$
Range	$(0, \infty)$
Intervals of Increase	$(-\infty, 0)$
Intervals of Decrease	$(0, \infty)$
Turning Points	none
Local Maxima	none
Local Minima	none
Global Maxima	none
Global Minima	none
Symmetry	y-axis

7. The square root function: $f(x) = \sqrt{x}$.

Domain	$[0, \infty)$
Range	$[0, \infty)$
Intervals of Increase	$[0, \infty)$
Intervals of Decrease	none
Turning Points	none
Local Maxima	none
Local Minima	$x = 0$
Global Maxima	none
Global Minima	$x = 0$
Symmetry	none

8. The cube root function: $f(x) = \sqrt[3]{x}$.

Domain	$(-\infty, \infty)$
Range	$(-\infty, \infty)$
Intervals of Increase	$(-\infty, \infty)$
Intervals of Decrease	none
Turning Points	none
Local Maxima	none
Local Minima	none
Global Maxima	none
Global Minima	none
Symmetry	origin
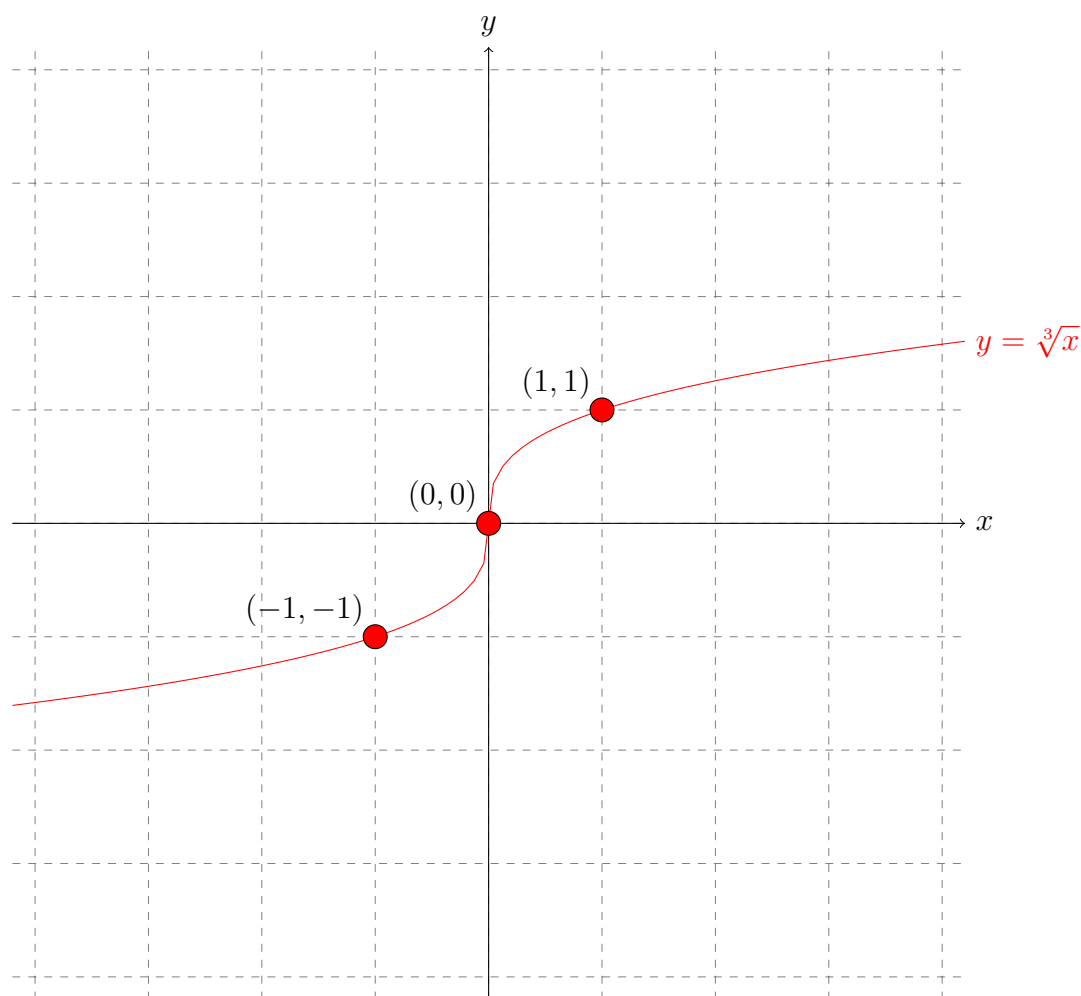
9. The absolute value function: $f(x) = |x|$.

Domain	$(-\infty, \infty)$
Range	$[0, \infty)$
Intervals of Increase	$[0, \infty)$
Intervals of Decrease	$(-\infty, 0]$
Turning Points	$x = 0$
Local Maxima	none
Local Minima	$x = 0$
Global Maxima	none
Global Minima	$x = 0$
Symmetry	y-axis

We are already familiar with linear functions from our work in the first week on lines. Linear functions are simply functions of the form $f(x) = ax + b$ where a and b are real numbers, not both zero. Note that if $a = 0$, then we have $f(x) = b$, which graphs as the the horizontal line $y = b$. The only lines that are not functions are the vertical lines, $x = c$, c any real number, as they (dramatically) fail the vertical line test.

As these functions are the most simple and we have worked on lines already, we will work on word problems instead of examining them directly. The following examples should be enough to show you how to solve the problems on your worksheet.

1. Let x denote a temperature on the Celsius scale, and let y denote the corresponding temperature on the Fahrenheit scale.

   (a) Find a linear function relating x and y; use the facts that $32°F$ corresponds to $0°C$ and $212°F$ corresponds to $100°C$.

   As this is a linear function, we know it must have the form $f(x) = ax + b$. We simply need to find a and b. However, a is just the slope and we have a formula for that given two points on the graph of the function (as the graph is a straight line). The two points given are $(0, 32)$ and $(100, 212)$. The slope is given by,

   $$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{212 - 32}{100 - 0} = \frac{180}{100} = \frac{9}{5} \ .$$

   So the function is of the form $f(x) = \frac{9}{5}x + b$. To find b we need only input one of the points,

   $$32 = \frac{9}{5}(0) + b \iff b = 32 \ .$$

   We now believe that the answer if $f(x) = \frac{9}{5}x + 32$, but we must input the other point to verify this. We have

   $$f(100) = \frac{9}{5}(100) + 32 = 180 + 32 = 212 \ ,$$

   so our answer is correct.

   (b) What Celsius temperature corresponds to $98.6°F$?

   This amounts to solving the equation $f(x) = 98.6$.

   $$\begin{aligned} 98.6 &= \frac{9}{5}x + 32 \\ 66.6 &= \frac{9}{5}x \\ 5(66.6) &= 9x \\ 333 &= 9x \\ x &= \frac{333}{9} = 37 \ , \end{aligned}$$

   and we conclude that $37°C$ is the equivalent of $98.6°F$.

(c) Find a number z for which $z^oF = z^oC$.

This amounts to solving the equation $z = f(z)$. We have,

$$z = \frac{9}{5}z + 32$$
$$-32 = \frac{4}{5}z$$
$$-160 = 4z$$
$$z = -40 \ .$$

2. (a) A biologist measured the population of a colony of fruit flies over a period of 39 days. After 12 days there were 105. After 18 days there were 225. Find the linear function whose graph passes through the two points given in the table. We compute the slope,

$$a = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{225 - 105}{18 - 12}$$
$$= \frac{120}{6} = 20 \ ,$$

and the y-intercept,

$$105 = 20(12) + b$$
$$105 = 240 + b$$
$$b = -135 \ .$$

So the linear function is $f(x) = 20x - 135$, and we check our work,

$$f(18) = 20(18) - 135 = 360 - 135 = 225.$$

(b) Find the population after 20 and 39 days.

$$f(20) = 20(20) - 135 = 400 - 135 = 265,$$
$$f(39) = 20(39) - 135 = 780 - 135 = 655.$$

This information was taken from an actual experiment and the true result after 39 days was actually 938. Linear functions are actually not very helpful in predicting real world solutions. However, they make a good starting point for us.

(c) Assuming our linear function actually gave good predictions of the actual fruit fly population, when would the population reach 1000?

This amounts to solving the equation $1000 = f(x)$, and we have

$$1000 = 20x - 135$$
$$20x = 1135$$
$$x = 56.75 \ .$$

The population would reach 1000 in about 57 days.

Find the linear functions satisfying the given conditions:

1. $f(3) = 2$ and $f(-3) = -4$.

2. $f(0) = 0$ and $f(1) = \sqrt{2}$.

3. $g(2) = 1$ and the graph of g is parallel to the line $6x - 3y = 2$.

4. $g(2) = 1$ and the graph of g is perpendicular to the line $6x - 3y = 2$.

5. The x and y-intercepts of f are 5 and -1, respectively.

6. During the 1990s the percentage of TV households viewing cable and satellite TV programs increased while the percentage viewing network affiliate shows generally decreased. The prime-time ratings for the network affiliates in 1993 was 40.9. The ratings fell in 1995 to 37.3.

   (a) Find the equation for the linear function (x =year, $y = f(x)$ =rating) whose graph passes through the two points given.

   (b) What should the ratings be in 2000?

   (c) What year would we expect the ratings to reach 30?

7. The population of Florida was 11,350,000 in 1985 and reached 13,000,000 in 1990.

   (a) Find the equation for the linear function (x =year, $y = f(x)$ =population) whose graph passes through the two points given.

   (b) What should the population be in 2000?

   (c) What year would we expect the population to reach 25,000,000?

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Find the linear functions satisfying the given conditions:

1. $f(3) = 2$ and $f(-3) = -4$.

   We have the points $(3, 2)$ and $(-3, -4)$. The slope of the line connecting these two points is

   $$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - (-4)}{3 - (-3)} = \frac{6}{6} = 1.$$

   We may then input one of the points to calculate b for our linear function $f(x) = ax + b$.

   $$2 = 1(3) + b \Rightarrow 2 = 3 + b \Rightarrow b = -1,$$

   so $f(x) = x - 1$, and we check our work

   $$f(-3) = -3 - 1 = -4.$$

2. $f(0) = 0$ and $f(1) = \sqrt{2}$.

   We have the points $(0, 0)$ and $(1, \sqrt{2})$. The slope of the line connecting these two points is:

   $$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\sqrt{2} - 0}{1 - 0} = \sqrt{2}$$

   We may then input one of the points to calculate b for our linear function $f(x) = ax + b$.

   $$0 = \sqrt{2}(0) + b \Rightarrow 0 = 0 + b \Rightarrow b = 0 \,,$$

   and so $f(x) = \sqrt{2}x$. We check our work,

   $$f(1) = \sqrt{2}(1) = \sqrt{2}.$$

3. $g(2) = 1$ and the graph of g is parallel to the line $6x - 3y = 2$.

   $6x - 3y = 2 \Rightarrow 3y = 6x - 2 \Rightarrow y = 2x - \frac{2}{3}$ and parallel lines have the same slope, so $a = 2$.

   We then have the point $(2, 1)$ to input: $1 = 2(2) + b \Rightarrow 1 = 4 + b \Rightarrow b = -3$ for a final answer of $f(x) = 2x - 3$.

4. $g(2) = 1$ and the graph of g is perpendicular to the line $6x - 3y = 2$.

   Perpendicular lines have slopes whose product is -1. The slope of the line is 2 (from the preceding problem), so we have: $m_1 m_2 = -1 \Rightarrow 2a = -1 \Rightarrow a = -\frac{1}{2}$

   We then have the point $(2, 1)$ to input: $1 = -\frac{1}{2}(2) + b \Rightarrow 1 = -1 + b \Rightarrow b = 2$ for a final answer of $f(x) = -\frac{1}{2}x + 2$.

5. The x and y-intercepts of f are 5 and -1, respectively.

   We have the points $(5, 0)$ and $(0, -1)$. The slope of the line connecting these two points is:

   $$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - (-1)}{5 - 0} = \frac{1}{5}$$

   We may then input one of the points to calculate b for our linear function $f(x) = ax + b$, to obtain

   $$0 = \frac{1}{5}(5) + b \Rightarrow 0 = 1 + b \Rightarrow b = -1,$$

   so $f(x) = \frac{1}{5}x - 1$, and we check our work

   $$f(0) = \frac{1}{5}(0) - 1 = -1 .$$

6. During the 1990s the percentage of TV households viewing cable and satellite TV programs increased while the percentage viewing network affiliate shows generally decreased. The primetime ratings for the network affiliates in 1993 was 40.9. The ratings fell in 1995 to 37.3.

   (a) Find the equation for the linear function ($x =$ year, $y = f(x) =$ rating) whose graph passes through the two points given.

   $$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{37.3 - 40.9}{1995 - 1993} = \frac{-3.6}{2} = -1.8 = -\frac{9}{5}$$

   We may then input one of the points to calculate b for our linear function $f(x) = ax + b$.

   $$37.3 = -\frac{9}{5}(1995) + b \Rightarrow 37.3 = -3591 + b \Rightarrow b = 3628.3 \text{ so } f(x) = -\frac{9}{5}x + 3628.3$$

   and doublecheck

   $$f(1993) = -\frac{9}{5}(1993) + 3628.3 = -3587.4 + 3628.3 = 40.9 .$$

   (b) What should the ratings be in 2000?

   $$f(2000) = -\frac{9}{5}(2000) + 3628.3 = -3600 + 3628.3 = 28.3 .$$

(c) What year would we expect the ratings to reach 30?

This amounts to solving the equation

$$30 = -\frac{9}{5}x + 3628.3 \Rightarrow \frac{9}{5}x = 3598.3 \Rightarrow x = 1999 \text{(approximately)}.$$

7. The population of Florida was 11,350,000 in 1985 and reached 13,000,000 in 1990.

(a) Find the equation for the linear function (x =year, $y = f(x)$ =population) whose graph passes through the two points given.

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{13,000,000 - 11,350,000}{1990 - 1985} = \frac{1,650,000}{5} = 330,000$$

We may then input one of the points to calculate b for our linear function $f(x) = ax + b$.

$13,000,000 = 330,000(1990) + b \Rightarrow 13,000,000 = 656,700,000 + b$

$\Rightarrow b = -643,700,000$ so $f(x) = 330,000x - 643,700,000$

and doublecheck $f(1985) = 330,000(1985) - 643,700,000 = 655,050,000 - 643,700,000 = 11,350,000$✓

(b) What should the population be in 2000?

$f(2000) = 330,000(2000) - 643,700,000 = 660,000,000 - 643,700,000 = 16,300,000$

(c) What year would we expect the population to reach 25,000,000?

This amounts to solving the equation: $25,000,000 = 330,000x - 643,700,000$

$\Rightarrow 330,000x = 668,700,000 \Rightarrow x = 2026$ (approximately).

Compute and simplify:

1. $f(x) = x^2 + x + 1,$ $\frac{f(x+h)-f(x)}{h} =$

2. $g(x) = x^2 + 4x + 4,$ $\frac{g(b)-g(a)}{b-a} =$

3. $f(x) = x^2, g(x) = x^3 + 1, h(x) = \frac{1}{x}$

   (a) $f(g(x))$
   (b) $g \circ f(x)$
   (c) $h(f(2))$
   (d) $f \circ g \circ h(2)$
   (e) $f \circ h(g(x))$

4. $f(x) = \sqrt{3x+1}, g(x) = 3x + 1, h(x) = \frac{1}{x^2}$

   (a) $x^3 f \cdot h(x)$
   (b) $(g(x))^2$
   (c) $\frac{h}{g}(x)$
   (d) $g \cdot (h(x) \circ f(x))$
   (e) $\frac{h}{g}(x) \circ (g \circ h(x))$

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Compute and simplify:

1. $f(x) = x^2 + x + 1$, $\frac{f(x+h)-f(x)}{h} =$

   Answer 1.

   $$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 + (x+h) + 1 - (x^2 + x + 1)}{h} \\ &= \frac{x^2 + 2xh + h^2 + x + h + 1 - x^2 - x - 1}{h} \\ &= \frac{2xh + h^2 + h}{h} \\ &= 2x + h + 1 \end{aligned}$$

2. $g(x) = x^2 + 4x + 4$, $\frac{g(b)-g(a)}{b-a} =$

   Answer 2.

   $$\begin{aligned} \frac{g(b) - g(a)}{b - a} &= \frac{b^2 + 4b + 4 - (a^2 + 4a + 4)}{b - a} \\ &= \frac{b^2 + 4b + 4 - a^2 - 4a - 4}{b - a} \\ &= \frac{b^2 - a^2 + 4(b - a)}{b - a} \\ &= \frac{(b + a)(b - a) + 4(b - a)}{b - a} \\ &= \frac{(b - a)(b + a + 4)}{b - a} \\ &= b + a + 4 \end{aligned}$$

3. $f(x) = x^2, g(x) = x^3 + 1, h(x) = \frac{1}{x}$

   (a) $f(g(x))$

   Answer 3.
   $$f(g(x)) = (g(x))^2 = (x^3 + 1)^2 = x^6 + 2x^3 + 1$$

   (b) $g \circ f(x)$

   Answer 4.
   $$g \circ f(x) = (f(x))^3 + 1 = (x^2)^3 + 1 = x^6 + 1$$

   (c) $h(f(2))$

   Answer 5.
   $$h(f(2)) = \frac{1}{f(2)} = \frac{1}{4}$$

   (d) $f \circ g \circ h(2)$

Answer 6.

$$
\begin{aligned}
f \circ g \circ h(4) &= (g \circ h(2))^2 \\
&= ((h(2))^3 + 1)^2 \\
&= \left(\left(\frac{1}{2} \right)^3 + 1 \right)^2 \\
&= \left(\frac{1}{8} + 1 \right)^2 \\
&= \left(\frac{9}{8} \right)^2 \\
&= \frac{81}{64}
\end{aligned}
$$

(e) $f \circ h(g(x))$

Answer 7.

$$
f \circ h(g(x)) = (h(g(x)))^2 = \left(\frac{1}{g(x)} \right)^2 = \left(\frac{1}{x^3 + 1} \right)^2 = \frac{1}{x^6 + 2x^3 + 1}
$$

4. $f(x) = \sqrt{3x + 1}, g(x) = 3x + 1, h(x) = \frac{1}{x^2}$

(a) $x^3 f \cdot h(x)$

Answer 8.

$$
x^3 f \cdot h(x) = x^3 \cdot \sqrt{3x + 1} \cdot \frac{1}{x^2} = x\sqrt{3x + 1}
$$

(b) $(g(x))^2$

Answer 9.

$$
(g(x))^2 = (3x + 1)^2 = 9x^2 + 6x + 1
$$

(c) $\frac{h}{g}(x)$

Answer 10.

$$
\frac{h}{g}(x) = \frac{\frac{1}{x^2}}{3x + 1} = \frac{1}{x^2(3x + 1)} = \frac{1}{3x^3 + x^2}
$$

(d) $g \cdot h(x) \circ f(x)$

Answer 11.

$$
g \cdot h(x) \circ f(x) = g(x) \cdot \frac{1}{(f(x))^2} = \frac{g(x)}{(\sqrt{3x + 1})^2} = \frac{3x + 1}{|3x + 1|}
$$

(e) $\frac{h}{g}(x) \circ (g \circ h(x))$

Answer 12.

$$
\begin{aligned}
\frac{h}{g}(x) \circ (g \circ h(x)) &= \frac{1}{x^2(3x+1)} \circ \left(\frac{3}{x^2} + 1\right) \\
&= \frac{1}{(\frac{3}{x^2}+1)^2 \cdot (3(\frac{3}{x^2}+1)+1)} \\
&= \frac{1}{(\frac{9}{x^4} + \frac{6}{x^2} + 1)(\frac{9}{x^2} + 4)} \\
&= \frac{1}{\frac{81}{x^6} + \frac{54}{x^4} + \frac{9}{x^2} + \frac{36}{x^4} + \frac{24}{x^2} + 4} \\
&= \frac{1}{\frac{81}{x^6} + \frac{90}{x^4} + \frac{33}{x^2} + 4}
\end{aligned}
$$

Compute:

1. $f \circ g(x), f(x) = |x|, g(x) = x^2 + 1$

2. $g \circ f \circ g(x), f(x) = 2x + 1, g(x) = x^2$

3. $f \circ h \circ f(x), f(x) = 3x, h(x) = \frac{1}{2x+1}$

4. Write as a composition of two functions:

   (a) $\frac{1}{x+1}$

   (b) $\sqrt[3]{x^2 + 2x}$

   (c) $27x^3$

   (d) $\frac{2x+2}{2x+1}$

5. Write as a composition of three functions:

   (a) $*x^2 + 2\sqrt{x^2 + 1}$

   (b) $\frac{1}{\sqrt{x-1}}$

   (c) $*\frac{|2x+1|+(2x+1)^2}{(2x+1)^4}$

   We say that a function $f(x)$ is the *inverse* of a function $g(x)$ if

   $$f \circ g(x) = x = g \circ f(x)$$

6. Verify that the given functions are inverses of each other.

   (a) $f(x) = x^3, g(x) = \sqrt[3]{x}$

   (b) $f(x) = \frac{1}{x}, g(x) = \frac{1}{x}$

   (c) $f(x) = \frac{x+1}{x-1}, g(x) = \frac{x+1}{x-1}$

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18	A	B	C	D
19	A	B	C	D
28	A	B	C	D

Compute:

1. $f \circ g(x)$, $f(x) = |x|$, $g(x) = x^2 + 1$

   Answer 1.
   $$f \circ g(x) = |g(x)| = |x^2 + 1| = x^2 + 1$$

2. $g \circ f \circ g(x)$, $f(x) = 2x + 1$, $g(x) = x^2$

   Answer 2.
   $$g \circ f \circ g(x) = (f \circ g(x))^2 = (2g(x) + 1)^2 = (2x^2 + 1)^2 = 4x^4 + 4x^2 + 1$$

3. $f \circ h \circ f(x)$, $f(x) = 3x$, $h(x) = \frac{1}{2x+1}$

   Answer 3.
   $$f \circ h \circ f(x) = 3(h \circ f(x)) = 3\left(\frac{1}{2f(x) + 1}\right) = 3\left(\frac{1}{2 \cdot 3x + 1}\right) = \frac{3}{6x + 1}$$

4. Write as a composition of two functions:

   (a) $\frac{1}{x+1}$

   Answer 4. Let $f(x) = \frac{1}{x}$ and let $g(x) = x + 1$. Then
   $$f \circ g(x) = \frac{1}{g(x)} = \frac{1}{x + 1}$$

   (b) $\sqrt[3]{x^2 + 2x}$

   Answer 5. Let $f(x) = \sqrt[3]{x}$ and let $g(x) = x^2 + 2x$. Then
   $$f \circ g(x) = \sqrt[3]{g(x)} = \sqrt[3]{x^2 + 2x}$$

   (c) $27x^3$

   Answer 6. Let $f(x) = 3x$ and let $g(x) = x^3$. Then
   $$g \circ f(x) = (f(x))^3 = (3x)^3 = 27x^3$$

(d) $\frac{2x+2}{2x+1}$

Answer 7. Let $f(x) = \frac{x}{x-1}$ and let $g(x) = 2x + 2$. Then

$$f \circ g(x) = \frac{g(x)}{g(x) - 1} = \frac{2x + 2}{2x + 2 - 1} = \frac{2x + 2}{2x + 1}$$

5. Write as a composition of three functions:

(a) * $x^2 + 2\sqrt{x^2 + 1}$

Answer 8. Let $f(x) = x^2 + 2x - 1, g(x) = \sqrt{x}, h(x) = x^2 + 1$. Then

$$
\begin{aligned}
f \circ g \circ h(x) &= (g \circ h(x))^2 + 2(g \circ h(x)) - 1 \\
&= (\sqrt{h(x)})^2 + 2(\sqrt{h(x)}) - 1 \\
&= (\sqrt{x^2 + 1})^2 + 2(\sqrt{x^2 + 1}) - 1 \\
&= x^2 + 1 + 2\sqrt{x^2 + 1} - 1 \\
&= x^2 + 2\sqrt{x^2 + 1}
\end{aligned}
$$

(b) $\frac{1}{\sqrt{x-1}}$

Answer 9. Let $f(x) = \frac{1}{x}, g(x) = \sqrt{x}, h(x) = x - 1$. Then

$$f \circ g \circ h(x) = \frac{1}{g \circ h(x)} = \frac{1}{\sqrt{h(x)}} = \frac{1}{\sqrt{x - 1}}$$

(c) * $\frac{|2x+1|+(2x+1)^2}{(2x+1)^4}$

Answer 10. Let $f(x) = \frac{\sqrt{x}+x}{x^2}, g(x) = x^2, h(x) = 2x + 1$. Then

$$
\begin{aligned}
f \circ g \circ h(x) &= \frac{\sqrt{g \circ h(x)} + g \circ h(x)}{(g \circ h(x))^2} \\
&= \frac{\sqrt{(h(x))^2} + (h(x))^2}{((h(x))^2)^2} \\
&= \frac{|h(x)| + (h(x))^2}{(h(x))^4} \\
&= \frac{|2x + 1| + (2x + 1)^2}{(2x + 1)^4}
\end{aligned}
$$

We say that a function $f(x)$ is the *inverse* of a function $g(x)$ if

$$f \circ g(x) = x = g \circ f(x)$$

6. Verify that the given functions are inverses of each other.

   (a) $f(x) = x^3, g(x) = \sqrt[3]{x}$

   Answer 11.
   $$f \circ g(x) = (\sqrt[3]{g(x)})^3 = (\sqrt[3]{x})^3 = x$$
   $$g \circ f(x) = \sqrt[3]{f(x)} = \sqrt[3]{x^3} = x$$

   Since $f(g(x)) = x = g(f(x))$ the functions $f(x)$ and $g(x)$ are indeed inverses of one another.

   (b) $f(x) = \frac{1}{x}, g(x) = \frac{1}{x}$

   Answer 12.
   $$f \circ g(x) = \frac{1}{g(x)} = \frac{1}{\frac{1}{x}} = x$$
   $$g \circ f(x) = \frac{1}{f(x)} = \frac{1}{\frac{1}{x}} = x$$

   Therfore, $f \circ g(x) = x = g \circ f(x)$, as desired.

   (c) $f(x) = \frac{x+1}{x-1}, g(x) = \frac{x+1}{x-1}$

   Answer 13.

   $$\begin{aligned}
   f \circ g(x) &= \frac{g(x)+1}{g(x)-1} \\
   &= \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} \\
   &= \frac{\frac{x+1}{x-1}+\frac{x-1}{x-1}}{\frac{x+1}{x-1}-\frac{x-1}{x-1}} \\
   &= \frac{\frac{x+1+(x-1)}{x-1}}{\frac{x+1-(x-1)}{x-1}} \\
   &= \frac{\frac{2x}{x-1}}{\frac{2}{x-1}} \\
   &= \frac{2x}{x-1} \cdot \frac{x-1}{2} \\
   &= x
   \end{aligned}$$

   Clearly, $g \circ f(x) = x$ by the same calculation and therfore $f \circ g(x) = x = g \circ f(x)$ as desired.

We have developed the tools to solve the general quadratic equation

$$ax^2 + bx + c = 0; a, b, c \in \mathrm{R}$$

and will now discuss the solutions to equations that can be reduced to a quadratic equation.

1. Solve: $x^4 - 6x^2 + 8 = 0$.

    Answer 1. Observe that $x^4 - 6x^2 + 8 = (x^2 - 4)(x^2 - 2) = 0$ and so either

    $$x^2 - 4 = 0 \Rightarrow x = \pm 2 \qquad \text{or} \qquad x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2}$$

    and the solution set is $\{-2, -\sqrt{2}, \sqrt{2}, 2\}$.

    Notice that we could have used the quadratic formula if we changed the variable by setting $t = x^2$. Then our equation can be rewritten as

    $$x^4 - 6x^2 + 8 = (x^2)^2 - 6(x^2) + 8 = t^2 - 6t + 8 = 0$$

    and we apply the quadratic formula with $a = 1, b = -6, c = 8$ and obtain

    $$t = \frac{6 \pm \sqrt{36 - 32}}{2} = \frac{6 \pm 2}{2} \Rightarrow t \in \{2, 4\}$$

    We have thus solved the reduced equation, but $\{2, 4\}$ is not the solution set to our original problem. Remember that $t = x^2$ so as above we need to solve $x^2 = 2$ and $x^2 = 4$, which will produce the set of solutions to the original equations, i.e. $\{-2, -\sqrt{2}, \sqrt{2}, 2\}$.

2. Rewrite each equation of quadratic type as a quadratic equation in t and give the change of variable.

    (a) $x^6 + 2x^3 + 1 = 0$

    (b) $(x - 2)^4 + (x - 2)^2 + 6 = 0$

    (c) $8x^{\frac{4}{3}} + 2x^{\frac{2}{3}} + 1 = 0$

    (d) $x^{-6} + x^{-3} + 7 = 0$

    (e) $2^{4x} + 4^x + 3 = 0$

    Answer 2.

    (a) $t^2 + 3t + 1 = 0, t = x^3$

    (b) $t^2 + t + 6 = 0, t = (x - 2)^2$

    (c) $8t^2 + 2t + 1 = 0, t = x^{\frac{2}{3}}$

    (d) $t^2 + t + 7 = 0, t = x^{-3}$

    (e) $t^2 + t + 3 = 0, t = 2^{2x}$

3. Find all real solutions of each equation.

   (a) $y^{-2} - \frac{7}{y} + 12 = 0$

   (b) $x^{\frac{2}{3}} = 9$

   (c) $x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 28 = 0$

Answer 3.

(a) Rewrite $y^{-2} - \frac{7}{y} + 12 = 0$ with the change of variable $t = y^{-2}$ and obtain $t^2 - 7t + 12 = 0$. This reduced equation factors as $(t - 3)(t - 4) = 0$ and whence its solution set it $\{3, 4\}$. The solutions to the original equation are obtained by solving $y^{-2} = t = 3$ and $y^{-2} = t = 4$. We have,

$$y^{-2} = 3 \iff y^2 = \frac{1}{3}$$
$$\iff y = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3} \ ,$$

and

$$y^{-2} = 4 \iff y^2 = \frac{1}{4}$$
$$\iff y = \pm\frac{1}{2} \ .$$

Therefore, the solution set is $\{-\frac{1}{2}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{1}{2}\}$.

(b) Rewrite $x^{\frac{2}{3}} = 9$ with the change of variable $t = x^{\frac{1}{3}}$ and obtain $t^2 = 9$. It is clear that the set of solution to the reduced equation is $\{-3, 3\}$. Then, to obtain the solutions to the original equation, we solve

$$x^{\frac{1}{3}} = t = -3 \iff x^{\frac{1}{3}} = -3$$
$$\iff x = -27 \ ,$$

and

$$x^{\frac{1}{3}} = t = 3 \iff x^{\frac{1}{3}} = 3$$
$$\iff x = 27 \ .$$

Therefore, the solution set is $\{-27, 27\}$.

(c) Rewrite $x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 28 = 0$ with the change of variable $t = x^{\frac{1}{3}}$, and obtain

$$t^2 + 3t - 28 = 0 \ .$$

Observe that $t^2 + 3t - 28 = (t+7)(t-4) = 0$ and therefore the solution set to the reduced equation is $\{-7, 4\}$. Then,

$$
\begin{aligned}
x^{\frac{1}{3}} = t = -7 &\iff x^{\frac{1}{3}} = -7 \\
&\iff x = -343 \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
x^{\frac{1}{3}} = t = 4 &\iff x^{\frac{1}{3}} = 4 \\
&\iff x = 64 \ .
\end{aligned}
$$

The original equation has the solutions $\{-343, 64\}$.

1. Factor into at least two factors.

    (a) $x^{\frac{4}{3}} - 3x^{\frac{2}{3}} - 28 = 0$

    (b) $4x^{-4} - 33x^{-2} - 27 = 0$

    (c) $x^6 - 10x^4 + 24x^2 = 0$

    (d) $x^4 - 81 = 0$

2. Find all real solutions of the following equations.

    (a) $x^6 + 2x^3 + 1 = 0$

    (b) $(x - 1)^4 + (x - 1)^2 - 3 = 0$

    (c) $8x^{\frac{4}{3}} + 2x^{\frac{2}{3}} + 2 = 0$

    (d) $6x^{-6} + 3x^{-3} - 2 = 0$

    (e) $2^{2x} - 2^{x+1} - 8 = 0$

3. Solve and write the solutions in set notation.

    (a) $\sqrt{2x + 1} = 2$

    (b) $\sqrt{x - 3} = x - 3$

    (c) $\sqrt{x + 1} = x - 1$

Sample Midterm | Sample Final

33   A   B   C   D

1. Factor into at least two factors.

   (a) $x^{\frac{4}{3}} - 3x^{\frac{2}{3}} - 28 = 0$

   Answer 1.
   $$0 = x^{\frac{4}{3}} - 3x^{\frac{2}{3}} - 28 = (x^{\frac{2}{3}} - 7)(x^{\frac{2}{3}} + 4)$$

   (b) $4x^{-4} - 33x^{-2} - 27 = 0$

   Answer 2.
   $$0 = 4x^{-4} - 33x^{-2} - 27 = (4x^{-2} + 3)(x^{-2} - 9) = (4x^{-2} + 3)(x^{-1} - 3)(x^{-1} + 3)$$

   (c) $x^6 - 10x^4 + 24x^2 = 0$

   Answer 3.
   $$0 = x^6 - 10x^4 + 24x^2 = x^2(x^4 - 10x^2 + 24) = x^2(x^2 - 6)(x^2 - 4) = x^2(x^2 - 6)(x - 2)(x + 2)$$

   (d) $x^4 - 81 = 0$
   $$0 = x^4 - 81 = (x^2 + 9)(x^2 - 9) = (x^2 + 9)(x + 3)(x - 3)$$

   Answer 4.

2. Find all real solutions of the following equations.

   (a) $x^6 + 2x^3 + 1 = 0$

   Answer 5. The reduced quadratic equation is $t^2 + 2t + 1 = 0$ with $t = x^3$. Factoring we obtain $t^2 + 2t + 1 = (t + 1)^2 = 0$ and whence the set of solutions to the reduced quadratic equation is $\{-1\}$. Solving $t = x^3 = -1$ we obtain $\{-1\}$ as the set of solutions to the original equation. [Note that since $t = -1$ is a root of degree 2 of the reduced quadratic, $x = -1$ is a root of degree 2 of the original equation.]

   Another way to solve this problem is to observe that $x^6 + 2x^3 + 1 = (x^3 + 1)^2 = 0$ and therefore the solutions satisfy $x^3 = -1$, which produces the solution set $\{-1\}$.

(b) $(x-1)^4 + (x-1)^2 - 3 = 0$

Answer 6. The reduced quadratic is $t^2 + t - 3 = 0$ with $t = (x-1)^2$. By the quadratic formula we have

$$t = \frac{-1 \pm \sqrt{1+12}}{2} = \frac{-1 \pm \sqrt{13}}{2}$$

The solution set of the reduced quadratic is therefore

$$\left\{ \frac{-1 + \sqrt{13}}{2}, \frac{-1 - \sqrt{13}}{2} \right\}$$

and to obtain the solution set to the original equation we solve $t = (x-1)^2 = \frac{-1 \pm \sqrt{13}}{2}$

$$(x-1)^2 = \frac{-1 \pm \sqrt{13}}{2} \quad \Longleftrightarrow \quad x - 1 = \pm\sqrt{\frac{-1 \pm \sqrt{13}}{2}}$$

$$\Longleftrightarrow \quad x = 1 \pm \sqrt{\frac{-1 \pm \sqrt{13}}{2}}$$

Observe that since $3 < \sqrt{13} < 4$ the above calculation produces two complex solutions. The solution set of the original equation is therefore

$$\left\{ 1 + \sqrt{\frac{-1 + \sqrt{13}}{2}}, 1 - \sqrt{\frac{-1 + \sqrt{13}}{2}} \right\}$$

(c) $8x^{\frac{4}{3}} + 2x^{\frac{2}{3}} + 2 = 0$

Answer 7. The reduced quadratic equation is $8t^2 + 2t + 2 = 0$ with $t = x^{\frac{2}{3}}$. Computing the discriminant shows that since it is $4 - 4(8)(2) < 0$ that the reduced quadratic has no real solutions and therefore the original equation does not have any real solutions. The solution set is therefore the empty set, $\emptyset$.

(d) $6x^{-6} + 3x^{-3} - 2 = 0$

Answer 8. The reduced quadratic equation is $6t^2 + 3t - 2 = 0$ with $t = x^{-3}$.
By the quadratic formula we obtain

$$t = \frac{-3 \pm \sqrt{9 + 48}}{12} = \frac{-3 \pm \sqrt{57}}{12}$$

The solution set of the reduced quadratic equation is therefore

$$\left\{ \frac{-3 + \sqrt{57}}{12}, \frac{-3 - \sqrt{57}}{12} \right\}$$

We need to solve $t = x^{-3}$ for each element of this solution set in order to obtain the solutions to the original equation.

$$x^{-3} = \frac{-3 \pm \sqrt{57}}{12} \iff x^3 = \frac{12}{-3 \pm \sqrt{57}}$$

$$\iff x = \sqrt[3]{\frac{12}{-3 \pm \sqrt{57}}}$$

The solution set of the original equation is therefore

$$\left\{ \sqrt[3]{\frac{12}{-3 + \sqrt{57}}}, \sqrt[3]{\frac{12}{-3 - \sqrt{57}}} \right\}$$

(e) $2^{2x} - 2^{x+1} - 8 = 0$

Answer 9. The reduced quadratic equation is $t^2 - 2t - 8 = 0$ with $t = 2^x$, which factors as $(t - 4)(t + 2) = 0$. This tells us that the original equation factors as $(2^x - 4)(2^x + 2) = 0$ and either $2^x = 4 \Rightarrow x = 2$ or $2^x = -2$, which is impossible. The solution set of the original equation is therefore $\{2\}$.

3. Solve and write the solutions in set notation.

   (a) $\sqrt{2x + 1} = 2$

   Answer 10.

$$\begin{aligned}
\sqrt{2x + 1} &= 2 \\
(\sqrt{2x + 1})^2 &= 2^2 \\
2x + 1 &= 4 \\
2x &= 3 \\
x &= \frac{3}{2}
\end{aligned}$$

   We think that the solution set is $\{\frac{3}{2}\}$, but it is necessary to verify it.

$$\sqrt{2 \cdot \frac{3}{2} + 1} = 2$$
$$\sqrt{3 + 1} = 2$$
$$\sqrt{4} = 2$$

   The solution is valid and therefore the solution set is $\{\frac{3}{2}\}$.

(b) $\sqrt{x-3} = x - 3$

Answer 11.

$$
\begin{aligned}
\sqrt{x-3} &= x - 3 \\
(\sqrt{x-3})^2 &= (x-3)^2 \\
x - 3 &= (x-3)^2 \\
0 &= (x-3)^2 - (x-3) \\
0 &= (x-3)(x-3-1) \\
0 &= (x-3)(x-4)
\end{aligned}
$$

We think that the solution set is $\{3, 4\}$, but is necessary to verify it.

$$
\begin{aligned}
\sqrt{4-3} &= 4 - 3 \\
\sqrt{1} &= 1
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{3-3} &= 3 - 3 \\
\sqrt{0} &= 0
\end{aligned}
$$

Both solutions are valid, therefore the solution set is $\{3, 4\}$.

(c) $\sqrt{x+1} = x - 1$

Answer 12.

$$
\begin{aligned}
\sqrt{x+1} &= x - 1 \\
(\sqrt{x+1})^2 &= (x-1)^2 \\
x + 1 &= x^2 - 2x + 1 \\
0 &= x^2 - 3x \\
0 &= x(x-3)
\end{aligned}
$$

We think that the solution set is $\{0, 3\}$, but it is necessary to verify it.

$$
\begin{aligned}
\sqrt{0+1} &= 0 - 1 \\
\sqrt{1} &\neq -1
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{3+1} &= 3 - 1 \\
\sqrt{4} &= 2
\end{aligned}
$$

$x = 0$ is not a valid solution, therefore the solution set is $\{3\}$.

1. In the "Functions: Examples" worksheet from Week 6 do the following:

   (a) Determine whether each function is one-to-one.

   (b) Graph the inverse of each one-to-one function in part (a). [Hint: Reflect about $y = x$.]

   (c) For each one-to-one function compute the inverse function $f^{-1}(x)$.

   (d) Verify algebraically that each inverse function from part (c) is indeed the inverse. That is, check that

   $$f(x) \circ f^{-1}(x) = x = f^{-1}(x) \circ f(x)$$

   (e) Verify that the domain of $f(x)$ is the range of $f^{-1}(x)$ and that the range of $f(x)$ is the domain of $f^{-1}(x)$.

2. For each $f(x)$ compute $f^{-1}(x)$ and find the range of $f(x)$.

   (a) $f(x) = \frac{1}{x-1} + 3$

   (b) $f(x) = \sqrt[3]{x} + 1$

   (c) ** $f(x) = 27x^3 + 27x^2 + 9x + 1$

3. The following functions are not one-to-one. For each function state the largest subset of the domain on which the given function is one-to-one, then compute the inverse function where it exists.

   (a) $f(x) = x^2$

   (b) $f(x) = |x|$

   (c) $f(x) = \frac{1}{x^2}$

Sample Midterm	Sample Final
	10 A B C D
	11 A B C D
	17 A B C D
	18 A B C D
	28 A B C D

1. In the "Functions: Examples" worksheet from Week 5 do the following:

    (a) Determine whether each function is one-to-one.

    Answer 1. The one-to-one functions, i.e. the ones which pass the horizontal line test are

        i. $f(x) = ax + b; a, b \in \mathrm{R}; a \neq 0$

        ii. $f(x) = x^3$

        iii. $f(x) = \frac{1}{x}$

        iv. $f(x) = \sqrt{x}$

        v. $f(x) = \sqrt[3]{x}$

    (b) Graph the inverse of each one-to-one function in part (a). [Hint: Reflect about $y = x$.]

    Answer 2. See graphs beginning on page 103.

    (c) For each one-to-one function compute the inverse function $f^{-1}(x)$.

    Answer 3. Set $f(x) = y$, replace y with x and solve for y:

        i. $f(x) = ax + b; a, b \in \mathrm{R}; a \neq 0$. Solve for y: $x = ay + b; a, b \in \mathrm{R}; a \neq 0$

    $$f^{-1}(x) = \frac{x - b}{a}$$

        ii. $f(x) = x^3$. Solve for y: $x = y^3$

    $$f^{-1} = \sqrt[3]{x}$$

        iii. $f(x) = \frac{1}{x}$. Solve for y: $x = \frac{1}{y}$

    $$f^{-1}(x) = \frac{1}{x}$$

        iv. $f(x) = \sqrt{x}$. Solve for y: $x = \sqrt{y}$

    $$f^{-1}(x) = x^2$$

        v. $f(x) = \sqrt[3]{x}$. Solve for y: $x = \sqrt[3]{y}$

    $$f^{-1}(x) = x^3$$

(d) Verify algebraically that each inverse function from part (c) is indeed the inverse. That is, check that

$$f(x) \circ f^{-1}(x) = x = f^{-1}(x) \circ f(x)$$

Answer 4.

    i.

$$f(x) \circ f^{-1}(x) = a\left(\frac{x-b}{a}\right) + b = x - b + b = x$$

$$f^{-1}(x) \circ f(x) = \frac{(ax+b)-b}{a} = \frac{ax}{a} = x$$

  ii. $f(x) = x^3$

$$f(x) \circ f^{-1}(x) = \left(\sqrt[3]{x}\right)^3 = x$$

$$f^{-1}(x) \circ f(x) = \sqrt[3]{x^3} = x$$

 iii. $f(x) = \frac{1}{x}$

$$f(x) \circ f^{-1}(x) = \frac{1}{\frac{1}{x}} = x$$

$$f^{-1}(x) \circ f(x) = \frac{1}{\frac{1}{x}} = x$$

 iv. $f(x) = \sqrt{x}$

$$f(x) \circ f^{-1}(x) = \sqrt{x^2} = x$$

$$f^{-1}(x) \circ f(x) = \left(\sqrt{x}\right)^2$$

  v. $f(x) = \sqrt[3]{x}$

$$f(x) \circ f^{-1}(x) = \sqrt[3]{x^3} = x$$

$$f^{-1}(x) \circ f(x) = \left(\sqrt[3]{x}\right)^3$$

(e) Verify that the domain of $f(x)$ is the range of $f^{-1}(x)$ and that the range of $f(x)$ is the domain of $f^{-1}(x)$.

Answer 5. See the table given for each one-to-one function.

2. For each $f(x)$ compute $f^{-1}(x)$ and find the range of $f(x)$.

(a) $f(x) = \frac{1}{x-1} + 3$

Answer 6. The inverse function is obtained by solving for y in $x = \frac{1}{y-1} + 3$.

$$x = \frac{1}{y-1} + 3 \quad \Longleftrightarrow \quad x - 3 = \frac{1}{y-1}$$

$$\Longleftrightarrow \quad y - 1 = \frac{1}{x-3}$$

$$\Longleftrightarrow \quad y = \frac{1}{x-3} + 1$$

Thus, $f^{-1}(x) = \frac{1}{x-3} + 1$ and the domain of $f^{-1}(x)$ is the range of $f(x)$ is the union $(-\infty, 3) \cup (3, \infty)$.

(b) $f(x) = \sqrt[3]{x} + 1$

Answer 7. The inverse function is obtained by solving for y in $x = \sqrt[3]{y} + 1$.

$$x = \sqrt[3]{y} + 1 \iff x - 1 = \sqrt[3]{y}$$
$$\iff y = (x - 1)^3$$

Thus, $f^{-1}(x) = (x - 1)^3$ and the domain of $f^{-1}(x)$ is the range of $f(x)$ is the entire real line, i.e. $(-\infty, \infty)$.

(c) ** $f(x) = 27x^3 + 27x^2 + 9x + 1$

Answer 8. Note that $f(x) = 27x^3 + 27x^2 + 9x + 1 = (3x + 1)^3$. Then the inverse function is obtained by solving for y in $x = (3y + 1)^3$.

$$x = (3y + 1)^3 \iff \sqrt[3]{x} = 3y + 1$$
$$\iff \sqrt[3]{x} - 1 = 3y$$
$$\iff y = \frac{1}{3}\sqrt[3]{x} - \frac{1}{3}$$

Thus, $f^{-1}(x) = \frac{1}{3}\sqrt[3]{x} - \frac{1}{3}$ and the domain of $f^{-1}(x)$ is the range of $f(x)$ is the entire real line, i.e. $(-\infty, \infty)$.

3. The following functions are not one-to-one. For each function state the largest subset of the domain on which the given function is one-to-one, then compute the inverse function where it exists.

Remark 9. Sometimes there is not one way to choose a valid domain, but some choices are more natural than others. In either case, be sure to carefully state the domain and range.

(a) $f(x) = x^2$

Answer 10. The two possible choices for the domain are $(-\infty, 0]$ and the usual choice, $[0, \infty)$. The inverse function of $f(x) = x^2$ restricted to $[0, \infty)$ is $f^{-1}(x) = \sqrt{x}$, as we have already seen. The more interesting case is the inverse function of $f(x) : (-\infty, 0] \to [0, \infty)$. We know that the domain of $f(x)$ is the range of $f^{-1}(x)$ and in the course of the usual computation we have

$$x = y^2 \iff y = \pm\sqrt{x}$$

and we must choose the negative square root. Therefore, the inverse function of $f(x) = x^2$ restricted to $(-\infty, 0]$ is $f^{-1}(x) = -\sqrt{x}$.

(b) $f(x) = |x|$

Answer 11. The two possible choices for the domain are $(-\infty, 0]$ and the usual choice, $[0, \infty)$. By definition of the absolute value function, on each possible domain we are dealing with a linear function. Thus, $f(x) = |x|$ restricted to $[0, \infty)$ is just $f(x) = x$ and is its own inverse, as we have already seen. The inverse function of $f(x) = |x|$ restricted to $(-\infty, 0]$ is the inverse function of $f(x) = -x$ which maps $(-\infty, 0]$ to $[0, \infty)$. It must be $f^{-1}(x) = -x$, i.e. its own inverse (put $a = -1, b = 0$ in the formula for the inverse of a linear function).

(c) $f(x) = \frac{1}{x^2}$

Answer 12. The two possible choices for the domain are $(-\infty, 0)$ and the usual choice, $(0, \infty)$. If we consider the one-to-one function $f(x) = \frac{1}{x^2} : (0, \infty) \to (0, \infty)$, then its inverse function is given by $f^{-1}(x) = \frac{1}{\sqrt{x}} = \frac{\sqrt{x}}{x}$. When looking for the inverse we are solving for y in the equation $x = \frac{1}{y^2}$ and so

$$x = \frac{1}{y^2} \iff y^2 = \frac{1}{x}$$
$$\iff y = \pm\frac{1}{\sqrt{x}}$$
$$\iff y = \pm\frac{\sqrt{x}}{x}$$

and we are forced to choose between the positive and negative square roots. If the domain of $f(x)$ is taken to be $(-\infty, 0)$, then the inverse function is $f^{-1}(x) = -\frac{\sqrt{x}}{x}$

Note 13. It would be excellent practice to graph these inverse functions and to verify algebraically that they are indeed inverses of one another! [Hint: $\sqrt{x^2} = |x|$]

1. The constant function: $f(x) = c, c \in \mathrm{R}$.

Domain	$(-\infty, \infty)$
Range	$\{c\}$
Intervals of Increase	none
Intervals of Decrease	none
Turning Points	none
Local Maxima	$c, x = c$
Local Minima	$c, x = c$
Global Maxima	$c, x = c$
Global Minima	$c, x = c$
Symmetry	y-axis
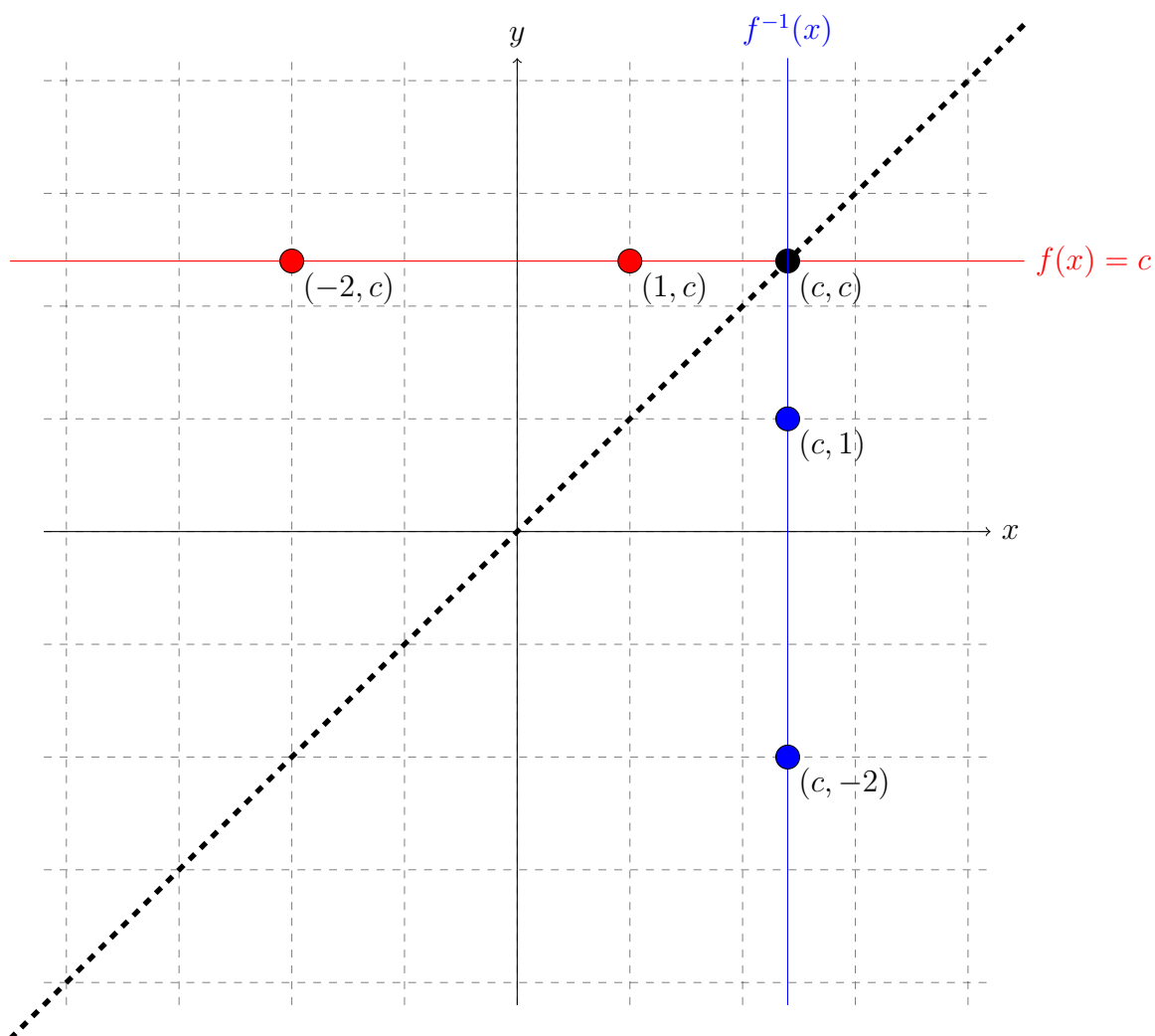
Recall that the reflection about the line $y = x$ of a horizontal line is a vertical line. Observe that the function $f(x) = c$ is not one-to-one and hence $f^{-1}(x)$ is not a function.

2. The linear function: $f(x) = ax + b; a, b \in \mathrm{R}$.

Domain	$(-\infty, \infty)$
Range	$\begin{cases} \{b\}, & a = 0; \\ (-\infty, \infty), & a \neq 0. \end{cases}$
Intervals of Increase	$\begin{cases} \text{none}, & a \leq 0; \\ (-\infty, \infty), & a > 0. \end{cases}$
Intervals of Decrease	$\begin{cases} \text{none}, & a \geq 0; \\ (-\infty, \infty), & a < 0. \end{cases}$
Turning Points	none
Local Maxima	$\begin{cases} \text{none}, & a \neq 0; \\ b, & a = 0. \end{cases}$
Local Minima	$\begin{cases} \text{none}, & a \neq 0; \\ b, & a = 0. \end{cases}$
Global Maxima	$\begin{cases} \text{none}, & a \neq 0; \\ b, & a = 0. \end{cases}$
Global Minima	$\begin{cases} \text{none}, & a \neq 0; \\ b, & a = 0. \end{cases}$
Symmetry	$\begin{cases} \text{none}, & a \neq 0; \\ y\text{-axis}, & a = 0. \end{cases}$

3. The square function: $f(x) = x^2$.

Domain	$(-\infty, \infty)$
Range	$[0, \infty)$
Intervals of Increase	$[0, \infty)$
Intervals of Decrease	$(-\infty, 0]$
Turning Points	$x = 0$
Local Maxima	none
Local Minima	$x = 0$
Global Maxima	none
Global Minima	$x = 0$
Symmetry	y-axis

Observe that $f(x) = x^2$ is not one-to-one on its domain, hence $f^{-1}(x)$ is not a function.

4. The cube function: $f(x) = x^3$.

Domain	$(-\infty, \infty)$
Range	$(-\infty, \infty)$
Intervals of Increase	$(-\infty, \infty)$
Intervals of Decrease	none
Turning Points	none
Local Maxima	none
Local Minima	none
Global Maxima	none
Global Minima	none
Symmetry	origin

5. The inverse function: $f(x) = \frac{1}{x}$.

Domain	$(-\infty, 0) \cup (0, \infty)$
Range	$(-\infty, 0) \cup (0, \infty)$
Intervals of Increase	none
Intervals of Decrease	$(-\infty, 0) \cup (0, \infty)$
Turning Points	none
Local Maxima	none
Local Minima	none
Global Maxima	none
Global Minima	none
Symmetry	origin

The function $f(x) = \frac{1}{x}$ is its own inverse and $f^{-1}(x) = \frac{1}{x}$.

6. The inverse square function: $f(x) = \frac{1}{x^2}$.

Domain	$(-\infty, 0) \cup (0, \infty)$
Range	$(0, \infty)$
Intervals of Increase	$(-\infty, 0)$
Intervals of Decrease	$(0, \infty)$
Turning Points	none
Local Maxima	none
Local Minima	none
Global Maxima	none
Global Minima	none
Symmetry	y-axis

Observe that $f(x) = \frac{1}{x^2}$ is not one-to-one on its domain, hence $f^{-1}(x)$ is not a function.

7. The square root function: $f(x) = \sqrt{x}$.

Domain	$[0, \infty)$
Range	$[0, \infty)$
Intervals of Increase	$[0, \infty)$
Intervals of Decrease	none
Turning Points	none
Local Maxima	none
Local Minima	$x = 0$
Global Maxima	none
Global Minima	$x = 0$
Symmetry	none
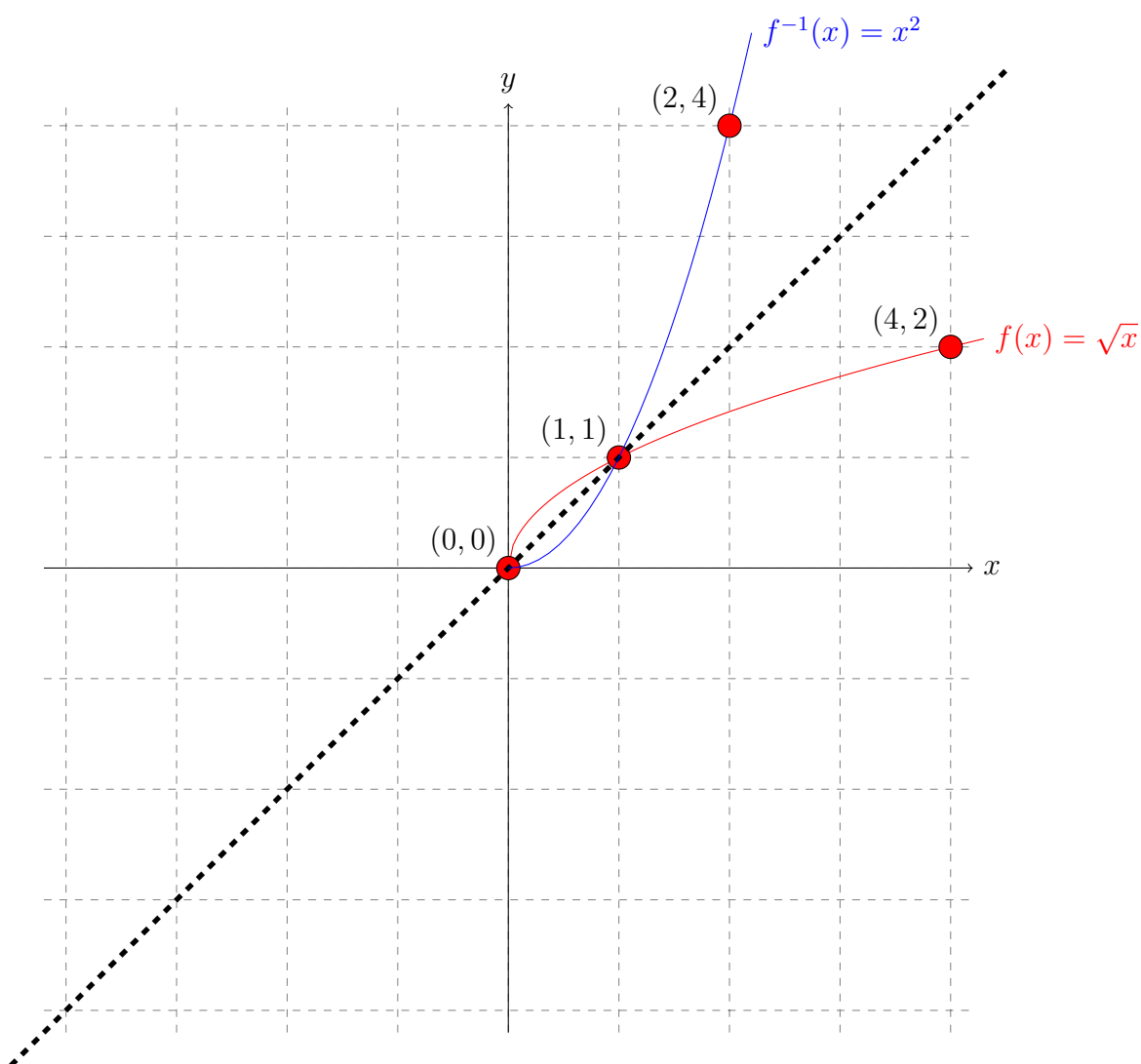
Note how the domain affects the graph of the inverse function and compare this to the above problem with $f(x) = x^2$.

8. The cube root function: $f(x) = \sqrt[3]{x}$.

Domain	$(-\infty, \infty)$
Range	$(-\infty, \infty)$
Intervals of Increase	$(-\infty, \infty)$
Intervals of Decrease	none
Turning Points	none
Local Maxima	none
Local Minima	none
Global Maxima	none
Global Minima	none
Symmetry	origin
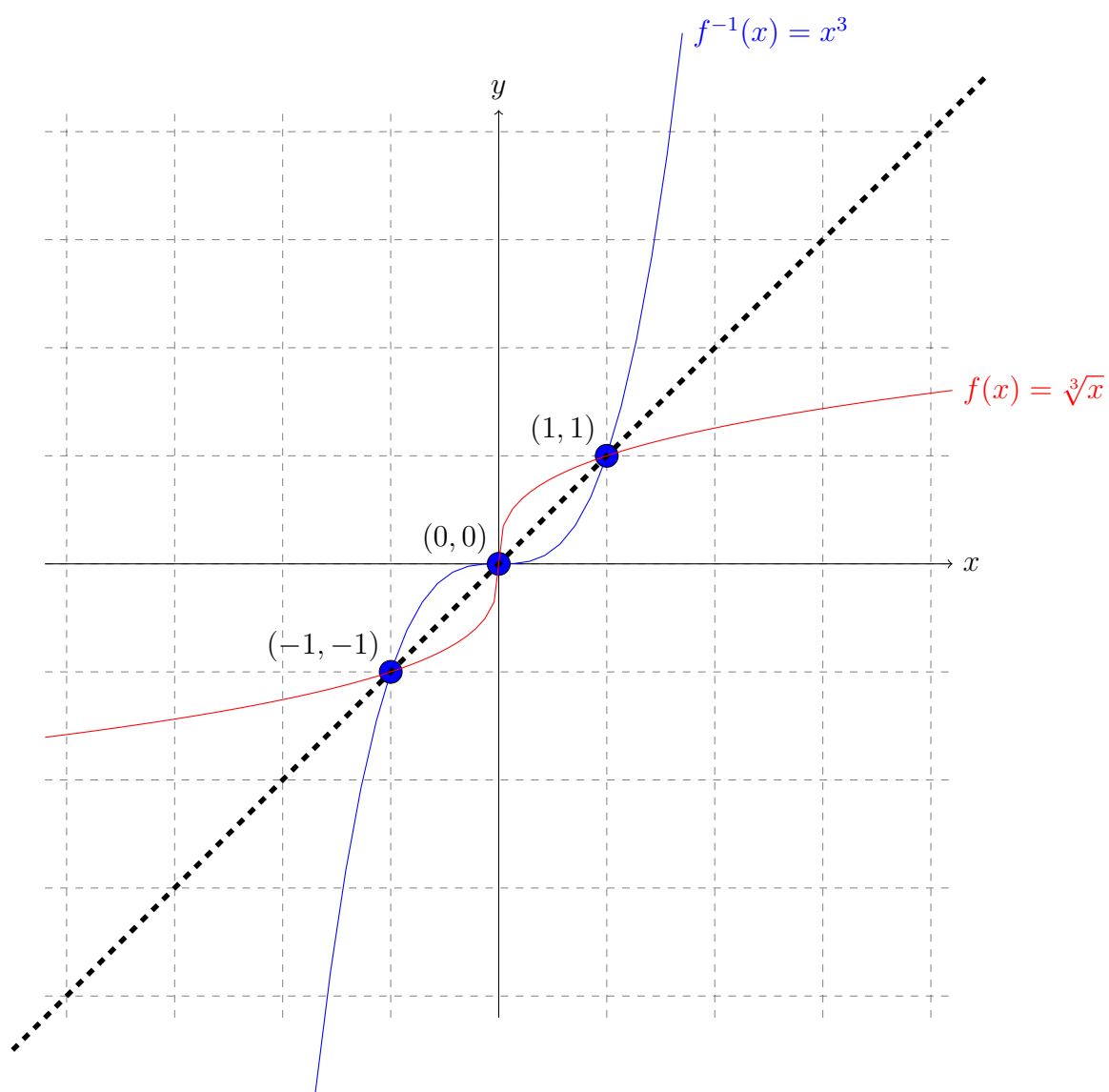
9. The absolute value function: $f(x) = |x|$.

Domain	$(-\infty, \infty)$
Range	$[0, \infty)$
Intervals of Increase	$[0, \infty)$
Intervals of Decrease	$(-\infty, 0]$
Turning Points	$x = 0$
Local Maxima	none
Local Minima	$x = 0$
Global Maxima	none
Global Minima	$x = 0$
Symmetry	y-axis

Note that $f(x) = |x|$ is not one-to-one and so $f^{-1}(x)$ is not a function.

Recall that we have begun the semester with the discussion of linear equations and have recently specialized to linear functions. Similarly, we will demand now that our quadratic equations be functions. There was but one class of linear equations that are not functions: the vertical lines. In order to guarantee that a quadratic equation passes the vertical line test and is therefore a function it must be of the form

$$f(x) = ax^2 + bx + c; a, b, c \in \mathrm{R}$$

This very special class of functions has an equally nice graphical representation. **The graph of a quadratic function is a parabola**. We have already developed the tools necessary to determine the basic characteristic of a parabola given the quadratic equation that represents it. Recall that by completing the square any quadratic equation may be rewritten in the form

$$a(x - h)^2 + k$$

and from here we make the following definitions

Definition 1.

1. The point with coordinates (h, k) is the vertex of the parabola.

2. The vertical line $x = h$ is the axis of symmetry of the parabola.

3. The x-intercepts are the roots of the parabola.

If $x = 0$ is the axis of symmetry, then the parabola is symmetric about the y-axis, but this need not always be the case. Both the axis of symmetry and the vertex are easily obtained from the quadratic equation by completing the square. In order to determine the roots of the parabola we calculate the x-intercepts, i.e. let $f(x) = 0$ and solve for x. This amounts to solving the quadratic equation and to do so we may use our favorite method.

The coefficients of x^2 play an important role in our analysis of $f(x)$. If $a > 0$, then the parabola opens up and the vertex is the point at which the global minimum occurs. If $a < 0$, then the parabola opens down and the vertex is the point at which the global maximum occurs. This observation suggests that in order to optimize a quantity that is expressible by a quadratic equation, it suffices to find the vertex of the parabola whose equation is the quadratic function $f(x)$. Then the $x - h$ is the value which optimizes $f(x)$ and $f(h) = k$ is the optimal value of $f(x)$.

Example 2. Let $\triangle$ be the triangle with one vertex on the positive x-axis, one vertex at the origin $(0,0)$ and one vertex on the line $y = -2x + 12$. Find the dimensions of the largest such triangle.

Answer 3. The area of $\triangle$ is a function of its base and height. Since x is the length of the base and y is the height the area function is $A_\triangle(x, y) = \frac{xy}{2}$. We do not know how to optimize functions of two variables, but fortunately we know that $y = -2x + 12$ and substituting for y, we obtain

$$A_\triangle(x) = \frac{x}{2}(-2x + 12) = -x^2 + 6x \ .$$

The above quadratic function is optimized by finding the coordinates of the vertex. Observe that $A_\triangle(0) = 0$ and that $a = -2$. This agrees with out intuition that a triangle of side length 0 has area 0 and that since we are looking to maximize the area, the parabola will open down. Completing the square we obtain

$$A_\triangle(x) = -(x-3)^2 + 9 \ .$$

The vertex of the parabola is the point $(3, 9)$ and so the maximum area is 9 when $x = 3$. The largest such triangle has therefore a base of length 3 units and a height of 6 units.

Note that there is another solutions to this problem. The triangle with vertices at the points (0,0), (0,12), (6,0) has an area equal to $\frac{1}{2}6 \cdot 12 = 36$. Be careful when deciding exactly what is to be done!

There is no best way to solve an optimization problem with a quadratic function, but the following procedure is always a good start:

1. Understand the problem. Draw a picture.

2. Identify the variables.

3. Obtain a two variable function which needs to be optimized.

4. Using the relationships between variables stated in the problem reduce the two variable function to a single variable quadratic function.

5. Optimize the quadratic function. Complete the square.

6. Answer the question.

1. Graph the following quadratic functions. Clearly mark and correctly scale the coordinate axes, the vertex, the axis of symmetry, the roots, and the y-intercept.

   (a) $f(x) = x^2 + 2x + 1$

   (b) $g(x) = -3x^2 - 6x + 2$

   (c) $h(x) = 5x^2 + 15x + 20$

2. What is the largest possible product of two numbers that add to 12?

3. What number exceeds twice its square by the greatest amount?

4. What is the largest possible area of a rectangle with a perimeter of 10 meters?

5. *Prove that the rectangle with the largest area is in fact a square.

6. **Recall that the distance on the plane between points $a = (x_1, y_1)$ and $b = (x_2, y_2)$ is given by the distance formula

$$D(a, b) = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

   What point on the graph of $f(x) = \sqrt{2x^2 + 1} + 1$ lies closest to the point $(1, 1)$? (Hint: Since $g(x) = \sqrt{x}$ is strictly increasing, to minimize $D(a, b)$ it suffices to minimize $(D(a, b))^2$.)

Sample Midterm	Sample Final
	30 A B C D
	31 A B C D
	32 A B C D
	34 A B C D
	35 A B C D

1. Graph the following quadratic functions. Clearly mark and correctly scale the coordinate axes, the vertex, the axis of symmetry, the roots, and the y-intercept.

(a) $f(x) = x^2 + 2x + 1$

Answer 1. Completing the square we obtain

$$\begin{aligned} x^2 + 2x + 1 &= (x^2 + 2x) + 1 \\ &= (x^2 + 2x + 1) + 1 - 1 \\ &= (x+1)^2 \end{aligned}$$

and we read off the desired information:

- $a = 1 > 0$ so the parabola opens up.
- $(-1, 0)$ is the vertex.
- $x = -1$ is the axis of symmetry.

Furthermore, setting $x = 0$ we obtain the y-intercept $(0, 1)$ and solving the quadratic equation $f(x) = 0$ yields one root $x = -1$. [Observe that the roots are the x-intercepts and as the vertex lies on the x-axis and the parabola opens up, $x = -1$ is the only root (of degree 2) of $f(x)$.]
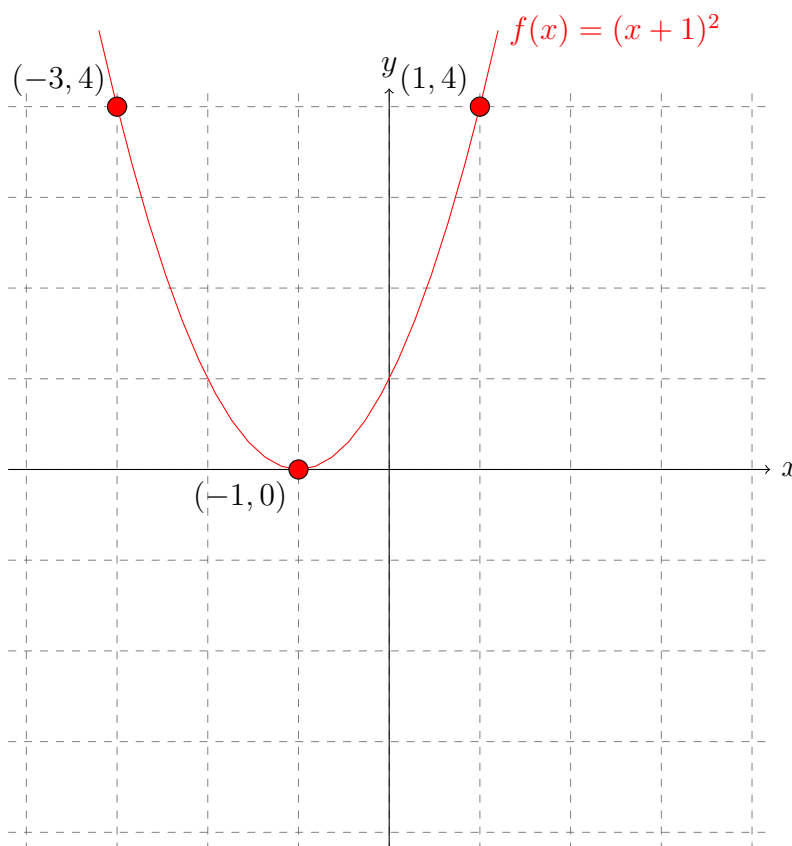
(b) $g(x) = -3x^2 - 6x + 2$

Answer 2. Completing the square we obtain

$$\begin{aligned} -3x^2 - 6x + 2 &= -3(x^2 + 2x) + 2 \\ &= -3(x^2 + 2x + 1) + 2 + 3 \\ &= -3(x+1)^2 + 5 \end{aligned}$$

and we read off the desired information:

- $a = -3 < 0$ so the parabola opens down.
- $(-1, 5)$ is the vertex.
- $x = -1$ is the axis of symmetry.

Furthermore, setting $x = 0$ we obtain the y-intercept $(0, 2)$. Observe that the vertex lies above the x-axis and the parabola opens down. There must be two distinct x-intercept, i.e. two real roots. Solving $f(x) = 0$ we obtain

$$f(x) = 0 = -3(x+1)^2 + 5 \Rightarrow (x+1)^2 = \frac{5}{3} \Rightarrow x + 1 = \pm\sqrt{\frac{5}{3}} \Rightarrow x = -1 \pm \frac{\sqrt{15}}{3}$$

The solutions set is therefore

$$\left\{ -1 + \frac{\sqrt{15}}{3}, -1 - \frac{\sqrt{15}}{3} \right\}$$

and since $3 < \sqrt{15} < 4$ it follows that $-1 - \frac{\sqrt{15}}{3} < 0$ and that $-1 + \frac{\sqrt{15}}{3} > 0$, which we can see on the graph below.
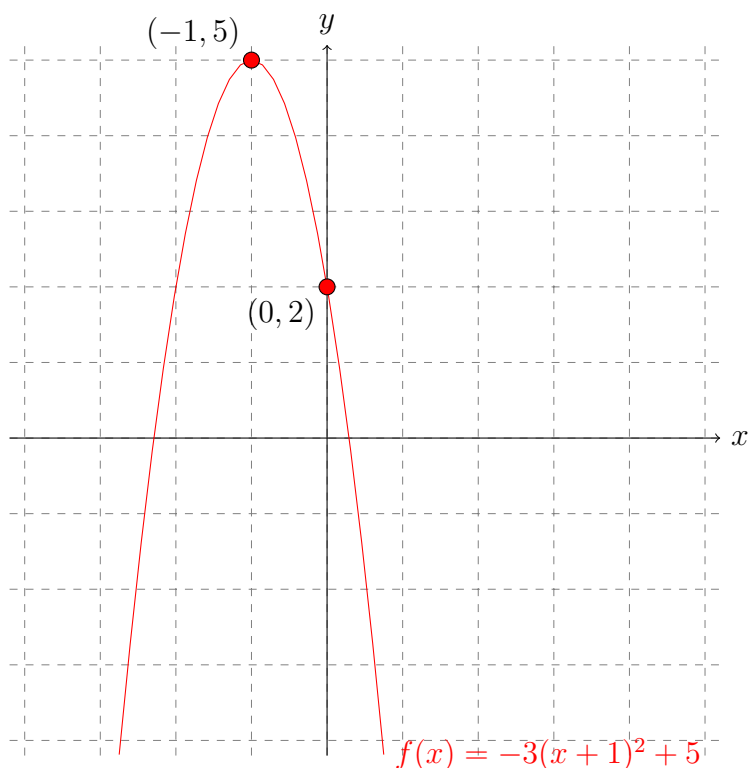
(c) $h(x) = 5x^2 + 15x + 20$

Answer 3. Completing the square we obtain

$$
\begin{aligned}
5x^2 + 15x + 20 &= 5(x^2 + 3x) + 20 \\
&= 5(x^2 + 3x + \frac{9}{4}) + 20 - \frac{45}{4} \\
&= 5(x + \frac{3}{2})^2 + \frac{35}{4}
\end{aligned}
$$

and we read off the desired information:

- $a = 5 > 0$ so the parabola opens up.
- $(-\frac{3}{2}, \frac{35}{4})$ is the vertex.
- $x = -\frac{3}{2}$ is the axis of symmetry.

Furthermore, since the parabola opens up and the vertex lies above the x-axis, the graph of $f(x)$ has no x-intercepts and consequently the quadratic equation $f(x) = 0$ has no real solutions. The y-intercept is $(0, 20)$ and the graph is given below.
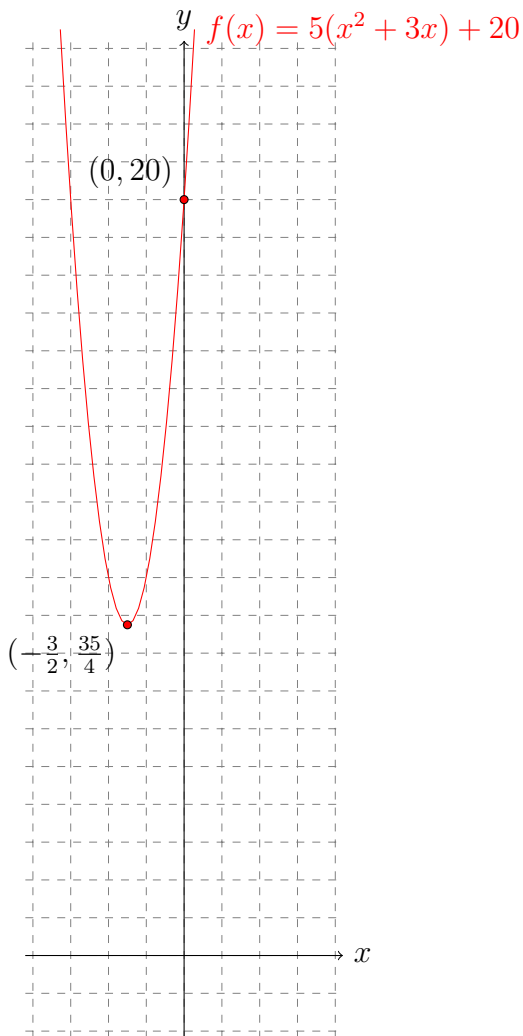
2. What is the largest possible product of two numbers that add to 12?

Answer 4. Let x and y be the two numbers. We are looking to maximize $P(x,y) = xy$ and we know that $x + y = 12$. This allows us to obtain a single variable quadratic function $p(x) = x(12 - x) = -x^2 + 12x$. Completing the square we obtain $p(x) = -(x - 6)^2 + 36$ and whence $(6, 36)$ is the vertex of the parabola $p(x)$. The maximizing value is $x = 6$ (and so $y = 6$) and $p(6) = 36$. Therefore, the largest possible product of two numbers that add to 12 is 36.

3. What number exceeds twice its square by the greatest amount?

Answer 5. Observe that squaring numbers greater than 1 increases them, so the solution should be a number strictly less than 1. Let x be the unknown number. Twice its square is then $2x^2$ and so we are looking to maximize the difference function $d(x) = x - 2x^2$. Completing the square we obtain $d(x) = -2(x - \frac{1}{4})^2 + \frac{1}{8}$. The largest difference is therefore $\frac{1}{8}$ and occurs for the number $x = \frac{1}{4}$. Therefore, $\frac{1}{4}$ is the number which exceeds twice its square by the greatest amount.

4. What is the largest possible area of a rectangle with a perimeter of 10 meters?

Answer 6. The area is a function of length x and width y. $A(x,y) = xy$, but we know that $P = 2x + 2y = 10$ and whence we have the area as a quadratic function $a(x) = x(5 - x) = -x^2 + 5x$. Completing the square we obtain $a(x) = -(x - \frac{5}{2})^2 + \frac{25}{4}$. Therefore, $\frac{5}{2}$ is the length of side x (and hence $y = \frac{5}{2}$) which produces the maximum area, $\frac{25}{4}$. Therefore, a rectangle with perimeter 10 has an area of at most $\frac{25}{4}$ square meters.

5. *Prove that the rectangle with the largest area is in fact a square.

Proof 7. *Observe that this is exactly the case in the above example. Suppose that a rectangle has dimension x and y and a perimeter P. Then $P = 2x + 2y$ and the area is given by the function $A(x,y) = xy$, which we proceed to reduce to the quadratic function $a(x) = x(\frac{P}{2} - x) = -x^2 + \frac{Px}{2}$. Completing the square we have*

$$
\begin{aligned}
-x^2 + \frac{Px}{2} &= -\left(x^2 - \frac{Px}{2}\right) \\
&= -\left(x^2 - \frac{Px}{2} + \frac{P^2}{16}\right) + \frac{P^2}{16} \\
&= -\left(x - \frac{P}{4}\right)^2 + \frac{P^2}{16}
\end{aligned}
$$

Therefore, $\frac{P}{4}$ is the length of x which maximizes the area. But then since $x + y = \frac{P}{2}$

$$
\frac{P}{4} + y = \frac{P}{2} \Rightarrow y = \frac{P}{4} \Rightarrow x = y
$$

Our rectangle with fixed perimeter P is therefore a square and since we made no additional assumption when we chose the perimeter to be P, it follows that the same result holds for an arbitrary perimeter P and whence for all rectangles.

6. **Recall that the distance on the plane between points $a = (x_1, y_1)$ and $b = (x_2, y_2)$ is given by the distance formula

$$D(a, b) = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

What point on the graph of $f(x) = \sqrt{2x^2 + 1} + 1$ lies closest to the point $(1, 1)$? (Hint: Since $g(x) = \sqrt{x}$ is strictly increasing, to minimize $D(a, b)$ it suffices to minimize $(D(a, b))^2$.)

Answer 8. The distance between a fixed point (x_1, y_1) on the graph of $g(x)$ and $(x_2, y_2) = (1, 1)$ is given by

$$\sqrt{(1 - y_1)^2 + (1 - x_1)^2}$$

but we also know that $y_1 = 2x_1^2 + 1$ and with this substitution we obtain a relation depending just on x_1

$$\sqrt{\left(1 - \left(\sqrt{2x_1^2 + 1} + 1\right)\right)^2 + (1 - x_1)^2} = \sqrt{\left(-\sqrt{2x_1^2 + 1}\right)^2 + (1 - 2x_1 + x_1^2)}$$

$$= \sqrt{2x_1^2 + 1 + 1 - 2x_1 + x_1^2}$$

$$= \sqrt{3x_1^2 - 2x_1 + 2}$$

Letting x_1 vary over the domain of $g(x)$ we obtain the function $d(x) = \sqrt{3x^2 - 2x + 2}$ which we have no idea how to minimize. But following the hint it suffices to minimize the quadratic function $h(x) = 3x^2 - 2x + 2$. Completing the square we obtain $h(x) = 3(x - \frac{1}{3})^2 + \frac{5}{3}$ and whence it is clear that the minimum value of $d(x)$ is $\frac{5}{3}$ and therefore the minimum distance to the graph of $g(x)$ is $\sqrt{\frac{5}{3}} = \frac{\sqrt{15}}{3}$. We, however, need to answer the question and are interested in the coordinates of this point. From the equation of $h(x)$ we see that $x = \frac{1}{3}$ is the x-coordinate and since this point is supposed to lie on the graph of $g(x)$, the y-coordinate is obtained by calculating $g(\frac{1}{3}) = \sqrt{\frac{11}{9}} + 1 = \frac{3 + \sqrt{11}}{3}$. Therefore, the point on the parabola $g(x)$ closest to $(1, 1)$ is the point $(\frac{1}{3}, \frac{3 + \sqrt{11}}{3})$.

Before we begin working with the graphs of polynomial and rational functions we need to develop the tools with which we will extract the information necessary to draw our graphs. We will be using the following definition:

Definition 1. (*Polynomial function*) A polynomial function is a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_0 x^0$$

and satisfies these conditions

- The numbers $a_n, a_{n-1}, \ldots, a_0$, called the coefficients of $p(x)$, are real numbers,

- Each exponent of x is a nonnegative integer.

We call the term with the highest power of x the *leading term* of $p(x)$ and the coefficient of the leading term is called the *leading coefficient* of $p(x)$. The degree of the leading term is called the *degree of the polynomial*.

Example 2. $p(x) = 5x^2 - x + 10x^{10} - 1$ is a polynomial function with leading term $10x^{10}$. The leading coefficient if 10 and the degree of $p(x)$ is 10. It is understood that the coefficients of the powers of x not present are zero, i.e. writing $p(x)$ in the *standard form* given in the definition

$$p(x) = 5x^2 - x + 10x^{10} - 1 = 10x^{10} + 0x^9 + 0x^8 + 0x^7 + 0x^6 + 0x^5 + 0x^4 + 0x^3 + 5x^2 - 1x^1 - 1x^0$$

Definition 3. (*Rational Function*) A function which is the quotient of two polynomial functions is called a rational function. That is, both the numerator and the denominator must be polynomial functions.

Example 4. $f(x) = \frac{g(x)}{h(x)} = \frac{3x^3 + x + 1}{x - 1}$ is a rational function because both $g(x) = 3x^3 + x + 1$ and $h(x) = x - 1$ are polynomials.

A natural question is to ask whether the above $f(x)$ has also a leading term. The answer requires to express $f(x)$ as a polynomial and hence we must be able to divide one polynomial by another.

The **DIVISION LAW** states that if $f(x) = \frac{g(x)}{h(x)}$ is a rational function and the degree of $g(x)$ is at least as large as the degree of $h(x)$, then there are unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = \frac{g(x)}{h(x)} = q(x) + \frac{r(x)}{h(x)}$$

Moreover, the degree of $r(x)$ is strictly less than the degree of $h(x)$. We call the polynomial $h(x)$ the *divisor*, the polynomial $q(x)$ the *quotient*, and the polynomial $r(x)$ the *remainder*.

Let us apply the division law to the above example. We dive $3x^2 + x + 1$ by $x - 1$ using polynomial long division:

$$
\begin{array}{r}
3x + 4 \\
x - 1 \enclose{longdiv}{3x^2 + x + 1} \\
\underline{3x^2 - 3x} \\
4x + 1 \\
\underline{4x - 4} \\
5
\end{array}
$$

The quotient $3x + 4$ and the divisor $x - 1$ are degree one polynomials and the remainder 5 is a degree zero polynomial and

$$
\frac{3x^2 + x + 1}{x - 1} = 3x + 4 + \frac{5}{x - 1}
$$

The leading term of $f(x)$ is the leading term of the quotient and it is just the quotient of the leading terms of $g(x)$ and $h(x)$. If the remainder is zero, then we say that $h(x)$ divides $g(x)$ evenly. In this case $h(x)$ is a *factor* of $f(x)$.

The **FACTOR THEOREM** states that a is a root of a polynomial $p(x)$ if and only if $(x - a)$ is a factor of $p(x)$. That is, if and only if $(x - a)$ divides $p(x)$ evenly. Moreover, $p(c)$ is the remainder of the division of $p(x)$ by the linear polynomial $(x - c)$.

Example 5. Factor $t(x) = x^3 - x^2 - 2x + 2$ into linear factors provided that 1 is one root.

Since 1 is a root, it follows by the factor theorem that $x - 1$ divides $t(x)$ evenly and $t(1) = 0$. By polynomial long division we obtain

$$
\frac{x^3 - x^2 - 2x + 2}{x - 1} = x^2 - 2 \Rightarrow x^3 - x^2 - 2x + 2 = (x^2 - 2)(x - 1)
$$

and since we have a difference of squares $t(x)$ factors further as

$$
(x - \sqrt{2})(x + \sqrt{2})(x - 1)
$$

Observe that the roots of $t(x)$ are $\{-\sqrt{2}, 1, \sqrt{2}\}$.

1. Write the following polynomials in standard form:

   (a) $p(x) = x^3 - x + 1$

   (b) $g(x) = x^6 + x^2 - 11$

   (c) $f(x) = 5$

2. Perform the division. List the divisor, quotient, and remainder.

   (a) $\frac{x^6 - 64}{x - 2}$

   (b) $\frac{4x^3 + x + 1}{2x^2 + x + 1}$

   (c) $\frac{4x^2 + 3x + 7}{x^2 - 2x + 1}$

3. Factor and find all roots:

   (a) $x^3 + 8x^2 - 3x - 24$, if -8 is a root.

   (b) $2x^3 + x^2 - 5x - 3$, if $-\frac{3}{2}$ is a root.

   (c) $x^3 - 7x^2 - 4x + 28$, if 7 is a root.

	Sample Midterm	Sample Final
6		A B C D
22		A B C D

1. Write the following polynomials in standard form:

   (a) $p(x) = x^3 - x + 1$

   Answer 1.
   $$p(x) = x^3 + 0x^2 - x + 1x^0$$

   (b) $g(x) = x^6 + x^2 - 11$

   Answer 2.
   $$g(x) = x^6 + 0x^5 + 0x^4 + 0x^3 + x^2 + 0x - 11x^0$$

   (c) $f(x) = 5$

   Answer 3.
   $$f(x) = 5x^0$$

2. Perform the division. List the divisor, quotient, and remainder.

   (a) $\frac{x^6 - 64}{x - 2}$

   Answer 4.
   $$\frac{x^6 - 64}{x - 2} = x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 32$$

   where $x - 2$ is the divisor, $x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 32$ is the quotient, and the remainder is 0.

   (b) $\frac{4x^3 + x + 1}{2x^2 + x + 1}$

   Answer 5.
   $$\frac{4x^3 + x + 1}{2x^2 + x + 1} = 2x - 1 + \frac{2}{2x^2 + x + 1}$$

   where $2x^2 + x + 1$ is the divisor, $2x - 1$ is the quotient, and 2 is the remainder.

(c) $\frac{4x^2+3x+7}{x^2-2x+1}$

Answer 6.

$$\frac{4x^2 + 3x + 7}{x^2 - 2x + 1} = 4 + \frac{11x + 3}{x^2 - 2x + 1}$$

where $x^2 - 2x + 1$ is the divisor, 4 is the quotient, and $11x + 3$ is the remainder.

3. Factor and find all roots:

(a) $x^3 + 8x^2 - 3x - 24$, if -8 is a root.

Answer 7. Since -8 is a root $(x + 8)$ divides $x^3 + 8x^2 - 3x - 24$ evenly. The quotient is $x^2 - 3$ which factors as $(x^2 - 3) = (x - \sqrt{3})(x + \sqrt{3})$. Therefore,

$$x^3 + 8x^2 - 3x - 24 = (x + 8)(x - \sqrt{3})(x + \sqrt{3})$$

and the roots are $\{-8, -\sqrt{3}, \sqrt{3}\}$.

(b) $2x^3 + x^2 - 5x - 3$, if $-\frac{3}{2}$ is a root.

Answer 8. Since $-\frac{3}{2}$ is a root $2x + 3$ divides $2x^3 + x^2 - 5x - 3$ evenly. The quotient is $x^2 - x - 1$ which by the quadratic formula has the roots

$$x = \frac{1 \pm \sqrt{5}}{2}$$

Therefore,

$$2x^3 + x^2 - 5x - 3 = (2x + 3)\left(x - \frac{1 - \sqrt{5}}{2}\right)\left(x - \frac{1 + \sqrt{5}}{2}\right)$$

and the roots are

$$\left\{-\frac{3}{2}, \frac{1 - \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2}\right\}$$

(c) $x^3 - 7x^2 - 4x + 28$, if 7 is a root.

Answer 9. Since 7 is a root $x - 7$ divides $x^3 - 7x^2 - 4x + 28$ evenly. The quotient is $x^2 - 4$ which factors as $(x - 2)(x + 2)$. Therefore,

$$x^3 - 7x^2 - 4x + 28 = (x - 7)(x - 2)(x + 2)$$

and the roots are $\{-2, 2, 7\}$.

Polynomial long division is a tedious process which can be shortened considerably in the special case when the divisor is a linear factor. By the factor theorem, if we divide a polynomial $p(x)$ by the linear polynomial $x - c$, then $p(c)$ is the remainder and $p(c) = 0$ if and only if $(x - c)$ is a factor of $p(x)$.

This observation suggest that in order to factor a polynomial $p(x)$ of large degree it suffices to look for numbers c such that $p(c) = 0$. This is equivalent to long division by $(x-c)$, which can be extremely drawn out. The method of synthetic division accomplishes division by a linear factor quickly and is done as follows:

Example 1. Divide $\frac{x^6 - 64}{x - 2}$.

Begin as with polynomial long division but write only the coefficients of $x^6 - 64$ making sure to list all of them.

$$
\begin{array}{r}
[2] \quad 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ \text{-}64 \\
0 \\
\hline
1
\end{array}
$$

Carry the coefficient of the leading term as shown and then multiply by $c = 2$, add to the next column and repeat. Thus

$$
\begin{array}{r}
[2] \quad 1 \ 0 \ 0 \ 0 \ 0 \ \ 0 \ \ \text{-}64 \\
0 \ 2 \ 4 \ 8 \ 16 \ 32 \ 64 \\
\hline
1 \ 2 \ 4 \ 8 \ 16 \ 32 \ \ [0]
\end{array}
$$

The numbers below the line are the coefficients of a polynomial of one degree lower, i.e. $x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 32$ and the last number is the remainder. Therefore,

$$\frac{x^6 - 64}{x - 2} = x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 32$$

Observe that finding roots of any polynomial $p(x)$ is now reduced to finding a number c (that would play the role of 2 above) such that the remainder of the synthetic division, namely $p(c)$, is zero.

Example 2. Factor and find all roots of $x^4 - 6x^3 - 11x^2 + 24x + 28$.

We begin with small test values c. Test first $1, -1, 2, -2, \ldots$. We have $p(1) = 36$, but $p(-1) = 0$ and

```
[-1]   1    -6    -11    24    28
       0    -1     7      4   -28
      ─────────────────────────────
       1    -7    -4     28   [0]
```

in other words we have performed the division

$$\frac{x^4 - 6x^3 - 11x^2 + 24x + 28}{x + 1} = x^3 - 7x^2 - 4x + 28$$

Repeating this process again with the newly obtained polynomial we see that -2 works and hence

```
[-2]   1    -7    -4    28
       0    -2    18   -28
      ──────────────────────
       1    -9    14   [0]
```

in other words we have performed the division

$$\frac{x^3 - 7x^2 - 4x + 28}{x + 2} = x^2 - 9x + 14 = (x - 7)(x - 2)$$

Therefore,

$$\frac{x^4 - 6x^3 - 11x^2 + 24x + 28}{(x + 1)(x + 2)} = (x - 7)(x - 2)$$

and

$$x^4 - 6x^3 - 11x^2 + 24x + 28 = (x + 1)(x + 2)(x - 7)(x - 2)$$

and the roots are $\{-2, -1, 2, 7\}$.

1. Perform the division. List the quotient and remainder.

   (a) $\frac{3x^2-11x+5}{x-4}$

   (b) $\frac{5x^5+3x^3+1}{x+2}$

   (c) $\frac{9x^3+14x-6}{3x-2}$

2. What is the remainder of the division of $p(x)$ by $x-3$ if:

   (a) $p(x) = 3x^4 + 3x - 1$

   (b) $p(x) = 7x^5 - 500x + 3$

   (c) $p(x) = 4x^4 + x$

3. Find all roots.

   (a) $x^3 - 2x^2 - 5x + 6$

   (b) $x^4 + 2x^3 - 9x^2 - 2x + 8$

1. Perform the division. List the quotient and remainder.

   (a) $\frac{3x^2 - 11x + 5}{x - 4}$

   Answer 1.

[4]	3	-11	5
	0	12	4
	3	1	[9]

   Therefore, $3x^2 - 11x + 5 = (x - 4)(3x + 1) + 9$ where $3x + 1$ is the quotient and 9 is the remainder.

   (b) $\frac{5x^5 + 3x^3 + 1}{x + 2}$

   Answer 2.

[-2]	5	0	3	0	0	1
	0	-10	20	-46	92	- 184
	5	-10	23	-46	92	[-183]

   Therefore, $5x^5 + 3x^3 + 1 = (5x^4 - 10x^3 + 23x^2 - 46x + 92)(x + 2) - 183$ where $5x^4 - 10x^3 + 23x^2 - 46x + 92$ is the quotient and -183 is the remainder.

   (c) $\frac{9x^3 + 14x - 6}{3x - 2}$

   Answer 3.

$[\frac{2}{3}]$	9	0	14	-6
	0	6	4	12
	9	6	18	[6]

   Therefore, $9x^3 + 14x - 6 = (9x^2 + 6x + 18)(x - \frac{2}{3}) + 6$ where $9x^2 + 6x + 18$ is the quotient and 6 is the remainder.

2. What is the remainder of the division of $p(x)$ by $x - 3$ if:

   (a) $p(x) = 3x^4 + 3x - 1$

   Answer 4.

[3]	3	0	0	3	-1
	0	9	27	81	252
	3	9	27	84	[251]

   Therefore, the remainder is $p(3) = 251$.

   (b) $p(x) = 7x^5 - 500x + 3$

   Answer 5.

[3]	7	0	0	0	-500	3
	0	21	63	189	567	201
	7	21	63	189	67	[204]

   Therefore, the remainder is $p(3) = 204$.

   (c) $p(x) = 4x^4 + x$

   Answer 6.

[3]	4	0	0	1	0
	0	12	36	108	327
	4	12	36	109	[327]

   Therefore, the remainder is $p(3) = 327$.

3. Find all roots.

   (a) $x^3 - 2x^2 - 5x + 6$

   Answer 7.
   $$x^3 - 2x^2 - 5x + 6 = (x - 1)(x + 2)(x - 3)$$

   and therefore the roots are $\{-2, 1, 3\}$.

   (b) $x^4 + 2x^3 - 9x^2 - 2x + 8$

   Answer 8.
   $$x^4 + 2x^3 - 9x^2 - 2x + 8 = (x - 1)(x + 1)(x - 2)(x + 4)$$

   and therefore the roots are $\{-4, -1, 1, 2\}$.

Our goal is to understand and be able to graph the polynomial and rational functions we have just encountered. As usual we start with the basics. We will build upon the graphs of functions in the "Functions: Examples" worksheet. The graph of the square functoin $f(x) = x^2$ is well-known to us and we also know how to graph the general quadratic function. Since both are examples of parabolas, what is the relation between them?

Example 1. Describe how to obtain through a series of translations the graph of $g(x) = (x + 2)^2 - 1$ from the graph of $f(x) = x^2$.

Answer 2. The vertex of $g(x)$ is the point $(-2, -1)$ while the vertex of $f(x)$ is $(0, 0)$. In both cases the coefficient of x^2 is 1, so other than the location of the vertex the graphs are the same. Therefore, $g(x)$ is obtained from $f(x)$ through translation by shifting the graph of $f(x)$ to the left 2 and down 1.

Extending this idea to other functions we find that there is no vertex we can track, but any point will suffice and we work with the points which are the easiest to find, i.e. the y-intercepts, the x-intercepts, the roots, and the intercepts of asymptotes.

Here is a summary of translation operations. If $f(x)$ is function and $k > 0$, then

Operation	Result on the graph of $f(x)$
$f(x) + k$	Shift up k units.
$f(x) - k$	Shift down k units.
$f(x + k)$	Shift left k units.
$f(x - k)$	Shift right k units.

Example 3. Describe how to obtain from the graph of $f(x)$ through a series of translations the graph of $g(x)$.

1. $g(x) = f(x - 3)$

2. $g(x) = 2 + f(x)$

3. $g(x) = f(x + 1) + 1$

Answer 4.

1. Shift $f(x)$ 3 to the right.

2. Shift $f(x)$ 2 up.

3. Shift $f(x)$ 1 to the left and 1 up.

In addition to translation the graph of a function may be reflected about an axis. Here is a summary of reflection operations.

Operation	Result on the graph of $f(x)$
$-f(x)$	Reflect about the x-axis.
$f(-x)$	Reflect about the y-axis.

Observe that the parabola with vertex at $(0,0)$ has the y-axis as its axis of symmetry. It is already symmetric about the y-axis and refelecting about the y-axis has no effect on its graph, because $f(x) = x^2 = (-x)^2 = f(-x)$. Whenever $f(x) = f(-x)$ we call $f(x)$ an *even function*. Even functions are already symmetric about the y-axis. Typically, these are polynomials where every term has even degree.

An *odd function* is one where $f(-x) = -f(x)$. In other words a reflection about the y-axis is the same as a reflection about the x-axis. Thus, odd functions are symmetric about the origin. Typically, odd functions are polynomials where every term has odd degree.

Example 5. On the same coordinate plane graph $f(x) = x^3$ and $g(x) = 1 - (1-x)^3$.

Answer 6. First we determine how the graphs of $g(x)$ and $f(x)$ are related. Since $g(x)$ is an odd function we have $g(x) = 1 - (1-x)^3 = 1 - [-1(x-1)]^3 = 1 + (x-1)^3$. We see that a shift of 1 up and 1 to the right is necessary;, i.e. we first graph $f(x) = x^3$ and then shift this graph 1 up and 1 to the right. (See the graph on the next page.)

Example 7. List in the correct order the translations and reflections that need to applied to the graph of $f(x)$ in order to obtain the graph of $g(x)$, and say what $f(x)$ you are using.

1. $g(x) = \sqrt{1-x} + 4$

2. $g(x) = 5 + \frac{1}{(x-1)^2}$

3. $g(x) = 1 - |x + 10|$

4. $g(x) = -(2-x)^2 + 3$

Answer 8.

1. reflect about the y-axis, 1 right, 4 up; $f(x) = \sqrt{x}$.

2. 1 right, 5 up; $f(x) = \frac{1}{x^2}$.

3. reflect about the x-axis, 10 left, 1 up; $f(x) = |x|$.

4. reflect about the x-axis, 2 right, 3 up; $f(x) = x^2$.

1. For each function determine whether it is odd, even, or neither.

   (a) $f(x) = x^6 + 3x^4 + x^2$

   (b) $f(x) = x^6 + 3x^4 + 1$

   (c) $f(x) = x^3 + x$

   (d) $f(x) = x^3 + x + 1$

   (e) $f(x) = |x|$

2. Apply the given transformations to $f(x)$:

   (a) $f(x) = x + 1$; reflect about the x-axis, shift 4 up.

   (b) $f(x) = \sqrt{x} - 4$; shift 2 up and 1 to the left.

   (c) $f(x) = \frac{1-x}{3-x}$; reflect about y-axis, shift 3 down.

   (d) $f(x) = (x - 2)^2 + 7$; translate so the vertex is at $(5, 1)$

   (e) $f(x) = x^3$; translate so the x-intercept is $(-8, 0)$.

3. List in the correct order the translations and reflections that need to applied to the graph of $f(x)$ in order to obtain the graph of $g(x)$. Say what $f(x)$ is.

   (a) $g(x) = -\sqrt{x} + 4$

   (b) $g(x) = 1 - \frac{1}{x^2}$

   (c) $g(x) = |10 - x|$

   (d) $g(x) = -(2 - x)^2 + 3$

   (e) $g(x) = 2 - \sqrt[3]{x}$

4. Given piecewise function $h(x)$ do the following:

$$h(x) = \begin{cases} -x - 5, & -5 \le x \le -1; \\ 0, & x = 0; \\ -x + 5, & 1 \le x \le 5. \end{cases}$$

   (a) Graph $h(x)$ and label all endpoints.

   (b) Describe how to obtain $g(x) = 5 + h(x + 3)$. What happens to the point $(0, 0)$?

   (c) Graph $g(x)$ on the same coordinate plane as $h(x)$ and label all endpoints.

   (d) Observe that the translation $g(x)$ is obtained by adding $(-3, 5)$ coordinate-wise to every point on the graph of $h(x)$.

Sample Midterm | Sample Final

5   A   B   C   D

1. For each function determine whether it is odd, even, or neither.

    (a) $f(x) = x^6 + 3x^4 + x^2$

    Answer 1. $f(-x) = (-x)^6 + 3(-x)^4 + (-x)^2 = x^6 + x^4 + x^2 = f(x)$.
    This is an even function.

    (b) $f(x) = x^6 + 3x^4 + 1$

    Answer 2. $f(-x) = (-x)^6 + 3(-x)^4 + 1 = x^6 + 3x^4 + 1 = f(x)$.
    This is an even function.

    (c) $f(x) = x^3 + x$

    Answer 3. $f(-x) = (-x)^3 + (-x) = -x^3 - x = -(x^3 + x) = -f(x)$.
    This is an odd function.

    (d) $f(x) = x^3 + x + 1$

    Answer 4. $f(-x) = (-x)^3 + (-x) + 1 = -x^3 - x + 1 \neq -f(x)$.
    This function is neither even nor odd.

    (e) $f(x) = |x|$

    Answer 5. $f(-x) = |-x| = |x| = f(x)$.
    This function is even.

2. Apply the given transformations to $f(x)$:

    (a) $f(x) = x + 1$; reflect about the x-axis, shift 4 up.

    Answer 6. We obtain $-f(x) + 4 = -(x + 1) + 4 = -x + 3$.

    (b) $f(x) = \sqrt{x} - 4$; shift 2 up and 1 to the left.

    Answer 7. We obtain $f(x + 1) + 2 = \sqrt{x + 1} - 2$.

    (c) $f(x) = \frac{1-x}{3-x}$; reflect about y-axis, shift 3 down.

    Answer 8. We obtain $f(-x) - 3 = \frac{1+x}{3+x} - 3 = \frac{-2x-8}{3+x}$.

    (d) $f(x) = (x - 2)^2 + 7$; translate so the vertex is at $(5, 1)$.

    Answer 9. We obtain $f(x - 3) - 6 = (x - 5)^2 + 1$.

    (e) $f(x) = x^3$; translate so the x-intercept is $(-8, 0)$.

    Answer 10. The current x-intercept is at the origin, thus we need to shift 8 to the left. $f(x + 8) = (x + 8)^3$.

3. List in the correct order the translations and reflections that need to applied to the graph of $f(x)$ in order to obtain the graph of $g(x)$. Say what $f(x)$ is.

(a) $g(x) = -\sqrt{x} + 4$

Answer 11. $f(x) = \sqrt{x}$ and $g(x) = -f(x) + 4$; i.e., $g(x)$ is $f(x)$ reflected about the x-axis and shifted 4 up.

(b) $g(x) = 1 - \frac{1}{x^2}$

Answer 12. $f(x) = \frac{1}{x^2}$ and $g(x) = -f(x) + 1$; i.e., $g(x)$ is $f(x)$ reflected about the x-axis and shifted 1 up.

(c) $g(x) = |10 - x|$

Answer 13. $f(x) = |x|$ and $g(x) = f(-(x - 10))$; i.e., $g(x)$ is $f(x)$ reflected about the y-axis and shifted 10 right.

(d) $g(x) = -(2 - x)^2 + 3$

Answer 14. $f(x) = x^2$ and $g(x) = -f(-(x - 2)) + 3$; i.e., $g(x)$ is $f(x)$ reflected about the x- and y-axes, shifted 2 right and 3 up.

(e) $g(x) = 2 - \sqrt[3]{x}$

Answer 15. $f(x) = \sqrt[3]{x}$ and $g(x) = -f(x) + 2$; i.e., $g(x)$ is $f(x)$ reflected about the x-axis (or y-axis, since $f(x)$ is odd) and shifted 2 up.

4. Given piecewise function $h(x)$ do the following:

$$h(x) = \begin{cases} -x - 5, & -5 \le x \le -1; \\ 0, & x = 0; \\ -x + 5, & 1 \le x \le 5. \end{cases}$$

(a) Graph $h(x)$ and label all endpoints.

Answer 16.
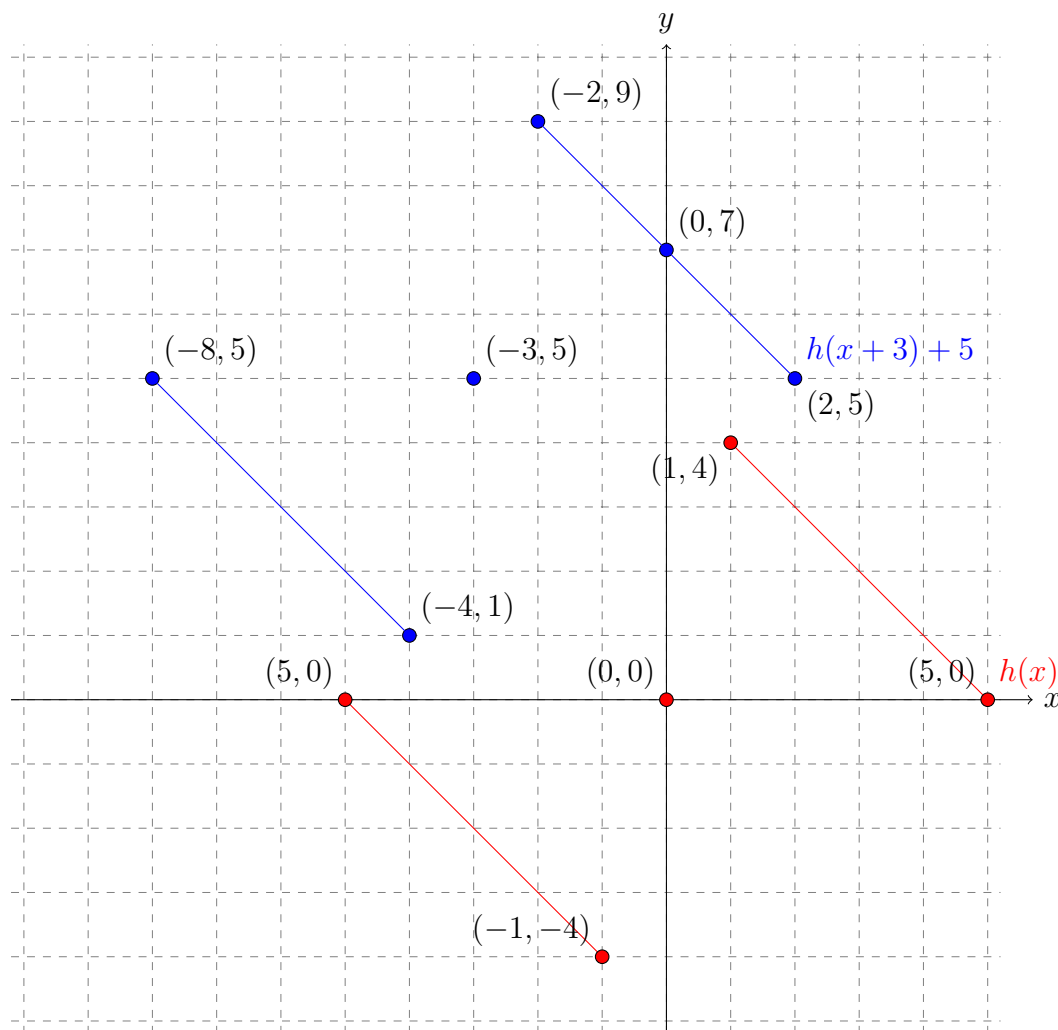
(b) Describe how to obtain $g(x) = 5 + h(x + 3)$. What happens to the point $(0, 0)$?

Answer 17. The graph of $g(x)$ is the graph of $h(x)$ shifted 5 up and 3 left. The point $(0, 0)$ under this translation becomes the point $(-3, 5)$.

(c) Graph $g(x)$ on the same coordinate plane as $h(x)$ and label all endpoints.

Answer 18. See the last page.

(d) Observe that the translation $g(x)$ is obtained by adding $(-3, 5)$ coordinate-wise to every point on the graph of $h(x)$.

Answer 19. Applying the shift to the origin we have $(0, 0) + (-3, 5) = (0 - 3, 0 + 5) = (-3, 5)$. Applying the shift to $(1, 4)$ we have $(1, 4) + (-3, 5) = (1 - 3, 4 + 5) = (-2, 9)$. Similarly for all other points.

1. Graph the following functions by applying the appropriate transformations. As usual correctly scale and label the axes, all intercepts, and asymptotes.

   (a) $f(x) = 5 - \sqrt{x + 2}$

   (b) $f(x) = \frac{1}{x+3} + 2$

   (c) $f(x) = -\frac{1}{(x-4)^2}$

   (d) $f(x) = 3 - |x - 7|$

2. *Stretches and Compressions of the x-axis.*

   (a) On the same coordinate plane graph $f(x) = x^2 - 4, g(x) = f(2x) = 4x^2 - 4$, $h(x) = f(4x) = 16x^2 - 4$.

   (b) What effect on the roots of $f(x)$ does multiplying the argument of f by $a > 1$ have? What happens to the vertex?

   (c) In general, if $a > 1$ then how does the graph of $f(ax)$ differ from that of $f(x)$?

   (d) Repeat the problem for $0 < a < 1$.

3. *Stretches and Compressions of the y-axis.*

   (a) On the same coordinate plane graph $f(x) = -x^2 + 4$, $g(x) = 4f(x) = -4x^2 + 16, h(x) = 16f(x) = -16x^2 + 64$.

   (b) What effect on the roots of $f(x)$ does multiplying the function by $a > 1$ have? What happens to the vertex?

   (c) In general, if $a > 1$ then how does the graph of $af(x)$ differ from that of $f(x)$?

   (d) Repeat the problem for $0 < a < 1$.

Answer 1. We first graph $f(x) = \sqrt{x}$ and then $5 - f(x+2)$; i.e., reflect the graph of $\sqrt{x}$ about the x-axis, shift 2 left and 5 up. (Refer to graph).
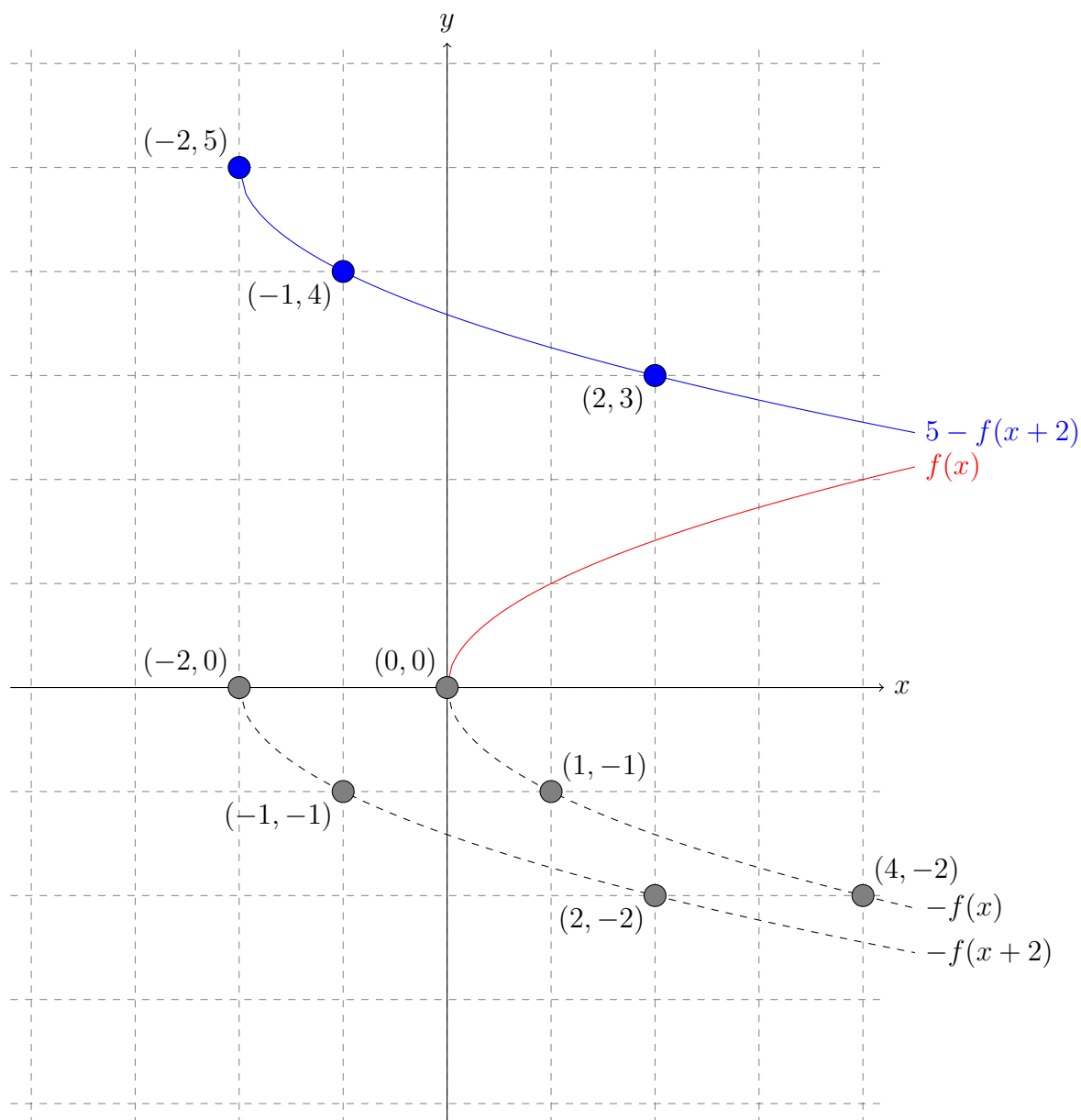
Answer 2. We first graph $f(x) = \frac{1}{x}$ and then $f(x+3) + 2$; i.e., shift the graph of $\frac{1}{x}$ 3 to the left and 2 up. (Refer to graph).
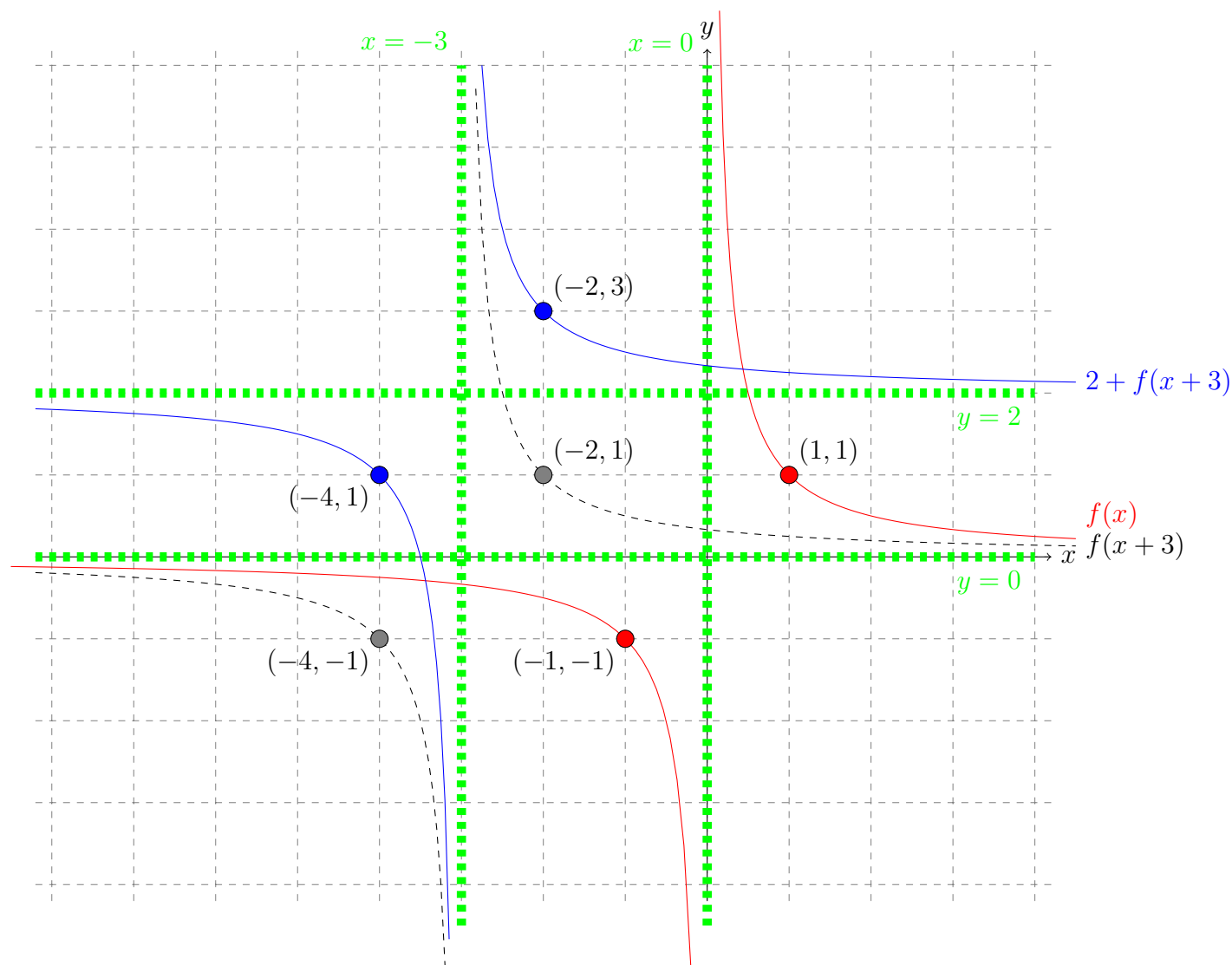
Answer 3. We first graph $f(x) = \frac{1}{x^2}$ and then $-f(x-4)$; i.e., reflect the graph of $f(x)$ about the x-axis and shift 4 right. (Refer to graph).
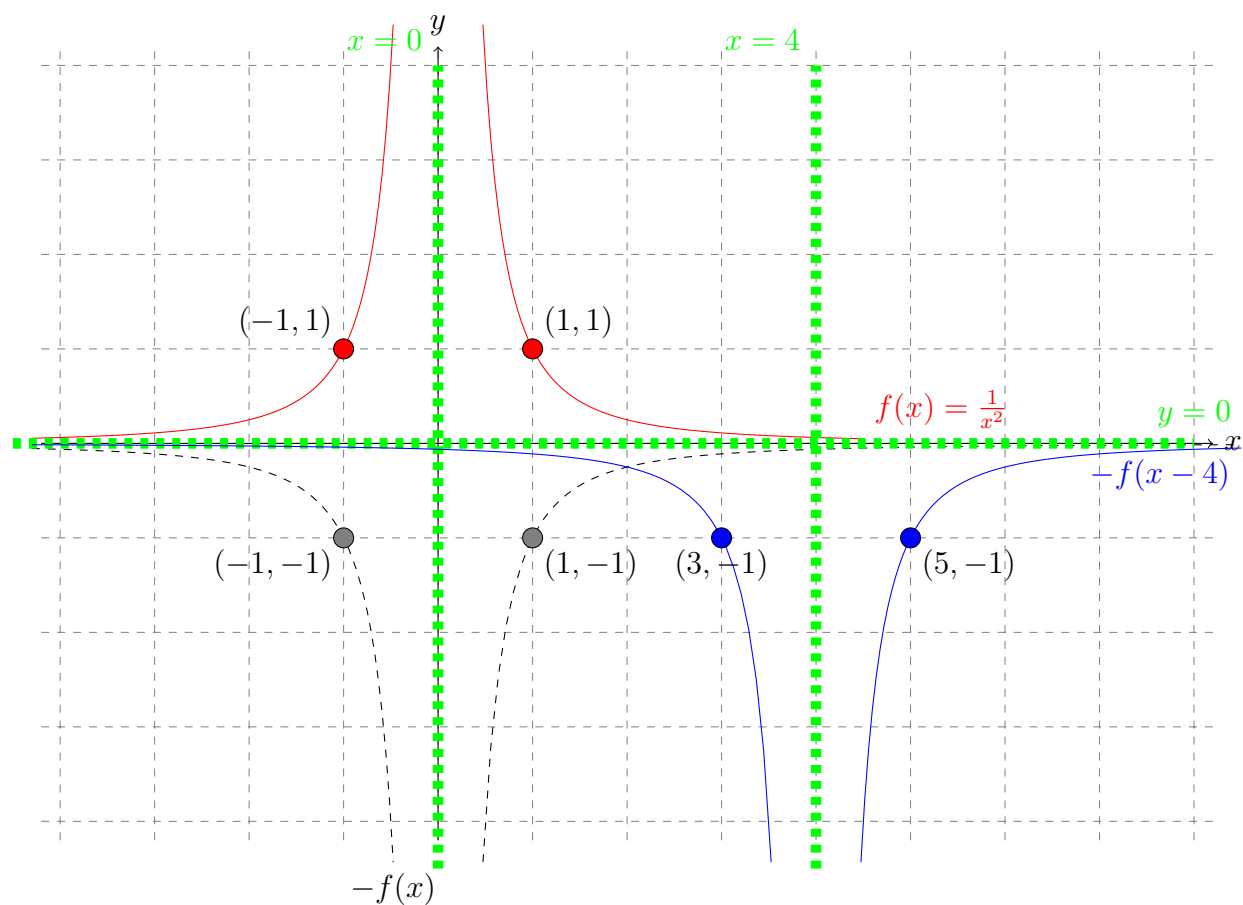
Answer 4. We first graph $f(x) = |x|$ and then $3 - f(x - 7)$; i.e., reflect the graph of $f(x)$ about the x-axis, then shift 7 right and 3 up. (Refer to graph).
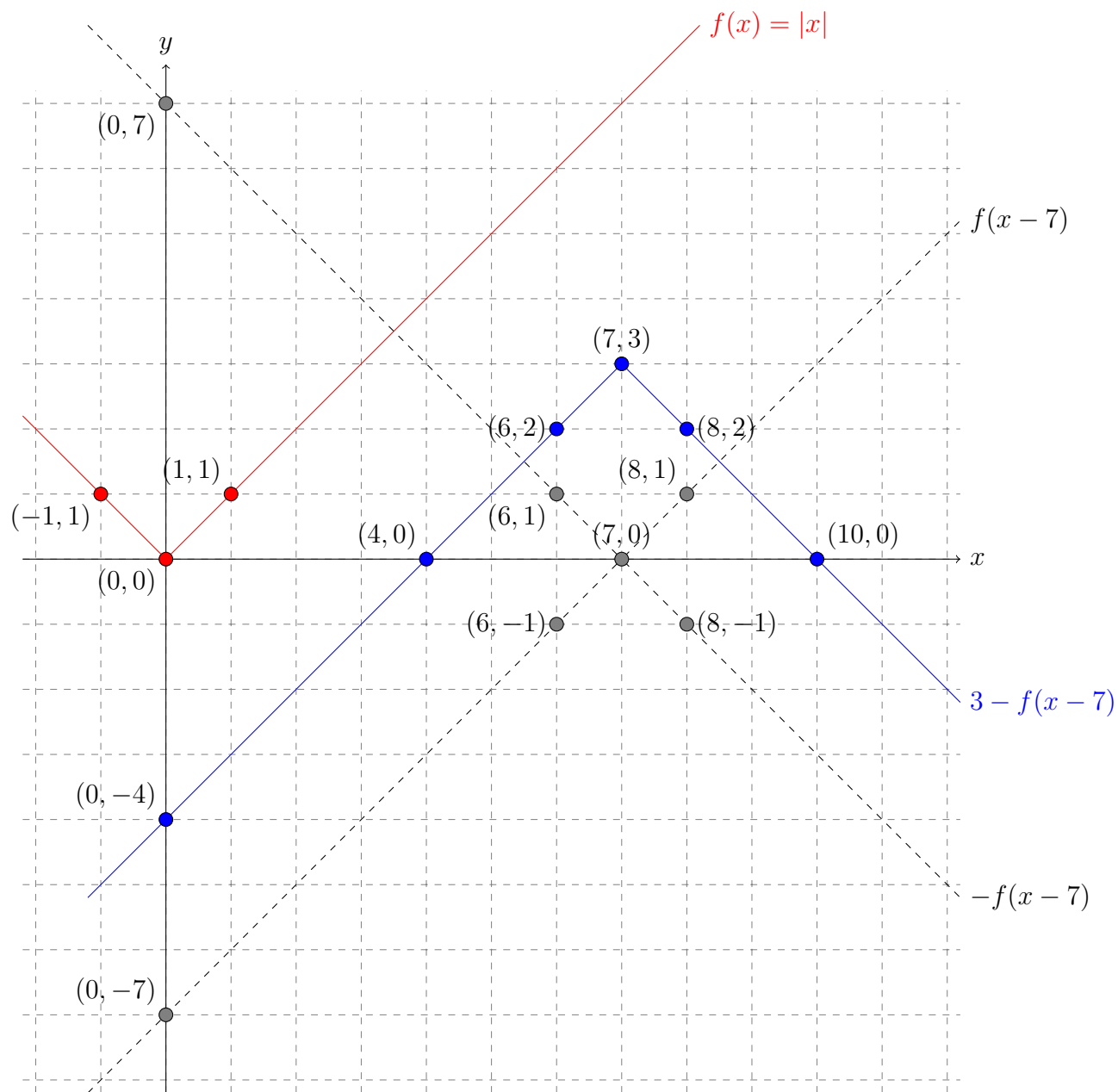
Stretches and Compressions of the x-axis.

1. On the same coordinate plane graph $f(x) = x^2 - 4$, $g(x) = f(2x) = 4x^2 - 4$, $h(x) = f(4x) = 16x^2 - 4$.

2. What effect on the roots of $f(x)$ does multiplying the argument of f by $a > 1$ have? What happens to the vertex?

   Answer 5. The vertex remains unchanged, but the x-axis is compressed. The roots are shifted closer to the axis of symmetry. (Refer to graph).
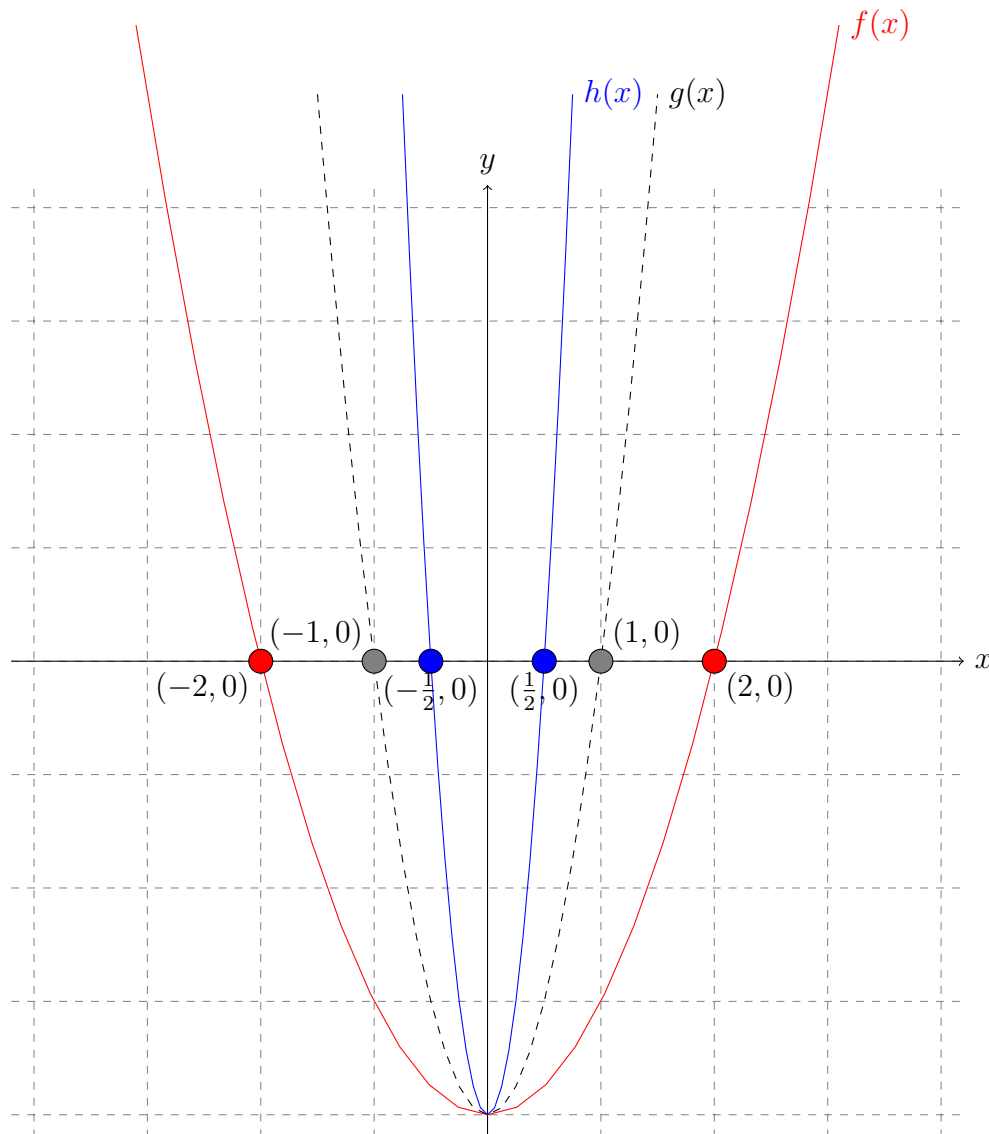
3. In general, if $a > 1$ then how does the graph of $f(ax)$ differ from that of $f(x)$?

   Answer 6. The entire graph is compressed with respect to the x-axis but remains unchanged with respect to the y-axis.

4. Repeat the problem for $0 < a < 1$.

   Answer 7. The effects are reversed. The graph remains unchanged with respect to the y-axis, but is stretched with respect to the x-axis. The vertex remains in place, but the roots are now farther apart. (Refer to graph).
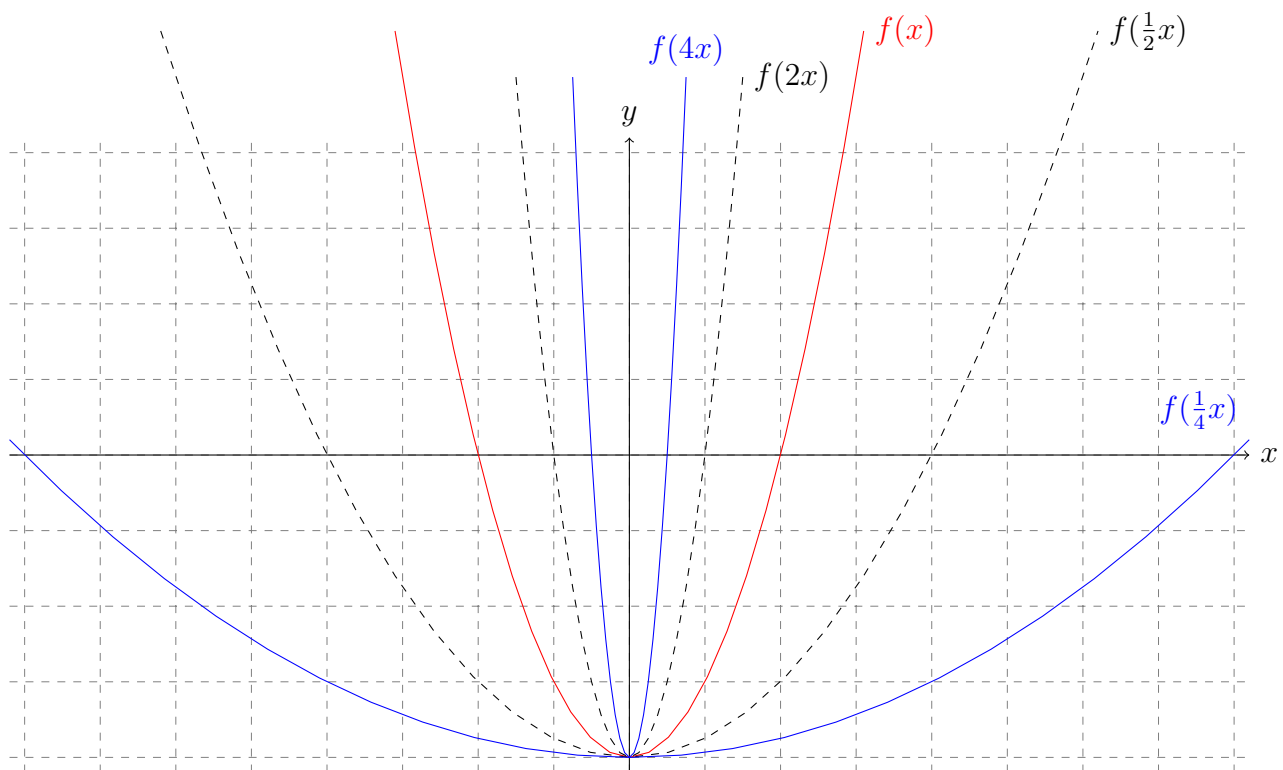
Stretches and Compressions of the y-axis.

1. On the same coordinate plane graph $f(x) = -x^2 + 4$,
    $g(x) = 4f(x) = -4x^2 + 16, h(x) = 16f(x) = -16x^2 + 64.$

2. What effect on the roots of $f(x)$ does multiplying the function by $a > 1$ have? What happens to the vertex?

    Answer 8. The roots remain in the same place, but the vertex moves away from the x-axis. The y-axis is stretched.

3. In general, if $a > 1$ then how does the graph of $af(x)$ differ from that of $f(x)$?

    Answer 9. The entire graph is stretched with respect to the y-axis, but remains unchanged with respect to the x-axis.

4. Repeat the problem for $0 < a < 1$.

    Answer 10. The effects are reversed. The graph remains unchanged with respect to the x-axis, but is compressed with respect to the y-axis. The roots remain in place, but the vertex moves toward the x-axis. (Refer to graph).
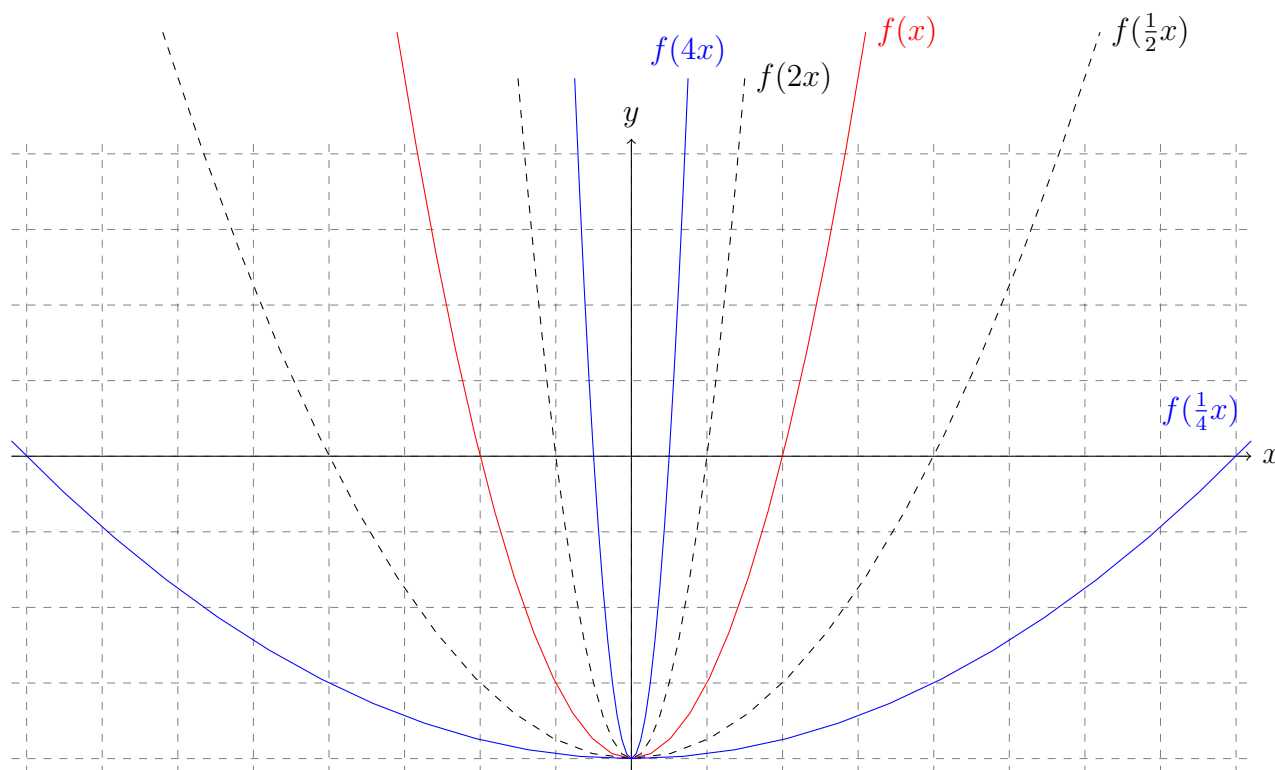
1. Find the leading term and use it determine the long-term behavior of each polynomial function.

    (a) $f(x) = x^2 + 3x + 1$

    (b) $g(x) = -3x + 1$

    (c) $p(x) = -x^4 + x^3 + x - 4$

    (d) $t(x) = (2x - 1)^2(3x + 2)^2(x - 1)(x + 2)$

    (e) $h(x) = (x^2 + 2x + 1)^2(2x + 3)^4$

2. Find all roots and their degrees. Describe the behavior of the graph at each root.

    (a) $f(x) = (x - 2)^2(x + 2)(x + 4)$

    (b) $g(x) = (x + 5)^4(x - 1)$

    (c) $p(x) = (x^2 - 3x + 2)(x^2 - x - 6)$

    (d) $h(x) = (x^2 + x + 1)(x - 3)^2(x + 1)^2$

    (e) $r(x) = (x^2 + 4)(x^3 + 1)(x + 1)^3$

3. Give the degree of each polynomial function. At most how many turning points does each graph have?

    (a) $f(x) = (x - 2)(x + 2)(x - 1)(x + 1)(x + 3)$

    (b) $g(x) = (x - 10)^2(x + 10)^2$

    (c) $p(x) = (x^3 + x^2 + x + 1)^3$

    (d) $t(x) = x(x^2 - 3x + 2)^2(x - 7)^2$

    (e) $h(x) = (5x^2 - 2)(x^2 + x + 1)^3$

4. Graph each polynomial function. As usual, correctly scale and label the graph and all axes. Label all roots with their degrees and mark all intercepts. The graph must be smooth and continuous.

    (a) $p(x) = -x(x - 2)(x - 3)$

    (b) $h(x) = (x + 4)^2(x - 1)^2(x - 5)$

    (c) $g(x) = -(x - 3)^3(x - 5)$

    (d) $s(x) = (x - 2)^2(x^2 + 4x + 4)(x^2 + 6x + 9)$

    (e) $f(x) = x(2 - x^2)(x + 1)(x - 3)$

5. *Working backwards.* Find a possible polynomial function for each graph with the given degree. The y-axis is left intentionally without scale.

   (a) degree 4

   (b) degree 2

   (c) degree 4 [Not the reflection of (B) about the x-axis.]

   (d) degree 6

   (e) degree 6 [Not the reflection of (A) about the x-axis.]

A

B

C

D

E
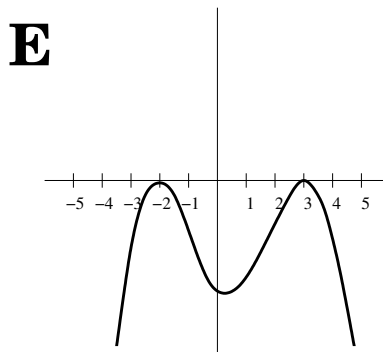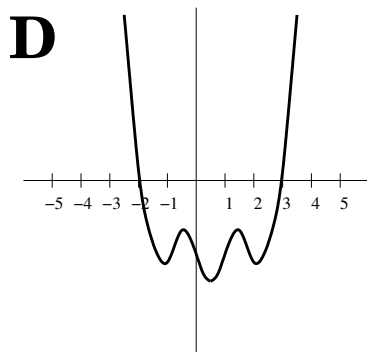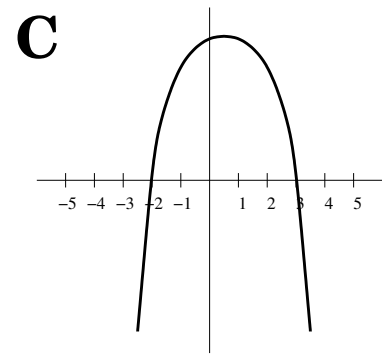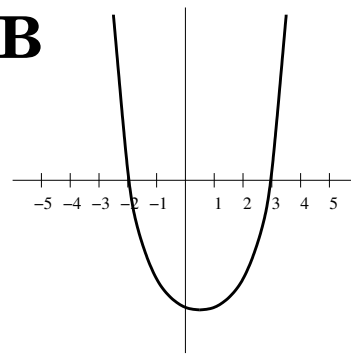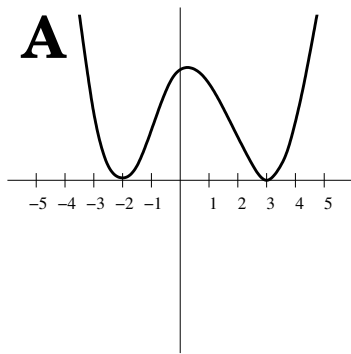
Sample Midterm	Sample Final			
	3	A	B	C	D
	4	A	B	C	D
	8				D
	20	A	B	C	D
	36	A	B	C	D

1. Find the leading term and use it determine the long-term behavior of each polynomial function.

   (a) $f(x) = x^2 + 3x + 1$

   Answer 1. The leading term of $f(x)$ is x^2 so $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

   (b) $g(x) = -3x + 1$

   Answer 2. The leading term of $g(x)$ is $-3x$ so $g(x) \to -\infty$ as $x \to \infty$ and $g(x) \to \infty$ as $x \to -\infty$.

   (c) $p(x) = -x^4 + x^3 + x - 4$

   Answer 3. The leading term of $p(x)$ is $-x^4$ so $p(x) \to -\infty$ as $x \to \pm\infty$.

   (d) $t(x) = (2x - 1)^2(3x + 2)^2(x - 1)(x + 2)$

   Answer 4. The leading term of $t(x)$ is the product of the leading terms of each factor: $(2x)^2 \cdot (3x)^2 \cdot x \cdot x = 36x^6$. Thus, $t(x) \to \infty$ as $x \to \pm\infty$.

   (e) $h(x) = (x^2 + 2x + 1)^2(2x + 3)^4$

   Answer 5. The leading term of $h(x)$ is the product of the leading terms of each factor: $(x^2)^2 \cdot (2x)^4 = 14x^8$. Thus, $h(x) \to \infty$ as $x \to \pm\infty$.

2. Find all roots and their degrees. Describe the behavior of the graph at each root.

   (a) $f(x) = (x - 2)^2(x + 2)(x + 4)$

   Answer 6. $f(x)$ is already completely factored so we read off the roots: $x = 2$ (degree 2), $x = -2$ (degree 1), $x = -4$ (degree 1). The graph of $f(x)$ crosses the x-axis at each degree 1 root and touches the x-axis at the degree 2 root $x = 2$. Moreover, since $f(1.99) > 0$ the graph touches the x-axis from above at $x = 2$.

   (b) $g(x) = (x + 5)^4(x - 1)$

   Answer 7. $g(x)$ is completely factored so we read off the roots: $x = -5$ (degree 4) and $x = 1$ (degree 1). The graph of $g(x)$ crosses the x-axis at $x = 1$ and touches the x-axis at $x = -5$. Moreover, since $g(-5.01) > 0$ the graph touches the x-axis from above at $x = -5$.

   (c) $p(x) = (x^2 - 3x + 2)(x^2 - x - 6)$

   Answer 8. We fist factor $p(x)$ completely and then read off the roots. $p(x) = (x^2 - 3x + 2)(x^2 - x - 6) = (x - 1)(x - 2)(x - 3)(x + 2)$. Thus, the roots are $x = -2, 1, 2, 3$ and since all are of degree 1 the graph of $p(x)$ will cross the x-axis at each root.

   (d) $h(x) = (x^2 + x + 1)(x - 3)^2(x + 1)^2$

   Answer 9. $h(x)$ is already completely factored so we read off the roots: $x = 3$ (degree 2) and $x = -1$ (degree 2). Both roots are of even degree and since $h(2.99) > 0$ and $h(-0.99) > 0$ the graph touches the x-axis from above at each root.

(e) $r(x) = (x^2 + 4)(x^3 + 1)(x + 1)^3$

Answer 10. We first factor $r(x)$ completely and the only factor which can be simplified is $(x^3+1) = (x+1)(x^2-x+1)$. Thus, $r(x) = (x^2+4)(x^2-x+1)(x+1)^4$ and so $x = -1$ (degree 4) is the only root. Since it is an even degree root the graph of $r(x)$ will touch the x-axis at $x = -1$ and since $r(-1.01) > 0$ it follows that the graph touch the x-axis form above.

3. Give the degree of each polynomial function. At most how many turning points does each graph have?

(a) $f(x) = (x - 2)(x + 2)(x - 1)(x + 1)(x + 3)$

Answer 11. $f(x)$ is a degree 5 polynomial so it can have at most 4 turning points.

(b) $g(x) = (x - 10)^2(x + 10)^2$

Answer 12. $g(x)$ is a degree 4 polynomial so it can have at most 3 turning points.

(c) $p(x) = (x^3 + x^2 + x + 1)^3$

Answer 13. $p(x)$ is a degree 9 polynomial so it can have at most 8 turning points.

(d) $t(x) = x(x^2 - 3x + 2)^2(x - 7)^2$

Answer 14. $t(x)$ is a degree 7 polynomial so it can have at most 6 turning points.

(e) $h(x) = (5x^2 - 2)(x^2 + x + 1)^3$

Answer 15. $h(x)$ is a degree 8 polynomial so it can have at most 7 turning points.

4. Graph each polynomial function. As usual, correctly scale and label the graph and all axes. Label all roots with their degrees and mark all intercepts. The graph must be smooth and continuous.

(a) $p(x) = -x(x-2)(x-3)$
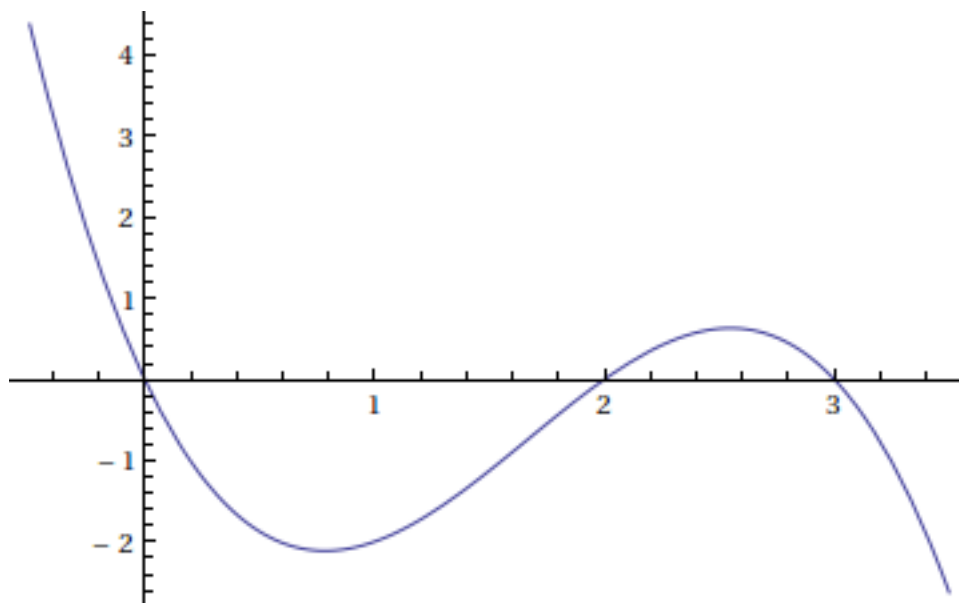
(b) $h(x) = (x+4)^2(x-1)^2(x-5)$
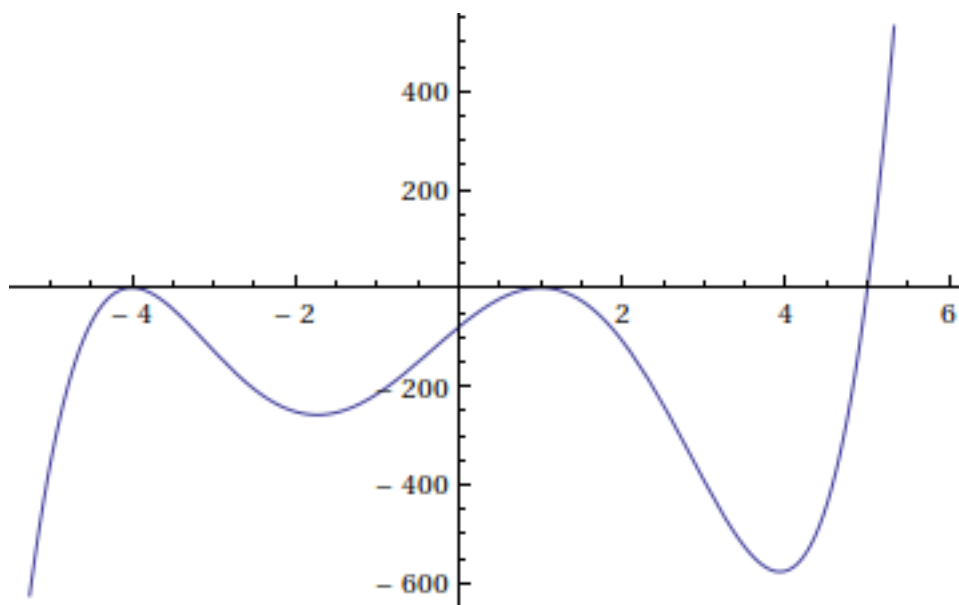
(c) $g(x) = -(x-3)^3(x-5)$
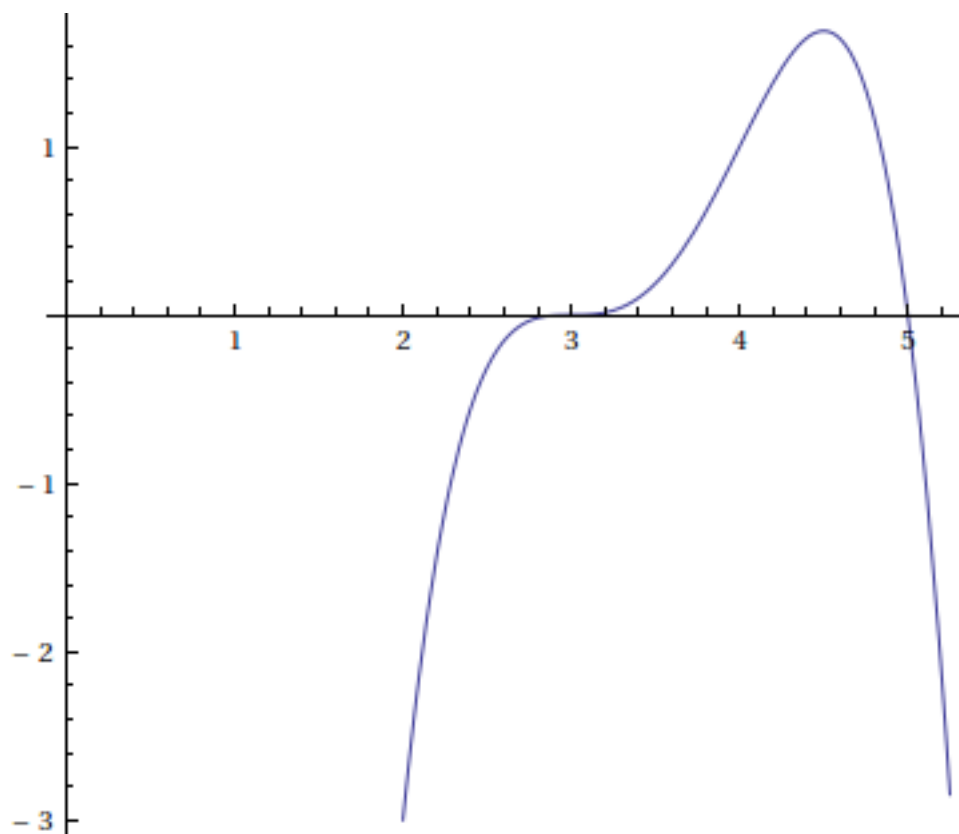
(d) $s(x) = (x-2)^2(x^2+4x+4)(x^2+6x+9)$
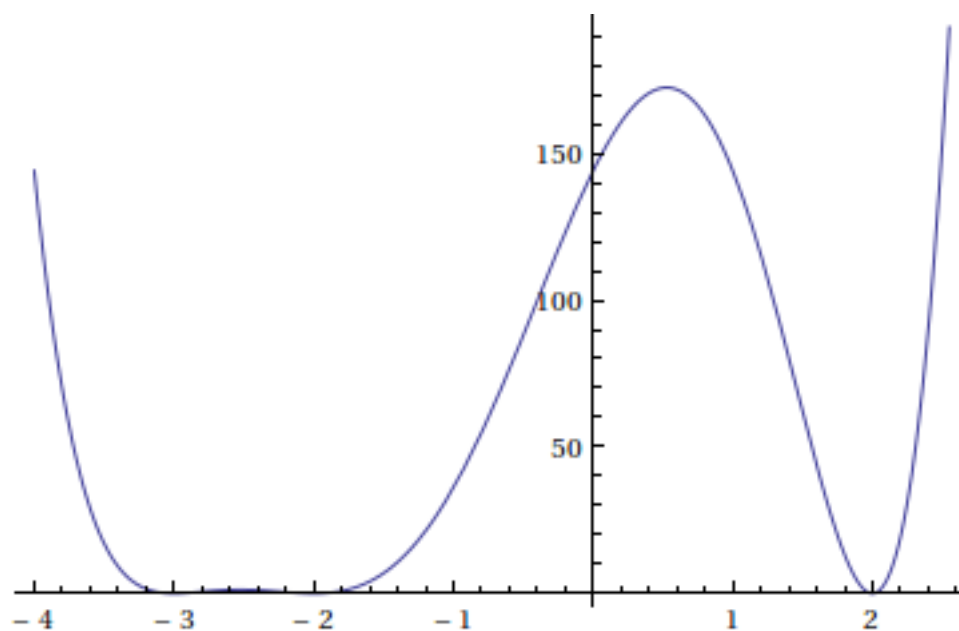
(e) $f(x) = x(2 - x^2)(x + 1)(x - 3)$
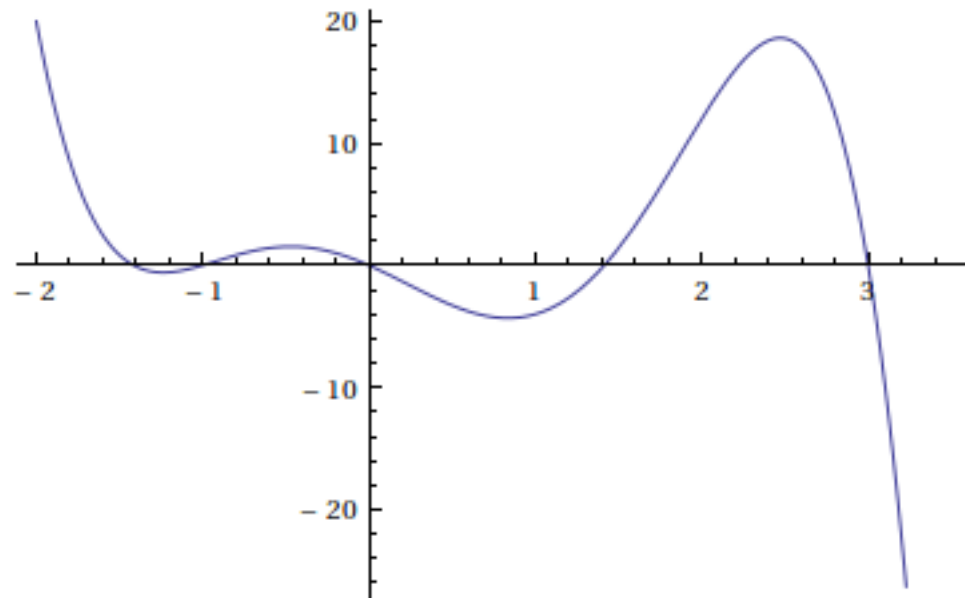
5. *Working backwards.* Find a possible polynomial function for each graph with the given degree. The y-axis is left intentionally without scale.

   (a) degree 4

   Answer 16. $(x+2)^2(x-3)^2$

   (b) degree 2

   Answer 17. $(x+2)(x-3)$

   (c) degree 4 [Not the reflection of (B) about the x-axis.]

   Answer 18. $-(x+2)(x-3)(x^2+1)$

   (d) degree 6

   Answer 19. $(x+2)(x-3)(x^2+1)^2$

   (e) degree 6 [Not the reflection of (A) about the x-axis.]

   Answer 20. $-(x+2)^2(x-3)^2(x^2+1)$
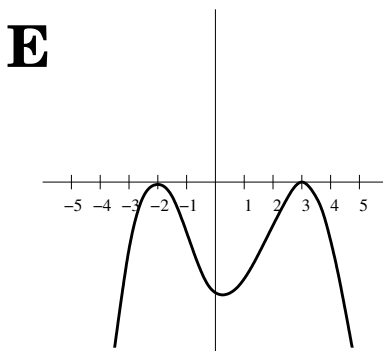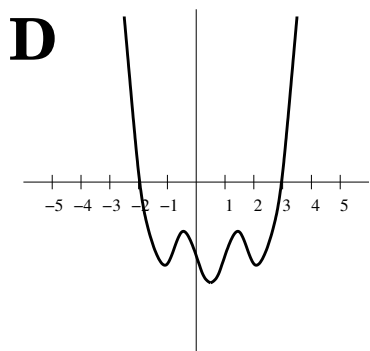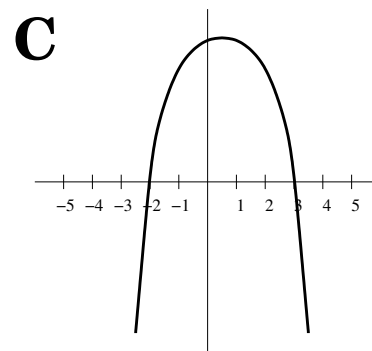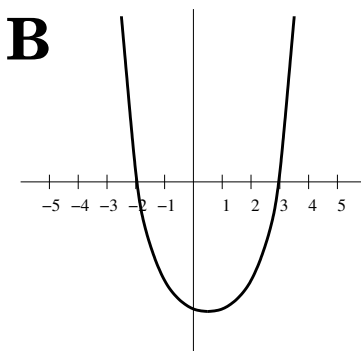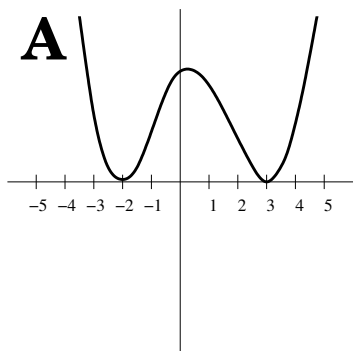
1. Graph each rational function.

   (a) $f(x) = \frac{1}{(x-2)(x+2)}$

   (b) $g(x) = \frac{x^2}{(x+3)(x-5)}$

   (c) $q(x) = \frac{(3x-1)(x+2)^2}{(x-2)(x-1)^2}$

   (d) $h(x) = \frac{x^2+x+1}{x+2}$

   (e) $t(x) = \frac{(x-1)(x+2)}{x}$

   Follow these steps to graph each of the above rational functions.

   (a) Find all roots and their degrees. Plot the roots and note the behavior of the graph at each x-intercept. Plot the y-intercept, if there is one.

   (b) Find the equations of all asymptotes and list the degrees of vertical asymptotes. Graph the asymptotes and using their degrees and the key number method determine the behavior of the graph at each vertical asymptote. Find all points, if any, where the horizontal or oblique asymptotes intersect the graph.

   (c) Find the leading term and use it determine the long-term behavior of each rational function.

   (d) Graph each rational function. As usual, correctly scale and label the graph and all axes. Label all roots and all asymptotes with their degrees and mark all intercepts. The graph must be smooth.

2. *Working backwards.* Find a possible rational function for each graph.
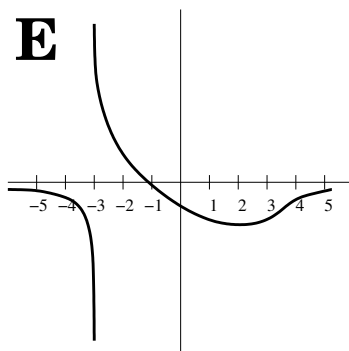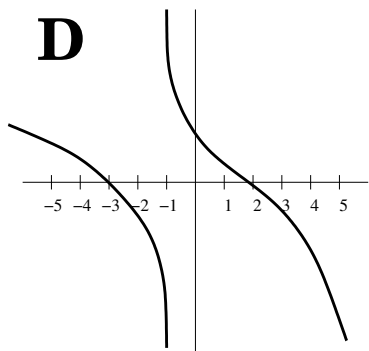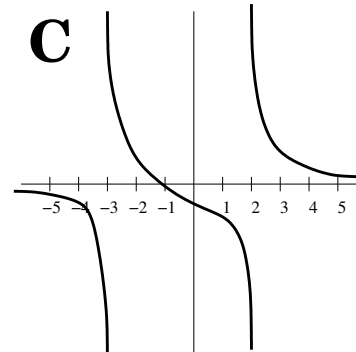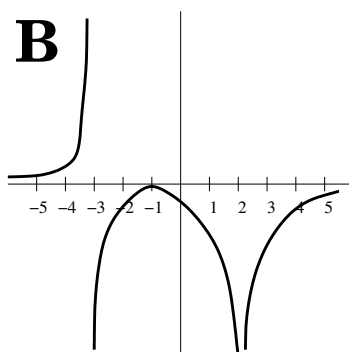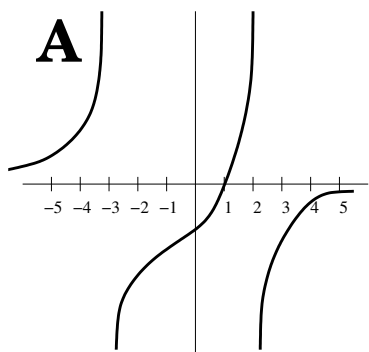
Sample Midterm		Sample Final			
	7	A	B	C	D
	8	A			D
	19	A	B	C	D
	21	A	B	C	D
	24	A	B	C	D
	27	A	B	C	D

1. Graph each rational function.

   (a) $f(x) = \frac{1}{(x-2)(x+2)}$

   Answer 1.

   - (Roots) There are no roots, hence the graph will have no x-intercepts. Letting $x = 0$, we have $f(0) = -\frac{1}{4}$, and so the point $(0, -\frac{1}{4})$ is the y-intercept.
   - (Asymptotes) The degree of the denominator is larger than the degree of the numerator, hence the graph has a horizontal asymptote $y = 0$. The vertical asymptotes are $x = \pm 2$ and both have degree 1. Since the y-intercept is below the x-axis and there are no roots, we expect the graph to lie below the x-axis in the region between the two asymptotes. Since there are no roots, the graph does not intersect its horizontal asymptote.
   - (Leading Term) The leading term is $\frac{1}{x^2}$, the graph of which tends to $y = 0$ as $x \to \pm\infty$.
   - Graph.
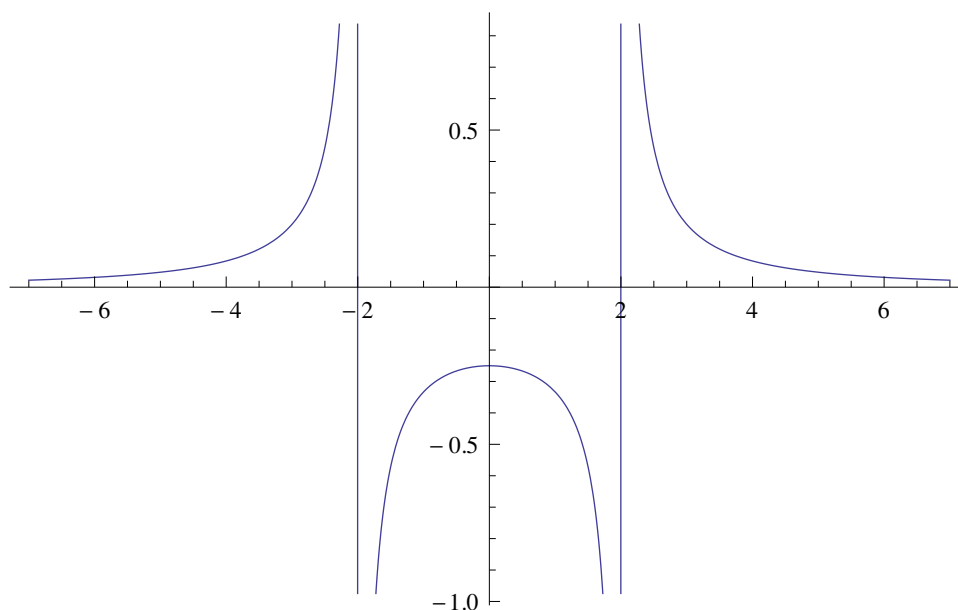
(b) $g(x) = \frac{x^2}{(x+3)(x-5)}$

Answer 2.

- (Roots) $x = 0$ is a root of degree 2, hence the graph of $g(x)$ touches the x-axis at the origin. This is also the y-intercept.
- (Asymptotes) The are vertical asymptotes of degree 1 at $x = -3$ and $x = 5$. Since the degree of the numerator equals the degree of the denominator there is a horizontal asymptote equal to the leading term; i.e., $y = 1$.
- (Leading Term) The leading term is 1 and so $g(x)$ behaves like $y = 1$ as $x \to \pm\infty$.
- Since we have a horizontal asymptote we check if the graph intersects it. We need to solve $g(x) = 1$.

$$g(x) = \frac{x^2}{(x+3)(x-5)} = 1 \iff x^2 = x^2 - 2x - 15 \iff x = -\frac{15}{2}$$

Therefore, $\left(-\frac{15}{2}, 1\right)$ is an intercept of $g(x)$ with its horizontal asymptote.
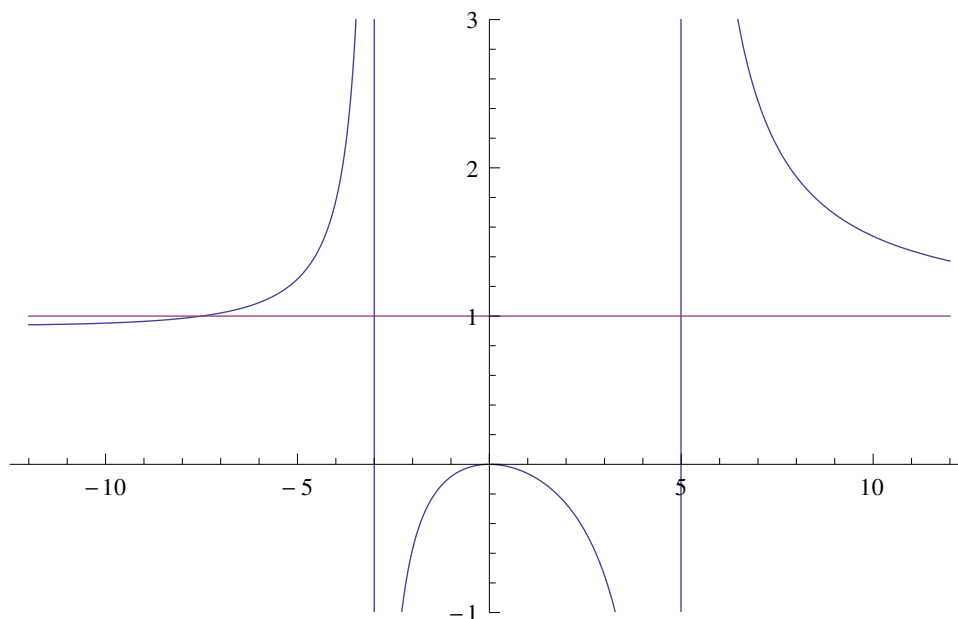
(c) $q(x) = \frac{(3x-1)(x+2)^2}{(x-2)(x-1)^2}$

Answer 3.

- (Roots) There is a degree 1 root at $x = \frac{1}{3}$ so the graph crosses here, and a degree 2 root at $x = -2$ so the graph touches here. $q(0) = 2$ so $(0, 2)$ is the y-intercept.
- (Asymptotes) There is a degree 1 vertical asymptote at $x = 2$ and a degree 2 vertical asymptote at $x = 1$. The equation of the horizontal asymptote is $y = 3$.
- (Leading Term) The leading terms is 3 and so $q(x)$ behaves like $y = 3$ as $x \to \pm\infty$.
- Graph. To see if the horizontal asymptote is crossed solve $q(x) = 3$

$$
\begin{aligned}
q(x) = 3 \quad &\Longleftrightarrow \quad (3x - 1)(x + 2)^2 = 3(x - 2)(x - 1)^2 \\
&\Longleftrightarrow \quad (3x - 1)(x^2 + 4x + 4) = (3x - 6)(x^2 - 2x + 1) \\
&\Longleftrightarrow \quad 23x^2 - 7x + 2 = 0
\end{aligned}
$$

Since the discriminant of this quadratic equation is negative there are no real solutions and hence no intercepts of the horizontal asymptote with $q(x)$.
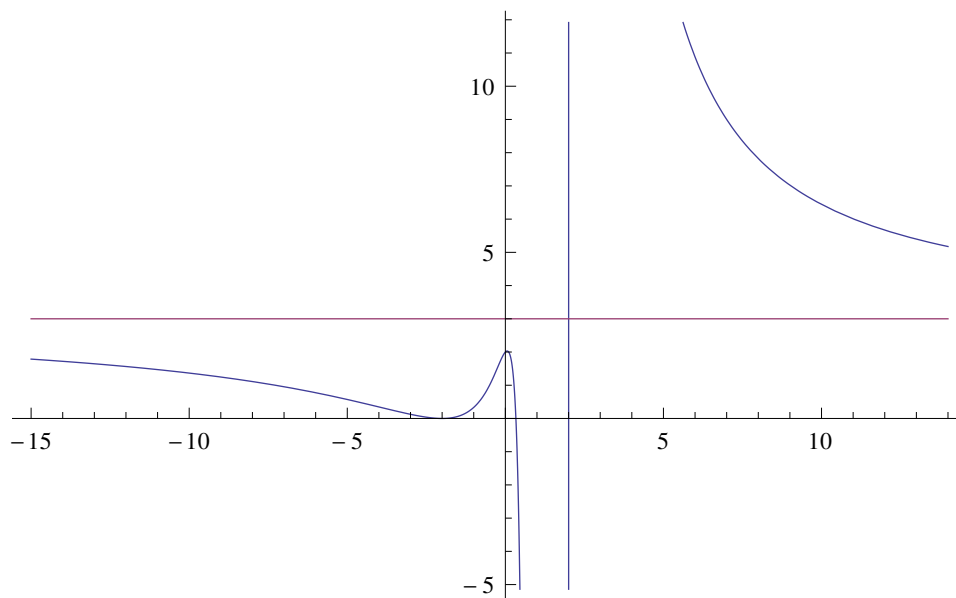
Note that the vertical asymptote $x = 1$ is not visible in the computer generated graph, so sketch it in. How does the graph of $q(x)$ look like on the interval $(1, 2)$ and why is it omitted from the picture? Sketch in the missing piece of the graph and explain why it does not appear in the above graph.

(d) $h(x) = \frac{x^2 + x + 1}{x + 2}$

Answer 4.

- (Roots) The numerator has no real solutions, hence there are no roots. $h(0) = \frac{1}{2}$, so $(0, \frac{1}{2})$ is the y-intercept.
- (Asymptotes) There is a vertical asymptote of degree 1 at $x = -2$. Since the degree of the numerator is 1 greater than the degree of the denominator there is an oblique asymptote. Long division yields $h(x) = x - 1 + \frac{3}{x+2}$ so the asymptote is the line $y = x - 1$.
- (Leading Term) The leading term is x, hence the graph behaves like the line with slope 1 and $h(x) \to \infty$ as $x \to \infty$ and $h(x) \to -\infty$ as $x \to -\infty$.
- (Graph) Refer to the graph of $h(x)$. To see if the graph intersects the oblique asymptote solve $h(x) = x$:

$$h(x) = x \iff x^2 + x + 1 = x^2 + 2x \iff x = 1$$

Therefore, $(1, 1)$ is the intercept of the graph of $h(x)$ with its oblique asymptote.
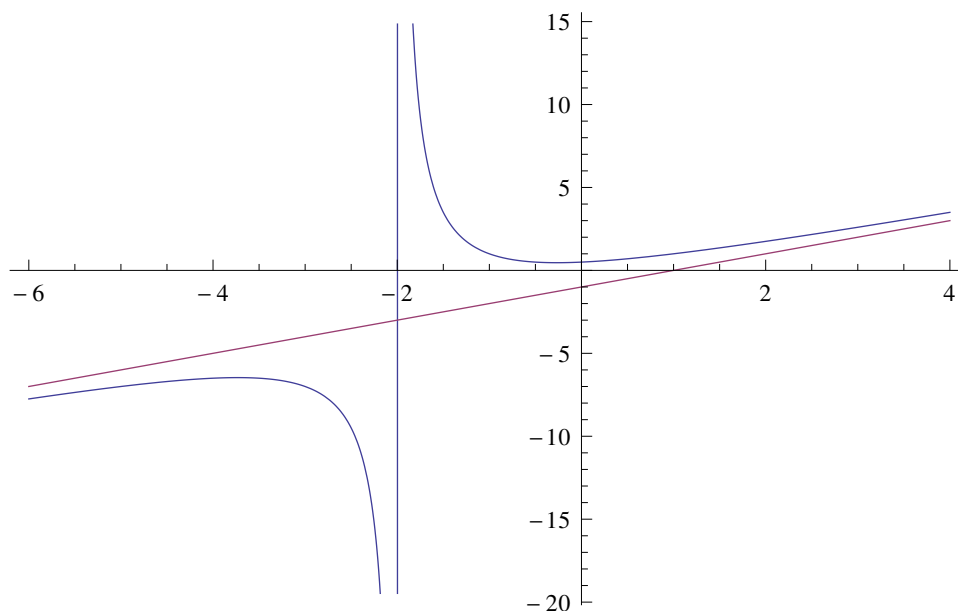
(e) $t(x) = \frac{(x-1)(x+2)}{x}$

Answer 5.

- (Roots) There are degree 1 roots at $x = 1$ and $x = -2$. There is no y-intercept as $t(0)$ is undefined.
- (Asymptotes) There is a degree 1 asymptote through the origin and an oblique asymptote. $t(x) = x + 1 - \frac{2}{x}$ so $y = x + 1$ is an oblique asymptote.
- (Leading Term) The leading term is x, hence the graph behaves like the line with slope 1 and $t(x) \to \infty$ as $x \to \infty$ and $t(x) \to -\infty$ as $x \to -\infty$.
- To see if the graph intersects the oblique asymptote solve $t(x) = x + 1$:

$$t(x) = x + 1 \iff (x-1)(x+2) = x(x+1) \iff -2 = 0$$

Therefore, there is no intercept of the graph with the oblique asymptote.
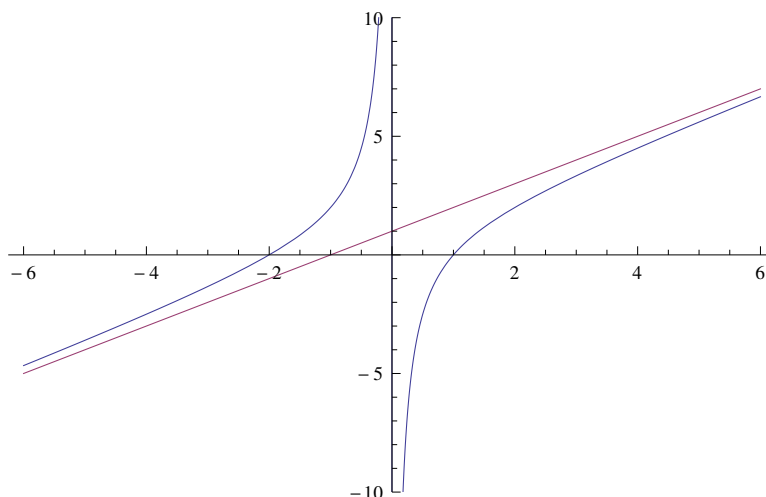
2. *Working backwards.* Find a possible rational function for each graph.
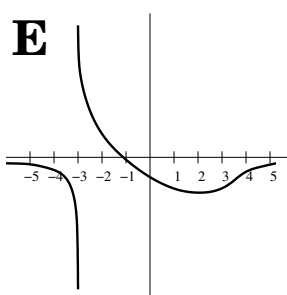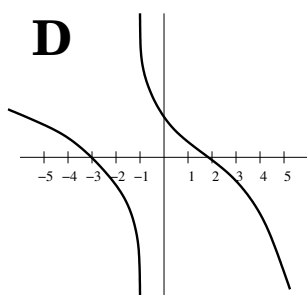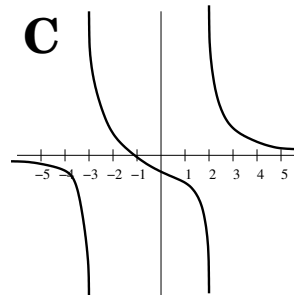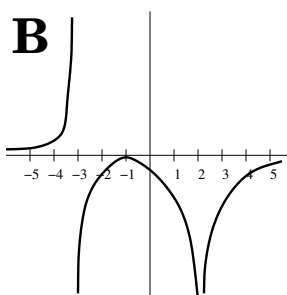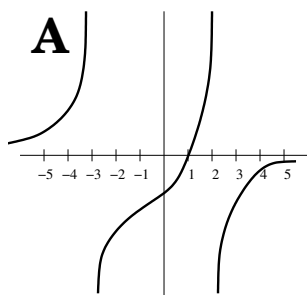
Answer 6.

(a) $-\dfrac{(x-1)}{(x-2)(x+3)}$

(b) $-\dfrac{(x+1)^2}{(x+3)(x-2)^2}$

(c) $\dfrac{x+1}{(x+3)(x-2)}$

(d) $-\dfrac{(x-2)(x+3)}{x+1}$

(e) $-\dfrac{x+1}{(x+3)(x^2+1)}$

- **Rules of Logarithms**

  1. $\log_a x = y \iff a^y = x$
  2. $a^{\log_a M} = M$
  3. $\log_a a = 1$
  4. $\log_a 1 = 0$
  5. $\log_a M^r = r \log_a M$
  6. $\log_a(M \cdot N) = \log_a M + \log_a N$
  7. $\log_a(\frac{M}{N}) = \log_a M - \log_a N$
  8. $\log_a M = \frac{\log_b M}{\log_b a}$

- **Common Mistakes**

  1. $\log_a(M - N) = \frac{\log_a M}{\log_a N}$
  2. $\log_a(M + N) = \log_a(MN)$
  3. $\frac{\log_b M}{\log_b a} = \frac{M}{a}$

- **Arrange from least to greatest**:

  1. $e, \ln e, \frac{1}{2}$
  2. $e^2, 1, \ln e^2$
  3. $\ln \frac{1}{e}, e^{-1}, 1$
  4. $4, \ln 4, e$

- **Simplify:**

  1. $\log_{10} 10^{-3}$
  2. $\log_3 27^{\frac{5}{3}}$
  3. $\log_2 2\sqrt{8}$
  4. $(\ln(e^2))^{-1}$
  5. $2^{\log_2 3} \cdot 3^{\log_3 2}$
  6. * $\log_2 3 \log_3 4 \log_4 8$
  7. * $e^{\log_{e^3} 27}$

- **Write as a sum or difference of logarithms without any exponents:**

  1. $\ln(x^2 - y^2)$
  2. $\log_2 \frac{3x^5}{y^8}$
  3. $\log_a \sqrt[5]{\frac{2x}{x^2-1}}$
  4. $\log_b \left(\sqrt[3]{\frac{1}{y^2}} \cdot \sqrt{\frac{x^2}{z}} \right)$

- **Combine into a single logarithm:**

  1. $\log_2 4x + \log_2 x + 2\log_2 x$

  2. $\frac{1}{3}\left[\ln 2 + \ln y - \ln y^2 - 4\ln y\right]$

  3. $\frac{1}{3}\log_a x^2 + \log_a \sqrt{x + y^2} - \log_a(x^2 + y)$

  4. $\frac{\ln x^3 + 1}{\ln 2} - \log_2(x^3 + 1)$ [Hint: Change of base.]

Sample Midterm	Sample Final			
	9	A B C D		
	13	A B C D		
	16	A B C D		
	23	A B C D		
	29	A B C D		
	40	A B C D		

- **Arrange from least to greatest**:

  1. $e, \ln e, \frac{1}{2}$

  $$\frac{1}{2} < \ln e = 1 < e \approx 2.718$$

  2. $e^2, 1, \ln e^2$

  $$1 < \ln e^2 = 2 < e^2 \approx 7.389$$

  3. $\ln \frac{1}{e}, e^{-1}, 1$

  $$\ln \frac{1}{e} = -1 < e^{-1} \approx 0.368 < 1$$

  4. $4, \ln 4, e$

  $$\ln 4 \approx 1.386 < e \approx 2.718 < 4$$

- **Simplify:**

  1. $\log_{10} 10^{-3} = -3$
  2. $\log_3 27^{\frac{5}{3}} = \log_3 (3^3)^{\frac{5}{3}} = \log_3 3^5 = 5$
  3. $\log_2 2\sqrt{8} = \log_2 2 \cdot 2\sqrt{2} = \log_2 2^{2+\frac{1}{2}} = \log_2 2^{\frac{5}{2}} = \frac{5}{2}$
  4. $\left(\ln(e^2)\right)^{-1} = 2^{-1} = \frac{1}{2}$
  5. $2^{\log_2 3} \cdot 3^{\log_3 2} = 3 \cdot 2 = 6$
  6. $\log_2 3 \log_3 4 \log_4 8$

  $$
  \begin{aligned}
  \log_2 3 \log_3 4 \log_4 8 &= \log_2 3(\log_3 4 \log_4 8) \\
  &= (\log_3 4 \log_4 8) \cdot \log_2 3 \\
  &= \log_2 3^{(\log_3 4 \log_4 8)} \\
  &= \log_2 3^{\log_3 4^{\log_4 8}} \\
  &= \log_2 8 \\
  &= 3
  \end{aligned}
  $$

  7. $e^{\log_{e^3} 27} = e^{\log_{e^3} 3^3} = e^{3 \log_{e^3} 3} = \left(e^3\right)^{\log_{e^3} 3} = 3$

- **Write as a sum or difference of logarithms without any exponents:**

  1. $\ln(x^2 - y^2) = \ln(x+y)(x-y) = \ln(x+y) + \ln(x-y)$
  2.

  $$
  \begin{aligned}
  \log_2 \frac{3x^5}{y^8} &= \log_2 3x^5 - \log_2 y^8 \\
  &= \log_2 3 + \log_2 x^5 - \log_2 y^8 \\
  &= \log_2 3 + 5\log_2 x - 8\log_2 y
  \end{aligned}
  $$

  3.

  $$
  \log_a \sqrt[5]{\frac{2x}{x^2-1}} = \frac{1}{5}\log_a \left(\frac{2x}{x^2-1}\right)
  $$

$$
\begin{aligned}
&= \frac{1}{5}[\log_a 2x - \log_a(x^2 - 1)] \\
&= \frac{1}{5}[\log_a 2x - \log_a(x+1)(x-1)] \\
&= \frac{1}{5}[\log_a 2 + \log_a x - (\log_a(x+1) + \log_a(x-1))] \\
&= \frac{1}{5}\log_a 2 + \frac{1}{5}\log_a x - \frac{1}{5}\log_a(x+1) - \frac{1}{5}\log_a(x-1)
\end{aligned}
$$

4.

$$
\begin{aligned}
\log_b\left(\sqrt[3]{\frac{1}{y^2}} \cdot \sqrt{\frac{x^2}{z}}\right) &= \frac{1}{3}\log_b \frac{1}{y^2} + \frac{1}{2}\log_b \frac{x^2}{z} \\
&= \frac{1}{3}\log_b y^{-2} + \frac{1}{2}[\log_b x^2 - \log_b z] \\
&= -\frac{2}{3}\log_b y + \frac{1}{2}[2\log_b x - \log_b z] \\
&= -\frac{2}{3}\log_b y + \log_b x - \frac{1}{2}\log_b z
\end{aligned}
$$

- **Combine into a single logarithm:**

1. $\log_2 4x + \log_2 x + 2\log_2 x = \log_2 4x + \log_2 x + \log_2 x^2 = \log_2(4x \cdot x \cdot x^2) = \log_2 4x^4$

2.

$$
\begin{aligned}
\frac{1}{3}[\ln 2 + \ln y - \ln y^2 - 4\ln y] &= \frac{1}{3}[\ln 2 + \ln y - (\ln y^2 + 4\ln y)] \\
&= \frac{1}{3}[\ln 2y - (\ln y^2 + \ln y^4)] \\
&= \frac{1}{3}[\ln 2y - \ln y^6] \\
&= \frac{1}{3}\ln\left(\frac{2y}{y^6}\right) \\
&= \ln\sqrt[3]{\frac{2y}{y^6}} \\
&= \ln\sqrt[3]{\frac{2}{y^5}}
\end{aligned}
$$

3.

$$
\begin{aligned}
\frac{1}{3}\log_a x^2 + \log_a \sqrt{x+y^2} - \log_a(x^2+y) &= \log_a x^{\frac{2}{3}} + \log_a \sqrt{x+y^2} - \log_a(x^2+y) \\
&= \log_a(x^{\frac{2}{3}} \cdot \sqrt{x+y^2}) - \log_a(x^2+y) \\
&= \log_a \frac{(x^{\frac{2}{3}} \cdot \sqrt{x+y^2})}{x^2+y}
\end{aligned}
$$

4. $\frac{\ln x^3 + 1}{\ln 2} - \log_2(x^3+1) = \log_2(x^3+1) - \log_2(x^3+1) = \log_2 1 = 0$

Whenever it is possible to work with a single base we do so, because then we may equate the coefficients.

Example 1. Solve $5^x = 125^{x+1}$

$$
\begin{aligned}
5^x &= 125^{x+1} \\
5^x &= (5^3)^{x+1} \\
5^x &= 5^{3(x+1)} \\
x &= 3(x+1) \\
x &= 3x + 3 \\
x &= -\frac{3}{2}
\end{aligned}
$$

What we have done really in line (4) was to secretly take the logarithm base 5 of both sides, i.e.

$$5^x = 5^{3(x+1)} \iff \log_5 5^x = \log_5 5^{3(x+1)} \iff x \log_5 5 = 3(x+1) \log_5 5 \iff x = 3(x+1)$$

This method allows us to solve equations where it is too difficult or impossible to reduce to a single base. In such an event we have to choose a base for the logarithm, and the most natural choice is the *natural* logarithm.

Example 2. Solve: $5^{2x+1} = 2^{3x+3}$

$$
\begin{aligned}
5^{2x+1} &= 2^{3x+3} \\
\ln 5^{2x+1} &= \ln 2^{3x+3} \\
(2x+1)\ln 5 &= (3x+3)\ln 2 \\
2x \ln 5 + \ln 5 &= 3x \ln 2 + 3 \ln 2 \\
2x \ln 5 - 3x \ln 2 &= 3 \ln 2 - \ln 5 \\
x(2 \ln 5 - 3 \ln 2) &= 3 \ln 2 - \ln 5 \\
x &= \frac{3 \ln 2 - \ln 5}{2 \ln 5 - 3 \ln 2}
\end{aligned}
$$

Sometimes neither of these approaches will work. The next example recalls an idea we have encountered before. It is an equation of quadratic type and the key step in solving it is a substitution of the from $y = a^x$.

Example 3. Solve: $3^{2x} - 4 \cdot 3^x + 4 = 0$

$$
\begin{aligned}
3^{2x} - 4 \cdot 3^x + 4 &= 0, \text{ substitute } y = 3^x \\
y^2 - 4y + 4 &= 0 \\
(y - 2)(y - 2) &= 0
\end{aligned}
$$

The solution is therefore $x = \frac{\ln 2}{\ln 3}$ because

$$y = 2 = 3^x \Rightarrow x = \frac{\ln 2}{\ln 3}$$

An essential step in solving these problems is to verify each solution. Note that $y = -2 = 3^x$ would not produce a solution, because the exponential function is always positive so there is no value of x for which $3^x \leq 0$.

- **Solve:**

  1. $3^x = 27$

  2. $3^x = 3^{2x+1}$

  3. $3^x = 9^{2x+1}$

  4. $a^y = a^4$

  5. $a^2 \cdot a^y = a^{1-y}$

  6. $e^{(x+1)(x-1)} = 1$

  7. $3^{x+1} = 7$

  8. $5^{2x} = 2^{x+3}$

  9. $3^{3x} = 7^{2x+1}$

  10. $* \ 2^{2x} + 2^x - 12 = 0$

  11. $* \ e^{4x} - 5e^{2x} + 6 = 0$

Sample Midterm	Sample Final			
	12	A	B	C	D
	26	A	B	C	D
	38	A	B	C	D

- **Solve:**

    1. $3^x = 27$

    $$3^x = 27 \iff 3^x = 3^3$$
    $$\iff x = 3$$

    2. $3^x = 3^{2x+1}$

    $$3^x = 3^{2x+1} \iff x = 2x + 1$$
    $$\iff x = -1$$

    3. $3^x = 9^{2x+1}$

    $$3^x = 9^{2x+1} \iff 3^x = (3^2)^{2x+1}$$
    $$\iff 3^x = 3^{4x+2}$$
    $$\iff x = 4x + 2$$
    $$\iff x = -\frac{2}{3}$$

    4. $a^y = a^4$

    $$a^y = a^4 \iff y = 4$$

    5. $a^2 \cdot a^y = a^{1-y}$

    $$a^2 \cdot a^y = a^{1-y} \iff a^{2+y} = a^{1-y}$$
    $$\iff 2 + y = 1 - y$$
    $$\iff y = -\frac{1}{2}$$

    6. $e^{(x+1)(x-1)} = 1$

    $$e^{(x+1)(x-1)} = 1 \iff \ln e^{(x+1)(x-1)} = \ln 1$$
    $$\iff (x+1)(x-1) = 0$$
    $$\iff x = \pm 1$$

    7. $3^{x+1} = 7$

    $$\begin{aligned} 3^{x+1} &= 7 \\ \ln 3^{x+1} &= \ln 7 \\ (x+1)\ln 3 &= \ln 7 \\ x \ln 3 + \ln 3 &= \ln 7 \\ x \ln 3 &= \ln 7 - \ln 3 \\ x &= \frac{\ln 7 - \ln 3}{\ln 3} \end{aligned}$$

8. $5^{2x} = 2^{x+3}$

$$
\begin{aligned}
5^{2x} &= 2^{x+3} \\
\ln 5^{2x} &= \ln 2^{x+3} \\
2x \ln 5 &= (x+3) \ln 2 \\
2x \ln 5 &= x \ln 2 + 3 \ln 2 \\
2x \ln 5 - x \ln 2 &= 3 \ln 2 \\
x(2 \ln 5 - \ln 2) &= 3 \ln 2 \\
x &= \frac{3 \ln 2}{2 \ln 5 - \ln 2}
\end{aligned}
$$

9. $3^{3x} = 7^{2x+1}$

$$
\begin{aligned}
3^{3x} &= 7^{2x+1} \\
\ln 3^{3x} &= \ln 7^{2x+1} \\
3x \ln 3 &= (2x+1) \ln 7 \\
3x \ln 3 &= 2x \ln 7 + \ln 7 \\
3x \ln 3 - 2x \ln 7 &= \ln 7 \\
x(3 \ln 3 - 2 \ln 7) &= \ln 7 \\
x &= \frac{\ln 7}{3 \ln 3 - 2 \ln 7}
\end{aligned}
$$

10. $2^{2x} + 2^x - 12 = 0$

$$
\begin{aligned}
2^{2x} + 2^x - 12 &= 0, \text{ substitute } y = 2^x \\
y^2 + y - 12 &= 0 \\
(y+4)(y-3) &= 0
\end{aligned}
$$

The solution is therefore $x = \log_2 3$ as $y = 3 = 2^x \Rightarrow x = \log_2 3$. There is no solution coming from $y = -4 = 2^x$, because the exponential function is always greater than zero and there is no value of x for which $2^x = -4$.

11. $e^{4x} - 5e^{2x} + 6 = 0$

$$
\begin{aligned}
e^{4x} - 5e^{2x} + 6 &= 0, \text{ substitute } y = e^{2x} \\
y^2 - 5y + 6 &= 0 \\
(y-3)(y-2) &= 0
\end{aligned}
$$

The solutions therefore come from the equations $y = 3 = e^{2x}$ and $y = 2 = e^{2x}$ both of which have solutions. Solving these we obtain two solutions to the original equation:

$$
x = \frac{1}{2}\ln 3 \text{ and } x = \frac{1}{2}\ln 2
$$

Last time we saw that equations involving exponential functions use the rules $\log_b(b^x) = x$ and $\log_b(a^x) = x \log a$. We used the first rule in the first example on the previous lesson and the second rule (in which the base of the log is not the same as the base of the exponential function) in the second example.

In equations involving logarithmic expressions, we use the rule $b^{\log_b x} = x$. The following examples should allow you to do the following exercises.

Examples

1.

$$\ln x = 3 \iff e^{\ln x} = e^3$$
$$\iff x = e^3.$$

2.

$$\log_4(x - 11) = 2 \iff {}^{\log_4(x-11)} = 4^2$$
$$\iff x - 11 = 16$$
$$\iff x = 27$$

3.

$$\log_{16} \frac{x+3}{x-1} = \frac{1}{2} \iff 16^{\log_{16} \frac{x+3}{x-1}} = 16^{1/2}$$
$$\iff \frac{x+3}{x-1} = 4$$
$$\iff x + 3 = 4(x - 1)$$
$$\iff x + 3 = 4x - 4$$
$$\iff 7 = 3x$$
$$\iff x = \frac{7}{3}$$

4.

$$\log_2(2x^2 - 4) = 5 \iff 2^{\log_2(2x^2 - 4)} = 2^5$$
$$\iff 2x^2 - 4 = 32$$
$$\iff 2x^2 - 36 = 0$$
$$\iff x^2 - 18 = 0$$
$$\iff (x + \sqrt{18})(x - \sqrt{18}) = 0$$
$$\iff x = \pm\sqrt{18} = \pm 3\sqrt{2}$$

and although you can't put a negative in a log, both of these work as it is $2x^2 - 4$ which is inputted into the log.

5.

$$\ln[\ln(\ln x)] = 1 \iff e^{\ln[\ln(\ln x)]} = e^1$$
$$\iff \ln(\ln x) = e$$
$$\iff e^{\ln(\ln x)} = e^e$$
$$\iff \ln x = e^e$$
$$\iff e^{\ln x} = e^{e^e}$$
$$\iff x = e^{e^e}$$

This can be a bit more difficult when the equations get more complicated. You need to remember carefully the exponential rules $a^b \cdot a^c = a^{b+c}$ and $(a^b)^c = a^{bc}$.

6.

$$\log_2(x+4) = 2 - \log_2(x+1) \iff 2^{\log_2(x+4)} = 2^{2-\log_2(x+1)}$$
$$\iff 2^{\log_2(x+4)} = 2^2 2^{-\log_2(x+1)}$$
$$\iff 2^{\log_2(x+4)} = 2^2 2^{\log_2(x+1)^{-1}}$$
$$\iff x + 4 = \frac{4}{x+1}$$
$$\iff (x+1)(x+4) = 4$$
$$\iff x^2 + 5x^4 = 4$$
$$\iff x^2 + 5x = 0$$
$$\iff x(x+5) = 0$$
$$\iff x = 0 \text{ or } x = -5$$

There is another way to do this question, though, using the log rules. You must check for extraneous (false) solutions and reject them. For example, $x = -5$ is not a valid solution, because if we substitute $x = -5$ into the original equation we are taking logarithms of negative numbers, which is not allowed. In other words, $x = -5$ lies outside of the domain of the function we are considering. When working with logs we should always do this but sometimes it can be too complicated as in the example following this one. The other way for this example is

$$\log_2(x+4) = 2 - \log_2(x+1) \iff \log_2(x+4) + \log_2(x+1) = 2$$
$$\iff \log_2[(x+4)(x+1)] = 2$$
$$\iff 2^{\log_2[(x+4)(x+1)]} = 2^2$$
$$\iff (x+4)(x+1) = 4$$
$$\iff x^2 + 5x + 4 = 4$$
$$\iff x^2 + 5x = 0$$
$$\iff x(x+5) = 0$$
$$\iff x = 0 \text{ or } -5$$

However, only 0 is an appropriate input to the the logaritms above. If one of your answers fails to go into any log of the original equation properly (i.e. putting a negative or a zero into a log), it must be rejected. So the only answer here is zero, as above.

7.

$$
\begin{aligned}
\ln(x+1) = 2 + \ln(x-1) &\iff e^{\ln(x+1)} = e^{2+\ln(x-1)} \\
&\iff e^{\ln(x+1)} = e^2 e^{\ln(x-1)} \\
&\iff x + 1 = e^2(x-1) \\
&\iff x + 1 = e^2 x - e^2 \\
&\iff e^2 + 1 = e^2 x - x \\
&\iff e^2 + 1 = x(e^2 - 1) \\
&\iff x = \frac{e^2+1}{e^2-1}
\end{aligned}
$$

In the following example we have a coefficient in front of the log that isn't a 1. We may still put these as exponents above the base of the log but it is easier if we deal with the coefficient first.

8.

$$
\begin{aligned}
\log_{10}(x+1) = 2\log_{10}(x-1) &\iff \log_{10}(x+1) = \log_{10}(x-1)^2 \\
&\iff 10^{\log_{10}(x+1)} = 10^{\log_{10}(x-1)^2} \\
&\iff x + 1 = (x-1)^2 \\
&\iff x + 1 = x^2 - 2x + 1 \\
&\iff 0 = x^2 - 3x \\
&\iff x(x-3) = 0 \\
&\iff x = 0 \text{ or } 3
\end{aligned}
$$

However, 0 can't be put into the right log so the only answer is 3.

Solve for x. Make sure your solutions are valid and belong to the domain of the logarithm given in the problem.

1. $\log_{10}(x+3) = 3$

2. $\ln(\ln x) = 2$

3. $\log_{10}(\log_{10} x) = 2$

4. $\log_2(\log_3 x) = -1$

5. $\log_3(x^2 - 2x) = 1$

6. $\log_6 x + \log_6(x+1) = 1$

7. $\log_6 x + \log_6(x+1) = 0$

8. $\log_9(x+1) = \frac{1}{2} + \log_9 x$

9. $2\log_9(2x-3) = 1$

Sample Midterm	Sample Final			
	14	A B C D
	37	A B C D
	39	A B C D

1.

$$\log_{10}(x+3) = 3 \iff 10^{\log_{10}(x+3)} = 10^3$$
$$\iff x + 3 = 1000$$
$$\iff x = 997$$

2.

$$\ln(\ln x) = 2 \iff e^{\ln(\ln x)} = e^2$$
$$\iff \ln x = e^2$$
$$\iff e^{\ln x} = e^{e^2}$$
$$\iff x = e^{e^2}$$

3.

$$\log_{10}(\log_{10} x) = 2 \iff 10^{\log_{10}(\log_{10} x)} = 10^2$$
$$\iff \log_{10} x = 100$$
$$\iff 10^{\log_{10} x} = 10^{100}$$
$$\iff x = 10^{100}$$

4.

$$\log_2(\log_3 x) = -1 \iff 2^{\log_2(\log_3 x)} = 2^{-1}$$
$$\iff \log_3 x = \frac{1}{2}$$
$$\iff 3^{\log_3 x} = 3^{1/2}$$
$$\iff x = \sqrt{3}$$

5.

$$\log_3(x^2 - 2x) = 1 \iff 3^{\log_3(x^2-2x)} = 3^1$$
$$\iff x^2 - 2x = 3$$
$$\iff x^2 - 2x - 3 = 0$$
$$\iff (x+1)(x-3) = 0$$
$$\iff x = -1 \text{ or } 3$$

Note that both solutions are valid.

6. Method 1:

$$\log_6 x + \log_6(x+1) = 1 \iff \log_6[x(x+1)] = 1$$
$$\iff 6^{\log_6[x(x+1)]} = 6^1$$
$$\iff x(x+1) = 6$$
$$\iff x^2 + x = 6$$
$$\iff x^2 + x - 6 = 0$$
$$\iff (x+3)(x-2) = 0$$
$$\iff x = -3 \text{ or } 2$$

However, only 2 may be inputted into both logs. Method 2:

$$\log_6 x + \log_6(x+1) = 1 \iff 6^{\log_6 x + \log_6(x+1)} = 6^1$$
$$\iff 6^{\log_6 x} 6^{\log_6(x+1)} = 6$$
$$\iff x(x+1) = 6 \ldots \text{ finish as in Method 1.}$$

7. Method 1:

$$\log_6 x + \log_6(x+1) = 0 \iff \log_6[x(x+1)] = 0$$
$$\iff 6^{\log_6[x(x+1)]} = 6^0$$
$$\iff x(x+1) = 1$$
$$\iff x^2 + x = 1$$
$$\iff x^2 + x - 1 = 0$$

Then using the quadratic formula we get

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$$

and only $x = \frac{-1+\sqrt{5}}{2}$ may be inputted into both logs. Method 2:

$$\log_6 x + \log_6(x+1) = 0 \iff 6^{\log_6 x + \log_6(x+1)} = 6^0$$
$$\iff 6^{\log_6 x} 6^{\log_6(x+1)} = 1$$
$$\iff x(x+1) = 1 \ldots \text{ finish as in Method 1.}$$

8.

$$\log_9(x+1) = \frac{1}{2} + \log_9 x \iff \log_9(x+1) - \log_9 x = \frac{1}{2}$$
$$\iff \log_9\left(\frac{x+1}{x}\right) = \frac{1}{2}$$
$$\iff 9^{\log_9\left(\frac{x+1}{x}\right)} = 9^{1/2}$$

$$\Longleftrightarrow \ \frac{x+1}{x} = 3$$
$$\Longleftrightarrow \ x + 1 = 3x$$
$$\Longleftrightarrow \ 1 = 2x$$
$$\Longleftrightarrow \ x = \frac{1}{2}$$

9.

$$
\begin{aligned}
2\log_9(2x-3) = 1 \ &\Longleftrightarrow \ \log_9(2x-3)^2 = 1 \\
&\Longleftrightarrow \ 9^{\log_9(2x-3)^2} = 9^1 \\
&\Longleftrightarrow \ (2x-3)^2 = 9 \\
&\Longleftrightarrow \ 4x^2 - 12x + 9 = 9 \\
&\Longleftrightarrow \ 4x^2 - 12x = 0 \\
&\Longleftrightarrow \ x^2 - 3x = 0 \\
&\Longleftrightarrow \ x(x-3) = 0 \\
&\Longleftrightarrow \ x = 0 \text{ or } 3
\end{aligned}
$$

However, 0 is not a valid answer, because $\log_9(2(0) - 3) = \log_9(-3)$ is not defined. The only solution is $x = 3$.

In our study of linear functions we learned to construct linear models, and we observed how inaccurate these models can be when applied to phenomena which naturally exhibit exponential growth. Recall the following examples:

1. Several fruit flies (*Drosophila melanogaster*) have found their way into your kitchen and plot to reproduce exponentially. You first count only 5, but after 2 days you find 15. Find a linear function $f(x)$ which expresses your kitchen fruit fly population at time x. After how many days can you expect to be overwhelmed by a swarm of no less than 100 flies?

2. The half-life of cocaine is 1 hour. Supposing that you have ingested the minimum lethal dose of 1.2 grams, how long will you feel the effects of the drug, i.e. how long before you metabolize all but the usual effective dose of 0.080 grams? Round your answer to the nearest minute.

We will now solve the same problems using the more appropriate exponential growth models. The exponential function

$$N(t) = N_0 e^{kt}$$

represents the amount of growth (or decay) of the initial amount N_0 at a time t. The constant k, called the growth (or decay) constant, must be specified or obtained from the conditions stated in the problem.

Let us consider the first problem. We begin by writing the exponential growth function and assigning names to variables. Let $N(t) = N_0 e^{kt}$ represent the number of fruit flies at time t measured in days. We know from the problem that $N_0 = 5$, thus we have the equation

$$N(t) = 5 e^{kt}$$

The growth constant k is yet unknown to us, but we do know that $N(2) = 15$; i.e., after 2 days there are 15 fruit flies. We now have the initial amount and one data point which is enough information to solve for k.

$$N(2) = 15 = 5e^{2k} \iff 3 = e^{2k} \tag{27}$$
$$\iff \ln 3 = 2k \tag{28}$$
$$\iff \frac{\ln 3}{2} = k \tag{29}$$

The final equation with all parameters is given by

$$N(t) = 5 e^{\frac{t \ln 3}{2}}$$

The question asked for a prediction of the time when the fruit fly population reaches 100 specimens. We must solve the equation $N(t) = 100$.

$$N(t) = 100 = 5e^{\frac{t \ln 3}{2}} \iff 20 = e^{\frac{t \ln 3}{2}} \tag{30}$$
$$\iff \ln 20 = \frac{t \ln 3}{2} \tag{31}$$
$$\iff 2\frac{\ln 20}{\ln 3} = t \tag{32}$$

Thus, after $2\frac{\ln 20}{\ln 3} \approx 5.454$ days there will be at least 100 fruit flies.

How does this model compare to the linear model we have already seen? Recall that the linear function modeling the same problem was given by $f(x) = 5x + 5$ and solving $f(x) = 100$ yields the answer of 19 days before 100 fruit flies are present. Someone following the linear model may believe there is no danger of a fruit fly outbreak, as 100 fruit flies may be easily destroyed over a period of nearly 20 days. In reality, $N(19) = 170459.78$, and so the more accurate exponential model predicts 170,460 flies!

The aim of this exercise is to review most of the key notions we have encountered in Math 135. Complete each exercise thoroughly and conscientiously and you will be that much more prepared for the final examination.

1. Precisely state the definition of a function.

2. Explain how to obtain the inverse of a function $f^{-1}(x)$ from a given function $f(x)$ in each of the following cases:

   (a) If the domain and range of f are specified and the action of f that takes an element form the domain to an element in the range is described in words. No graph or equation is provided.

   (b) If $f(x)$ is given by an algebraic expression, but the graph is not known.

   (c) If only the graph of $f(x)$ is provided.

3. Graph the functions $f(x) = e^x$ and $g(x) = \ln x$ on the same set of coordinates.

   (a) State the domain and range of f and g.

   (b) Check using function composition that f is the inverse of g.

4. Graph the function $\varphi(x) = e - e^{1-x}$ by listing and graphing the four transformations on the graph of f from Question 3. State the domain and range. Mark all intercepts and asymptotes.

5. Compute the inverse of $\varphi(x)$ from Question 4. List the four transformations on the graph of g from Question 3 that will produce $\varphi^{-1}(x)$.

6. On the same coordinate plane graph both $\varphi(x)$ and $\varphi^{-1}(x)$. Mark all intercepts and asymptotes.

7. Check that $\varphi \circ \varphi^{-1}(x) = x$ and also that $\varphi^{-1} \circ \varphi(x) = x$, that the domain of φ^{-1} is the range of φ, and that the graphs of φ and φ^{-1} are reflections of one another with respect to the line $y = x$.

Sample Midterm | Sample Final
15   A   B   C   D

1. Given the graph of a line write its equation in point-slope and slope-intercept forms.

2. Express a shaded region of the number line in interval notation.

3. Determine the radius and center of a circle from its equation in standard form.

4. Given an equation of a line and a point in the plane, find the equation of a parallel line passing through that point.

5. Given an equation of a line and a point in the plane, find the equation of a perpendicular line passing through that point.

6. Find the domain of a function containing radicals.

7. Find the domain of a rational function containing radicals.

8. Decide if a point lies on the graph of a function given the point and the function equation.

9. Find the domain of a function from its graph.

10. Determine the number of distinct real solution of a quadratic equation.

11. Solve a quadratic equation.

12. Determine the x- and y-intercepts of a function given its equation.

13. Find the distance between two points in the plane.

14. Complete the square.

15. Determine if a correspondence is a function.

16. Given a graph of a relation determine whether it is a function.

17. Solve a word problem using a linear function.

18. Given the graph of a function determine where the relative maxima and minima occur.

19. Given a piecewise function determine its value at a point.

20. Compose two functions.

21. Find the domain of a composition of functions.

22. Determine the zeros of the function from its graph.

23. Solve an equation containing radicals.

24. Given two points find the equation of a line that passes through them.

25. Simplify a rational expression.

26. Given a general equation of a circle find its center and radius.

27. Compute the difference quotient of a function.

28. Express a function as a composition of two simpler functions.

29. Given two points in the plane find the coordinates of their midpoint.

30. Compute the distance between points on a number line.

31. Simplify an expression involving rational and integer exponents.

32. Solve a (compound) linear inequality and express the answer in interval notation.

33. Express an interval on the number line using absolute value notation.

34. Solve an inequality using the key number method.

35. Solve an inequality given its graph.

1. Write the equation of this line in point-slope and slope-intercept forms. (Integer points have been emphasized with a dot)
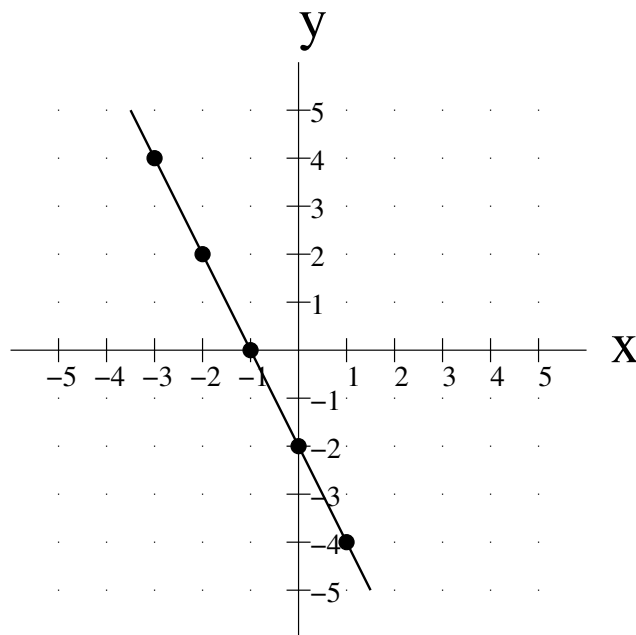
2. Express the shaded region as the union of two intervals. Explain why the shaded region cannot be represented as the intersection of two intervals. Finally, suppose that $x = -3$ is included in the region and express this new region using absolute value notation, i.e. find c and d such that $|x-c| \geq d$ represents the region $(-\infty, -3] \cup [5, \infty)$.
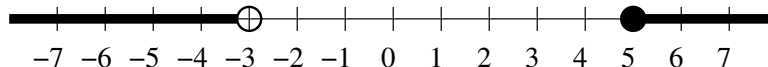
3. Determine the radius and center of a circle from its equation:

$$2x^2 + 4x + 2y^2 = 0$$

Does the point $(0,0)$ lie on the circle? Is the point $(2,1)$ an x-intercept? Find the y-intercepts.

4. Find the equation of the line perpendicular to the line pictured in problem 1 and passing through its x-intercept.

5. Find the domain of

$$g(x) = \frac{3\sqrt{3-x}}{2 - \sqrt{x+1}}$$

6. Find the domain of
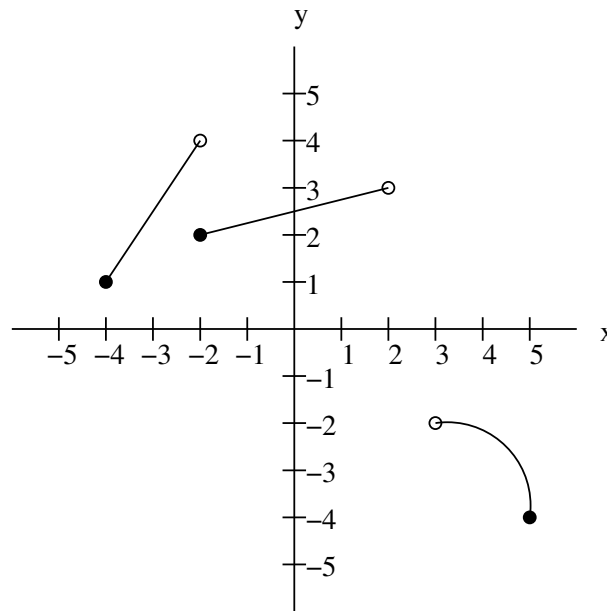
7. How many distinct real roots does this quadratic equation have?
   (Hint: compute the discriminant)

$$x^2 + 3x = \frac{7}{4}$$

8. Compute the distance between $(-1, 2)$ and $(5, 3)$ and find the coordinates of the midpoint of these two points.

9. Write the equation in problem 7 in the form $a(x - k)^2 = h$; that is, complete the square.

10. Which of these are functions?
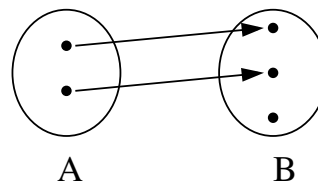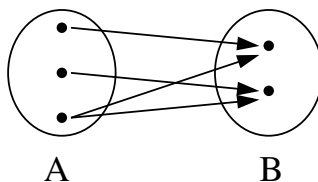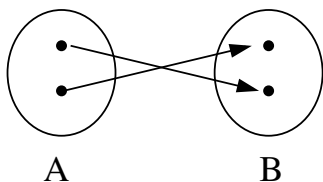
11. Which of these are functions?
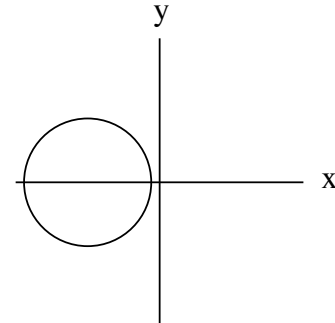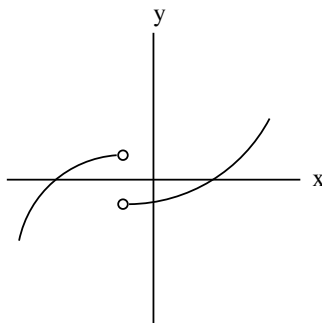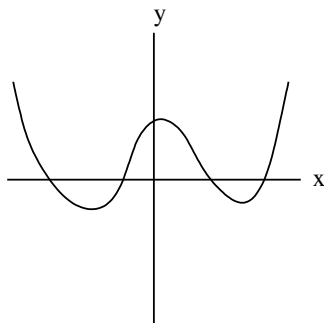
12. Several fruit flies (*Drosophila melanogaster*) have found their way into your kitchen and plot to reproduce exponentially. You first count only 5, but after 2 days you find 15. Find a linear function $f(x)$ which expresses your kitchen fruit fly population at time x. After how many days can you expect to be overwhelmed by a swarm of no less than 100 flies?

13. The half-life of cocaine is 1 hour. Supposing that you have ingested the minimum lethal dose of 1.2 grams, how long will you feel the effects of the drug, i.e. how long before you metabolize all but the usual effective dose of 0.080 grams? Round your answer to the nearest minute.

14. Pictured is the graph of $f(x)$. Find the x-coordinates of the relative maxima. Is there a global minimum? If so, at which x value does it occur? Is there a global maximum? If so, at which x value does it occur?
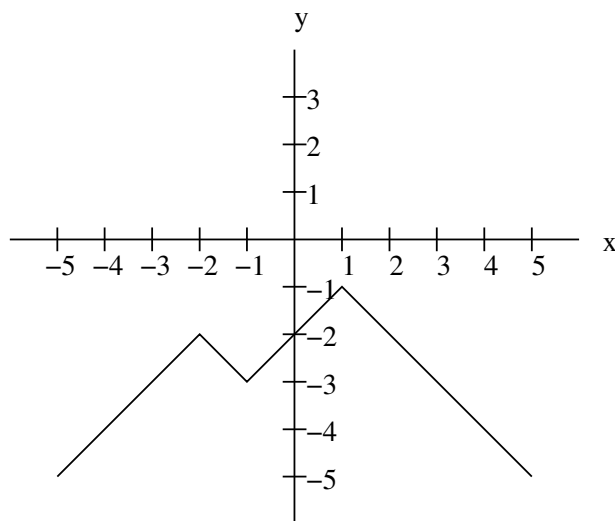
15. What is $h(3)$?

$$h(x) = \begin{cases} -3, & -3 \le x < 0; \\ x^3, & 0 < x < 3; \\ x, & 3 \le x < 10. \end{cases}$$

Is $h(x)$ a function? Find its domain and range.

16. Let $f(x) = \sqrt{3x^2 - 11}$ and $g(x) = x^3$. Compute $f(g(x))$ and $g \circ f(x)$. Compute the difference quotient of $g(x)$.

17. Solve

$$x - 3 = \sqrt{x} - 1$$

(Hint: remember to check your answers)

18. Simplify:

$$\frac{\frac{1}{x+y} - \frac{x+y}{y^2}}{\frac{1}{x} + \frac{1}{2y}}$$

19. Write

$$f(x) = \frac{x^3}{\sqrt{1 - x^3}}$$

as a composition of two or more simpler functions.

20. Solve and express the answer in interval notation:

$$-5 \le -x + 4 \le 11$$

21. Solve using the key number method:

$$\frac{4x}{(x-1)(x+3)} > 0$$

22. Given is the graph of

$$f(x) = \frac{4x}{(x-1)(x+3)}$$
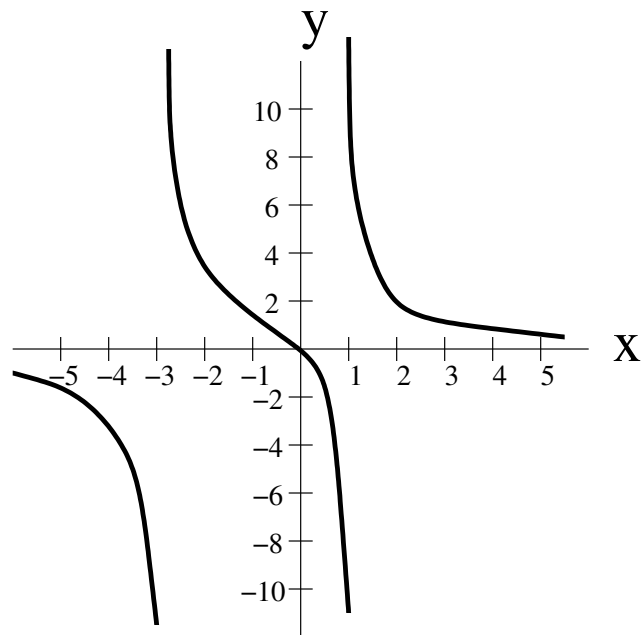
Use the graph of $f(x)$ to verify your answer to the previous problem.

Name:

UH ID:

Directions: Print your name and ID number. Clearly mark your answers. You may write on the exam, but will only be graded on the answer you mark. If you need to change answers, please completely erase or cross out the old answer and write and circle your final choice.

The passing score is 26.

Answers:
1.D	2.C	3.B	4.C	5.B	6.D	7.D	8.C	9.E	10.B
11.E	12.C	13.E	14.E	15.B	16.C	17.B	18.E	19.B	20.B
21.D	22.B	23.D	24.D	25.E	26.C	27.C	28.B	29.E	30.A
31.C	32.A	33.E	34.B	35.C	36.B	37.C	38.E	39.A	

What is an equation for the line graphed here?
(Points with integer coordinates have been emphasized with a dot.)

(A) $y = 3x + 4$

(B) $y = -4x + 3$

(C) $y = \frac{3}{4}x + 4$

(D) $y = -\frac{3}{4}x + 3$

(E) $y = -\frac{4}{3}x + 3$
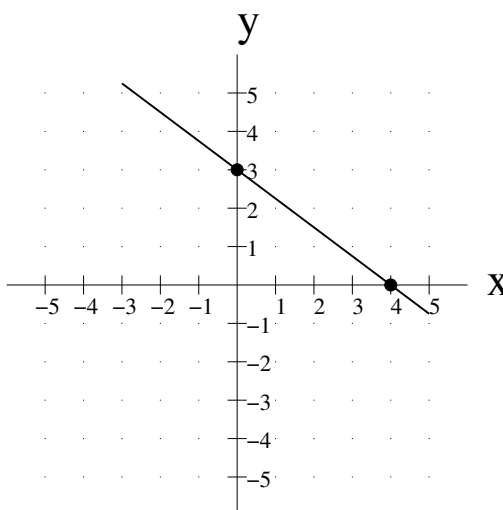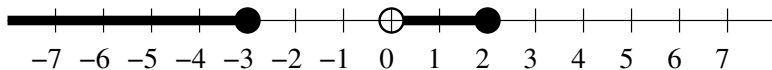
The intervals shown on this number line can be expressed in interval notation as:

(A) $[-3, \infty) \cup (0, 2]$

(B) $(-\infty, -3] \cap (0, 2]$

(C) $(-\infty, -3] \cup (0, 2]$

(D) $(-\infty, -3) \cup [0, 2]$

(E) $(-\infty, -3] \cap (0, 2]$

Here are three statements about the circle which is
the graph of the equation $(x + 3)^2 + (y - 2)^2 = 6$.

  I. The circle has center $(3, -2)$.
 II. The circle has radius $\sqrt{6}$.
III. The point $(0, 0)$ is on the circle.

Which of the three statements are true?

(A) Only I   (B) Only II   (C) Only III

(D) Only I and II   (E) Only I and III

Which of the following is a correct equation for the line that
is parallel to $3x + 6y = 9$ and passes through the point $(4, -1)$?
(A) $y + 1 = \frac{1}{2}(x - 4)$

(B) $y - 1 = \frac{1}{2}(x + 4)$

(C) $y + 1 = -\frac{1}{2}(x - 4)$

(D) $y + 1 = 2(x - 4)$

(E) $y - 1 = -2(x - 4)$

What is the domain of the function $f(x) = 4\sqrt{x - 1} + 3$?

(A) $x \neq 1$ (B) $x \geq 1$ (C) $x \geq 0$

(D) $x < 0$ (E) $x \geq 3$

Which of the these three equations have a graph that passes through the point $(-2, -1)$?

   I. $y = -1$
  II. $x^2 = x + 2y + 4$
 III. $y = 3x + 5$

(A) Only I   (B) Only II   (C) Only III

(D) Only I and III (E) All three

The graph of the function $f(x)$ is shown here. What is its domain?

(A) $(-1, 3)$

(B) $(-3, -1) \cup (-1, 2)$

(C) $(-1, 1) \cup (1, 3)$

(D) $(-3, 1) \cup (1, 2]$

(E) $(-3, 2]$
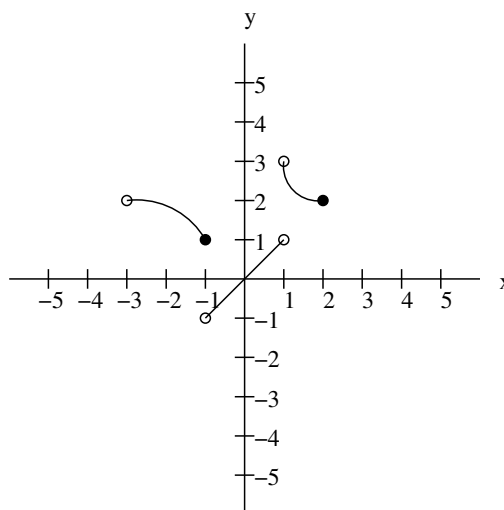
How many distinct, real-number solutions does
the equation $x^2 + 3x - 1 = -x - 2$ have?

(A) 0   (B) 1   (C) 2

(D) 3   (E) 4

Here are three statements about the graph of
the equation $y = x^2 - 9x + 18$:

  I. $(6, 0)$ is an x-intercept of the graph.
 II. $(-2, 0)$ is an x-intercept of the graph.
III. $(0, 18)$ is the only y-intercept of the graph.

Which of the three statements are true?

(A) Only I   (B) Only II   (C) Only III

(D) Only I and II   (E) Only I and III

What is the distance between the points $(4, 0.5)$ and $(5, 3.5)$?
(Radical expressions have not been reduced).

(A) 8   (B) $\sqrt{10}$ (C) 6

(D) $\sqrt{6}$ (E) 10

Which of the following is the correct result of completing the square on the expression $x^2 + 16x - 10$?

(A) $(x+8)^2 - 54$ (B) $(x+4)^2 - 6$ (C) $(x-8)^2 - 54$

(D) $(x-4)^2 - 6$ (E) None of these are correct

Shown here are three correspondences between domain sets A and range sets B. Which of these correspondences are functions?

I.   **II.**

A   B   A   B
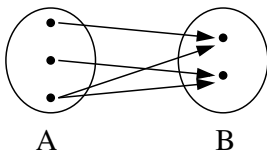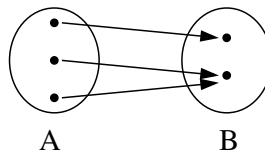
III. $A = [-4, 4]$, $B = [-4, 4]$ and an element x in A corresponds to an element y in B if $x^2 + y^2 = 4$.

(A) None of them   (B) Only I   (C) Only II

(D) Only III   (E) Only II and III

Shown here are the graphs of three relations in x and y. For which of these relations is y a function of x?

I.

II.

III.
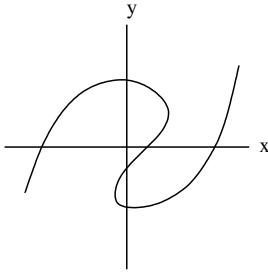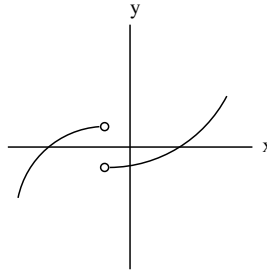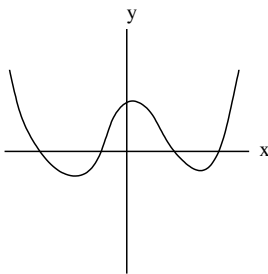
(A) Only I   (B) Only II   (C) Only III

(D) Only I and III (E) Only II and III

The population P of a certain bacterial colony is a linear function of time t. If the population was 300 at 1:00 PM and 1500 at 5:00 PM, what was the population at 3:00 PM?

(A) 1400   (B) 1000   (C) 800

(D) 1200   (E) 900

Find the x-values at which relative minima occur in the graph of $f(x)$ shown here.

(A) -1

(B) 2

(C) -1 and 2

(D) -1 and 3

(E) -1, 2, and 3
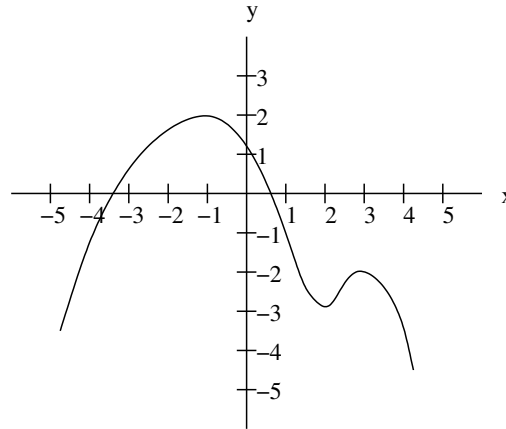
Define the piecewise function $h(x)$ as:

$$h(x) = \begin{cases} x^2 & \text{for } x < 0 \\ 2 - \sqrt{x} & \text{for } 0 \le x \le 4 \\ x + 1 & \text{for } x > 4 \end{cases}$$

What is $h(0)$?

(A) $h(0) = 0$ (B) $h(0) = 1$ (C) $h(0) = 2$

(D) $h(0) = \sqrt{2}$ (E) $h(0) = 4$

Solve the inequality $4x - 3 \geq 11 + 2x$.

(A) $x \leq \dfrac{7}{3}$ (B) $x \geq 7$ (C) $x \leq 7$

(D) $x \geq \dfrac{7}{3}$ (E) $x \leq 8$

Let $f(x) = 9 - x^2$ and $g(x) = -\sqrt{x} + 2$. What is the domain of $(g \circ f)(x) = -\sqrt{9 - x^2} + 2$?

(A) $[2, \infty)$ (B) $(2, \infty)$ (C) $(-\infty, \infty)$

(D) $(-3, 3)$ (E) $[-3, 3]$

Let $f(x) = 2x$ and $g(x) = \sqrt{x} - 3x$. What is $f(g(x))$?

(A) $2\sqrt{x} - 3x$ (B) $2\sqrt{x} - 6x$ (C) $2x(\sqrt{x} - 3x)$

(D) $\sqrt{2x} - 3x$ (E) $\sqrt{2x} - 6x$

Shown here is the graph of

$$y = x^3 + 4x^2 - 3x - 18$$

Solve the inequality:

$$x^3 + 4x^2 - 3x - 18 \leq 0$$

(A) $x \leq -3$ (B) $x \leq 2$ (C) $x \leq -3$ or $x \geq 2$

(D) $x \geq 2$ (E) $-3 \leq x \leq 2$

Simplify: $\dfrac{(x^{2/3}\, y)^2}{x^{3/2}\, y^4}$

(A) $x^8\, y^{-2}$ (B) $x^{-1/6}\, y^6$ (C) $x^{-1/6}\, y^{-2}$

(D) $x^{16}\, y^{-2}$ (E) $x^{7/6}\, y^{-1}$

Shown here are two numbers, A and B, on a number line where each mark represents 1 unit.

A   B
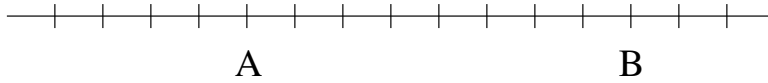
What is $|A - B|$?

(A) -8   (B) 8   (C) -7

(D) 7   (E) It cannot be determined

−7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7
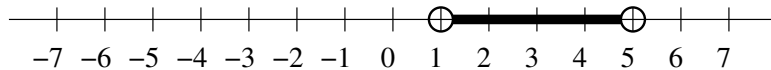
The interval shown on this number line can be expressed as:

(A) $|x + 3| < 2$ (B) $|x - 1| < 4$ (C) $|x - 2| < 3$

(D) $|x - 3| < 2$ (E) $|x - 3| < 3$

The graph of $g(x)$ is shown here. What are the zeros of $g(x)$?

(A) $-1, 0, 2, 4$

(B) $-1, 1, 4$

(C) $-1, 0$

(D) $-1, 4$

(E) -1
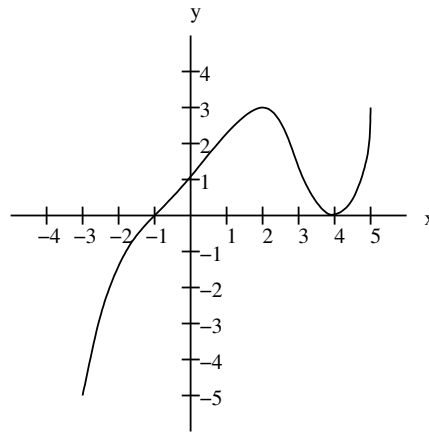
Shown here is the graph of $f(x) = \dfrac{1}{x}$. What are the coordinates of point P ?

(A) $(1, 3)$

(B) $\left(\frac{1}{3}, \frac{2}{3}\right)$

(C) $(3, 3)$

(D) $(3, 1)$

(E) $\left(3, \frac{1}{3}\right)$
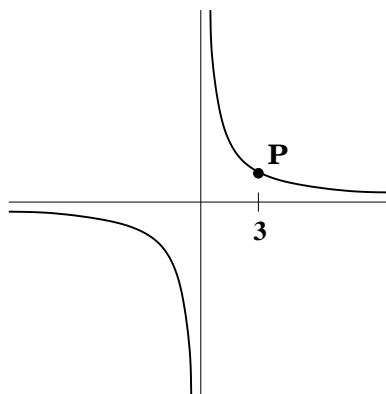
Solve the inequality $\frac{(x-1)(x+2)}{x-5} \leq 0$.

(A) $-2 \leq x < 5$ (B) $-2 \leq x \leq 1$

(C) $x \leq -2$ or $1 \leq x < 5$ (D) $-2 \leq x \leq 1$ or $x > 5$

(E) $x \leq -2$ or $x > 5$

Find the midpoint of the line segment between the points $(-3,\ -4)$ and $(7,\ 8)$.

(A) $(5,\ 6.5)$ (B) $(4,\ 5)$ (C) $(2,\ 2)$

(D) $(2,\ 2.5)$ (E) $(3,\ 2)$

Find functions f and g so that $(g \circ f)(x) = \sqrt{x^2 - 4} + 1$.

(A) $f(x) = \sqrt{x} + 1$ and $g(x) = x^2 - 4$

(B) $f(x) = x^2 - 4$ and $g(x) = \sqrt{x} + 1$

(C) $f(x) = \sqrt{x^2 - 4}$ and $g(x) = 1$

(D) $f(x) = \sqrt{x - 4}$ and $g(x) = x^2 + 1$

(E) $f(x) = x^2 + 1$ and $g(x) = \sqrt{x - 4}$

Find all real numbers that solve the equation
$x^2 + 2x + 1 = -6$.

(A) -4 and -2

(B) $-1 + 2\sqrt{2}$ and $-1 - 2\sqrt{2}$

(C) $1 + 2\sqrt{2}$ and $1 - 2\sqrt{2}$

(D) $-1 + 2\sqrt{-2}$ and $-1 - 2\sqrt{-2}$

(E) There are no real number solutions

Compute the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ for

$f(x) = x^2$, $x = 3$, and $h = \frac{1}{2}$.

(A) $\frac{13}{2}$ (B) $\frac{26}{4}$ (C) $\frac{1}{2}$

(D) 1 (E) $\frac{9}{2}$

Find the coordinates of the center and the radius of the
circle $x^2 + y^2 - 10x + 2y = 10$.

(A) Center $(-5, 1)$ Radius 6

(B) Center $(-5, 1)$ Radius 10

(C) Center $(5, -1)$ Radius 6

(D) Center $(5, -1)$ Radius 10

(E) Center $(5, -1)$ Radius $\sqrt{10}$

How many x-intercepts does the graph of the parabola
$y = -2x^2 + 2x - 5$ have?

(A) 0   (B) 1   (C) 2

(D) 3   (E) 4

Simplify $\dfrac{x + 1}{\sqrt{x} + 1}$.

(A) x (B) $x + 1$ (C) $\sqrt{x} - 1$

(D) $\sqrt{x} + 1$ (E) $\dfrac{(x+1)(\sqrt{x}-1)}{x-1}$

Find all values of x that solve the compound inequality:

$$-2 \leq -10 + 4x \leq 6$$

(A) $-2 \leq x \leq -4$ (B) $2 \leq x \leq 4$

(C) $x \leq -4$ or $x \geq -2$ (D) $x \leq 2$ or $x \geq 4$

(E) $-4 \leq x \leq -2$

Find all real number solutions to the equation
$x - 4\sqrt{x} - 5 = 0$.

(A) $x = 1$ and $x = 25$

(B) $x = \frac{1}{5}$ and $x = 2$

(C) $x = 25$

(D) $x = \sqrt{5}$

(E) There are no real number solutions

Find the equation of the line that passes through the
points $(-2, 3)$ and $(1, -5)$.

(A) $y = -\frac{3}{8}x + \frac{9}{4}$

(B) $y = -\frac{8}{3}x - \frac{7}{3}$

(C) $y = -\frac{3}{8}x - \frac{37}{8}$

(D) $y = \frac{8}{3}x + \frac{25}{3}$

(E) $y = \frac{3}{8}x - \frac{43}{8}$

What is the domain of the function $\dfrac{\sqrt{16 - x}}{\sqrt{x} - 4}$?

(A) $[0, \infty)$

(B) $[0, 16) \cup (16, \infty)$

(C) $[0, 16)$

(D) $(-\infty, 16]$

(E) $(-\infty, 16) \cup (16, \infty)$

Perform the addition: $\dfrac{x-5}{x+2} + \dfrac{5}{x}$

(A) $\dfrac{6x-5}{x^2-2x}$ (B) $\dfrac{x^2+10}{2x+2}$ (C) $\dfrac{6x-5}{2x-2}$

(D) $\dfrac{x^2+12x-4}{x^2-4x+1}$ (E) $\dfrac{x^2+10}{x^2+2x}$

Simplify: $\dfrac{\dfrac{1}{x^2}-\dfrac{1}{y}}{\dfrac{1}{x}+\dfrac{1}{y^2}}$

(A) $\dfrac{y^2-x^2y}{x^2+xy^2}$ (B) $\dfrac{x+y^2}{x^2+y}$ (C) $\dfrac{y}{x}$

(D) $\dfrac{1}{x^3+y^3}$ (E) $\dfrac{1}{(x^2+y)(x+y^2)}$

Name:

UH ID:

Directions: Print your name and ID number. Clearly mark your answers. You may write on the exam, but will only be graded on the answer you mark. If you need to change answers, please completely erase or cross out the old answer and write and circle your final choice.

The passing score is 26.

Answers:
1.A	2.D	3.E	4.A	5.A	6.B	7.D	8.C	9.E	10.E
11.A	12.E	13.D	14.C	15.C	16.B	17.D	18.A	19.D	20.E
21.C	22.E	23.A	24.B	25.A	26.E	27.E	28.C	29.C	30.E
31.C	32.C	33.B	34.C	35.E	36.A	37.C	38.B	39.E	

What is an equation for the line graphed here?
(Points with integer coordinates have been emphasized with a dot.)

(A) $y = \frac{2}{5}x + 1$

(B) $y = -\frac{2}{5}x + 1$

(C) $y = -\frac{5}{2}x + 1$

(D) $y = -\frac{5}{2}x - 1$

(E) $y = \frac{5}{2}x + 1$
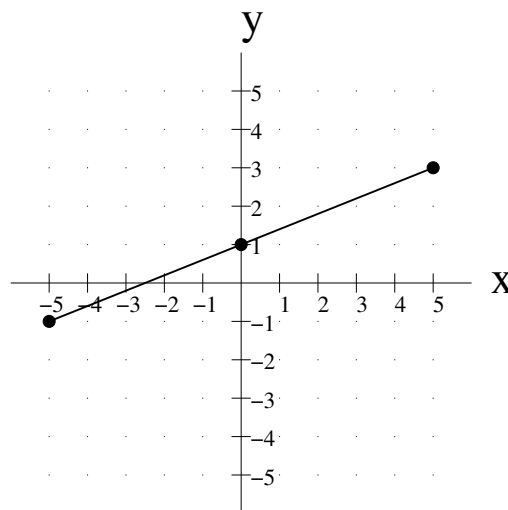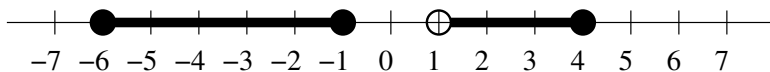
The intervals shown on this number line can be expressed in interval notation as:

(A) $[-6, -1) \cup (1, 4]$

(B) $(-6, 4] \cap (-1, 1]$

(C) $[-6, 4] \cup [-1, 1)$

(D) $[-6, -1] \cup (1, 4]$

(E) $(-6, -1] \cap [1, 4)$

Here are three statements about the circle which is
the graph of the equation $(x + 2)^2 + (y - 3)^2 = 13$.

  I. The circle has center $(-2, 3)$.
 II. The circle has radius 13.
III. The point $(1, 1)$ is on the circle.

Which of the three statements are true?

(A) Only I   (B) Only II   (C) Only III

(D) Only I and II   (E) Only I and III

Which of the following is a correct equation for the line that
is parallel to $-4x + y = 8$ and passes through the point $(-5, 6)$?
(A) $y - 6 = 4(x + 5)$

(B) $y - 6 = -\frac{1}{4}(x + 5)$

(C) $y + 6 = 4(x - 5)$

(D) $y + 6 = 4(x + 5)$

(E) $y + 6 = -\frac{1}{4}(x - 5)$

What is the domain of the function $f(x) = -2\sqrt{x + 4} - 5$?

(A) $x \geq -4$ (B) $x \geq -5$ (C) $x \geq 5$

(D) $x > -4$ (E) $x < -4$

Which of the these three equations have a graph that passes through the point $(-1, -1)$?

  I. $y = (x - 2)^2 + 4$
 II. $y + 4 = \frac{1}{2}(x + 7)$
III. $\sqrt{x^2 + x + 2} = y - 5$

(A) Only I   (B) Only II   (C) Only III

(D) Only I and II   (E) All three

The graph of the function $f(x)$ is shown here. What is its domain?

(A) $[-4, 2)$

(B) $[-4, 5]$

(C) $[-4, 4)$

(D) $[-4, 2) \cup (3, 5]$

(E) $[-4, -2) \cup [1, 4)$
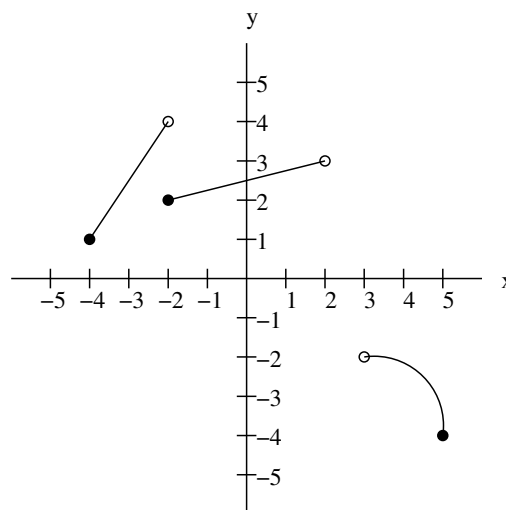
How many distinct, real-number solutions does
the equation $x^2 - 4x + 4 = 2$ have?

(A)　0　　　　　(B)　1　　　　　(C)　2

(D)　3　　　　　(E)　4

Here are three statements about the graph of
the equation $y = x^2 - 3x - 10$:

　I.　$(1, 0)$ is an x-intercept of the graph.
　II.　$(-2, 0)$ is an x-intercept of the graph.
III.　$(0, -10)$ is the only y-intercept of the graph.

Which of the three statements are true?

(A)　Only I　　　　　(B)　Only II　　　　　(C)　Only III

(D)　Only I and II　(E)　Only II and III

What is the distance between the points $(1, -5)$ and $(2, 1)$?

(A)　3　　　　　(B)　7　　　　　(C)　3.5

(D)　5　　　　　(E)　$\sqrt{37}$

Which of the following is the correct result of completing the square on the expression $x^2 + 2x + 4$?

(A) $(x+1)^2 + 3$ (B) $(x+1)^2 + 2$ (C) $(x+2)^2 - 1$

(D) $(x+2)^2 + 3$ (E) None of these are correct

Shown here are three correspondences between domain sets A and range sets B. Which of these correspondences are functions?

I.

II.

A   B   A   B
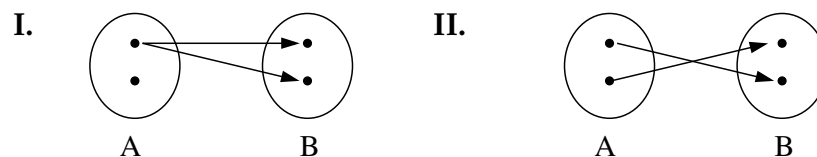
III. $A =$ the set of people in the world and $B =$ the set of days in the year. The correspondence assigns each person their birthday.

(A) None of them   (B) Only I   (C) Only II

(D) Only III   (E) Only II and III

Shown here are the graphs of three relations in x and y. For which of these relations is y a function of x?

I.

II.

III.
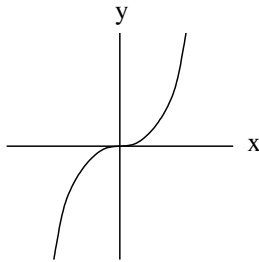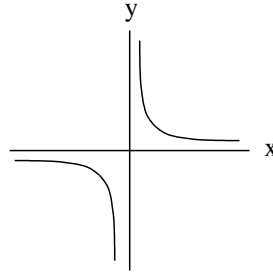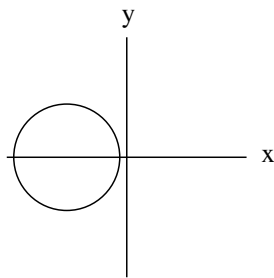
(A) Only I   (B) Only II   (C) Only III

(D) Only I and II   (E) Only I and III

The population P of a certain bacterial colony is a linear function of time t. If the population was 175 at 2:00 PM and 300 at 7:00 PM, what was the population at 5:00 PM?

(A) 225   (B) 230   (C) 250

(D) 260   (E) 275

Find the x-values at which relative minima occur in the graph of $f(x)$ shown here.

(A) 2

(B) 3

(C) $-2, 3$

(D) $-2, 0, 3$

(E) 0
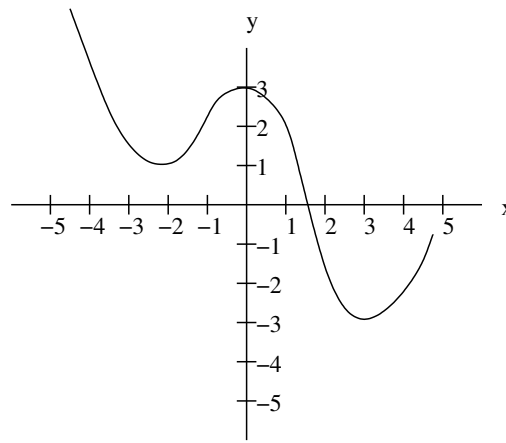
Define the piecewise function $h(x)$ as:

$$h(x) = \begin{cases} \dfrac{1}{x+1} & \text{for } x \leq -2 \\ 2x + 5 & \text{for } -2 < x \leq 3 \\ x^2 - 1 & \text{for } x > 3 \end{cases}$$

What is $h(-2)$?

(A) $h(-2) = \frac{1}{3}$ (B) $h(-2) = -1$ (C) $h(-2) = 3$

(D) $h(-2) = -2$ (E) $h(-2) = 1$

Solve the inequality $\frac{1}{2}x - 5 < 8 + 9x$.

(A) $x > -\dfrac{13}{8}$ (B) $x < \dfrac{13}{9}$ (C) $x > 2$

(D) $x > -\dfrac{26}{17}$ (E) $x < 12$

Let $f(x) = 9 - x^2$ and $g(x) = \sqrt{x} - \sqrt{2}$. What is the domain of
$(f \circ g)(x) = 9 - (\sqrt{x} - \sqrt{2})^2$?

(A) $[0, \infty)$ (B) $(0, \infty)$ (C) $(-\infty, \infty)$

(D) $(-\sqrt{2}, \sqrt{2})$ (E) $[-\sqrt{2}, \sqrt{2}]$

Let $f(x) = 7x$ and $g(x) = 7x + \sqrt{x}$. What is $g(f(x))$?

(A) $7x + \sqrt{x}$ (B) $7(x + \sqrt{x})$ (C) $7x + \sqrt{7x}$

(D) $49x + \sqrt{7x}$ (E) $7x + 7\sqrt{7x}$

Shown here is the graph of

$$y = \frac{4x}{(x-1)(x+3)}$$

Solve the inequality:

$$\frac{4x}{(x-1)(x+3)} > 0$$
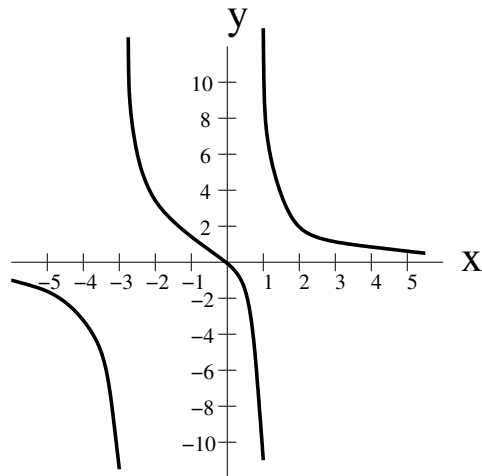
(A) $x > -3$ (B) $0 < x < 1$ (C) $x < -3$ or $x > 1$

(D) $x > 1$ (E) $-3 < x < 0$ or $x > 1$

Simplify: $\dfrac{x^{2/3}\, y^{3/2}\, x^{-3}}{(x^2\, y^3)^2}$

(A) $x^{-20/3}\, y^{-9/2}$ (B) $x^{-5}\, y^{-7/2}$ (C) $x^{-14/3}\, y$

(D) $x^{-19/3}\, y^{-9/2}$ (E) $x^{8/3}\, y^{-15/2}$

Shown here are two numbers, A and B, on a number line where each mark represents 1 unit.

A   B
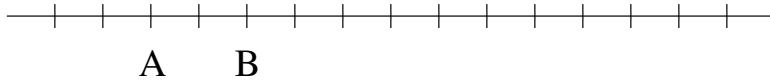
What is $|A - B|$?

(A) 3   (B) 5   (C) -2

(D) 1   (E) 2

−7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7
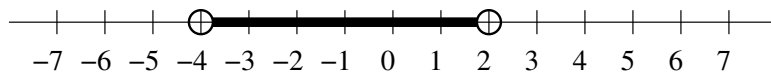
The interval shown on this number line can be expressed as:

(A) $|x + 1| < 3$ (B) $|x - 1| < 3$ (C) $|x - 4| < 6$

(D) $|x + 4| < 6$ (E) $|x - 6| < 1$

The graph of $g(x)$ is shown here. What are the zeros of $g(x)$?

(A) $-1, 0, 2, 4$

(B) $-3, 0, 4$

(C) $-3, 4$

(D) $-3, -1, 4$

(E) $-1, 2$
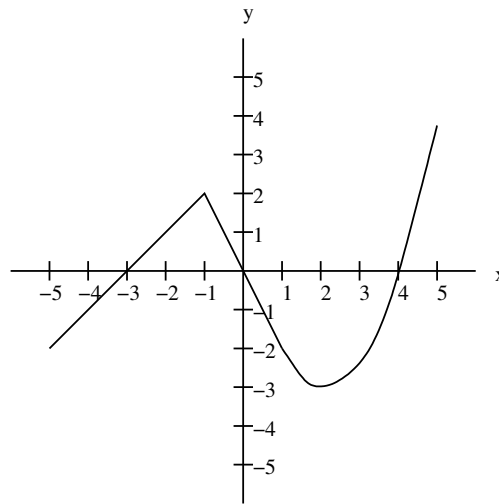
Shown here is the graph of $f(x) = \dfrac{1}{x}$. What are the coordinates of point P ?

(A) $\left(-2, -\frac{1}{2}\right)$

(B) $(-1, -2)$

(C) $(-2, -1)$

(D) $(-2, -2)$

(E) $\left(-2, \frac{1}{2}\right)$
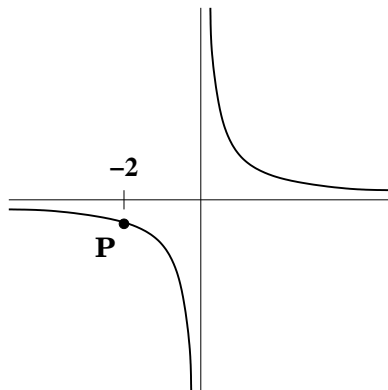
Solve the inequality $\frac{x+6}{(x-1)^2} \geq 0$.

(A) $x \leq -6$ (B) $x \geq -6$ (C) $-6 \leq x < 1$

(D) $x > 1$ (E) $-6 \leq x < 1$ or $x > 1$

Find the midpoint of the line segment between the points $(-2, -3)$ and $(11, 4)$.

(A) $(6.5, 0.5)$ (B) $(4.5, 3.5)$ (C) $(3.5, 4.5)$

(D) $(7, 1.5)$ (E) $(4.5, 0.5)$

Find functions f and g so that $(f \circ g)(x) = (x + 2)^4 - 3$.

(A) $f(x) = x + 2$ and $g(x) = x^2 - 3$

(B) $f(x) = x - 3$ and $g(x) = x + 2$

(C) $f(x) = x^2 - 3$ and $g(x) = (x + 2)^2$

(D) $f(x) = x^2 - 3$ and $g(x) = x + 2$

(E) $f(x) = (x + 2)^2$ and $g(x) = x - 3$

Find all real numbers that solve the equation
$x^2 - 4x + 1 = 0$.

(A) $\dfrac{2 + \sqrt{3}}{3}$ and $\dfrac{2 - \sqrt{3}}{3}$

(B) $\dfrac{1}{3}$ and 1

(C) $2 + \sqrt{3}$ and $2 - \sqrt{3}$

(D) 2 and 6

(E) There are no real number solutions

Compute the difference quotient $\dfrac{f(x + h) - f(x)}{h}$ for

$f(x) = 3x + 2$, $x = 3$, and $h = \frac{1}{3}$.

(A) $\frac{1}{3}$ (B) 1 (C) $\frac{5}{3}$

(D) 2 (E) 3

Find the coordinates of the center and the radius of the circle $x^2 + y^2 + 4x - 8y = 3$.

(A) Center $(4, -8)$ Radius 3

(B) Center $(4, -8)$ Radius $\sqrt{3}$

(C) Center $(-2, 4)$ Radius $\sqrt{23}$

(D) Center $(2, -4)$ Radius 23

(E) Center $(-4, 8)$ Radius 3

How many x-intercepts does the graph of the parabola $y = x^2 + 10x - 5$ have?

(A) 0   (B) 1   (C) 2

(D) 3   (E) 4

Simplify $\dfrac{x - 1}{\sqrt{x} + 1}$.

(A) $\sqrt{x} + 1$ (B) $\sqrt{x} - 1$ (C) x

(D) 1   (E) $\dfrac{(x - 1)(\sqrt{x} + 1)}{x + 1}$

Find all values of x that solve the compound inequality:

$$-10 \leq 6 + 2x \leq -4$$

(A) $5 \leq x \leq 8$ (B) $x \leq -8$ or $x \geq -5$

(C) $-8 \leq x \leq -5$ (D) $x \leq 5$ or $x \geq 8$

(E) $8 \leq x \leq 5$

Find all real number solutions to the equation
$x + 4\sqrt{x} + 6 = 0$.

(A) $x = -2$ and $x = -3$

(B) $x = 9$

(C) $x = 2$ and $x = 3$

(D) $x = 4$ and $x = 9$

(E) There are no real number solutions

Find the equation of the line that passes through the points $(6, -3)$ and $(2, -2)$.

(A) $y = -\frac{1}{4}x - \frac{3}{2}$

(B) $y = -\frac{5}{4}x + \frac{1}{2}$

(C) $y = -4x - 21$

(D) $y = -4x + 6$

(E) $y = -\frac{1}{4}x - \frac{15}{4}$

What is the domain of the function $\sqrt{9 - x} - \sqrt{x - 1}$?

(A) $[0, \infty)$

(B) $[0, 9]$

(C) $[1, 9]$

(D) $(-\infty, 9]$

(E) $[-1, 9]$

Perform the addition: $\dfrac{x+2}{x+1} - \dfrac{7}{x}$

(A) $\dfrac{8x+9}{x^2+x}$ (B) $\dfrac{x^2-5x-7}{x^2+x}$ (C) $\dfrac{x^2+4x+3}{x^2+5x}$

(D) $\dfrac{x+9}{2x+1}$ (E) $\dfrac{7x+14}{x^2+x}$

Simplify: $\dfrac{\dfrac{x}{y} + \dfrac{1}{x}}{\dfrac{1}{y} + \dfrac{y}{x}}$

(A) $\dfrac{(x^2-y)(x+y^2)}{x^2 y^2}$ (B) $\dfrac{(x^2-y)(x+y^2)}{xy^2}$

(C) $\dfrac{(x-1)(x+y)}{(y+1)(x-y)}$ (D) $x-y$

(E) $\dfrac{x^2+y}{x+y^2}$

Name:

UH ID:

Directions: Print your name and ID number. Clearly mark your answers. You may write on the exam, but will only be graded on the answer you mark. If you need to change answers, please completely erase or cross out the old answer and write and circle your final choice.

The passing score is 26.

Answers:

1.B	2.C	3.E	4.B	5.B	6.E	7.E	8.B	9.E	10.D
11.C	12.B	13.A	14.E	15.B	16.C	17.B	18.B	19.D	20.A
21.C	22.B	23.B	24.E	25.C	26.E	27.C	28.E	29.A	30.A
31.E	32.C	33.B	34.C	35.D	36.A	37.C	38.A	39.A	

What is an equation for the line graphed here?
(Points with integer coordinates have been emphasized with a dot.)

(A) $y = 2x - 2$

(B) $y = -2x - 2$

(C) $y = -\frac{1}{2}x - 2$

(D) $y = \frac{1}{2}x - 1$

(E) $y = -\frac{1}{2}x - 1$
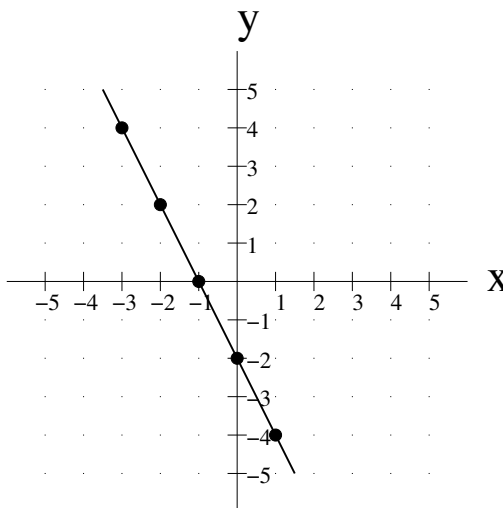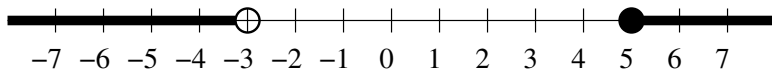
The intervals shown on this number line can be expressed in interval notation as:

(A) $(-\infty, -3) \cap (5, \infty)$

(B) $(-\infty, -3) \cup (5, \infty)$

(C) $(-\infty, -3) \cup [5, \infty)$

(D) $(-\infty, -3] \cap [5, \infty)$

(E) $(-\infty, -3] \cup [5, \infty)$

Here are three statements about the circle which is
the graph of the equation $(x - 4)^2 + (y + 1)^2 = 25$.

  I. The circle has center $(4, -1)$.
 II. The circle has radius 5.
III. The point $(0, 2)$ is on the circle.

Which of the three statements are true?

(A) Only I   (B) Only II   (C) Only III

(D) Only I and II (E) All three are true

Which of the following is a correct equation for the line that
is perpendicular to $-4x + y = 8$ and passes through the point $(-5, 6)$?
(A) $y - 6 = 4(x + 5)$

(B) $y - 6 = -\frac{1}{4}(x + 5)$

(C) $y + 6 = 4(x - 5)$

(D) $y + 6 = 4(x - 5)$

(E) $y + 6 = -\frac{1}{4}(x - 5)$

What is the domain of the function $f(x) = 5\sqrt{x + 3} - 3$?

(A) $x > -3$ (B) $x \geq -3$ (C) $x \geq 1$

(D) $x > -1$ (E) $x \geq -1$

Which of the these three equations have a graph that passes through the point $(-4, 1)$?

  I. $x^2 + y^2 + x = 16$

 II. $x + 4 = 0$

III. $(x + y)^2 = y - 2x$

(A) Only I   (B) Only II   (C) Only III

(D) Only I and III   (E) Only II and III

Math 135   Question 6 of 39   Sample C

The graph of the function $f(x)$ is shown here. What is its domain?

(A) $(-3, 2]$

(B) $[-4, 4]$

(C) $(-4, 4)$

(D) $(-3, 3)$

(E) $(-4, -1] \cup (2, 4)$
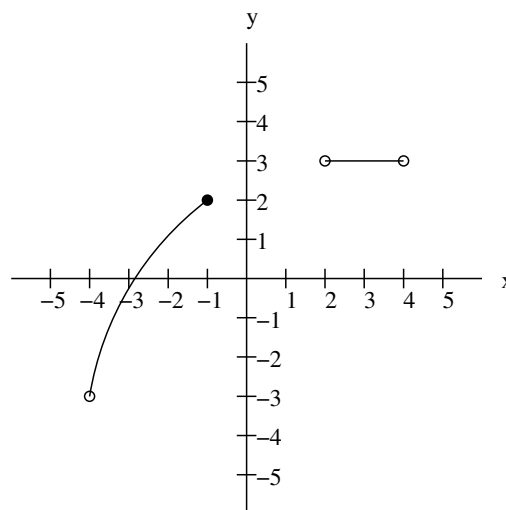
Math 135   Question 7 of 39   Sample C

How many distinct, real-number solutions does
the equation $x^2 - 2x + 5 = 4$ have?

(A) 0   (B) 1   (C) 2

(D) 3   (E) 4

Here are three statements about the graph of
the equation $y = 2x^2 - 2x - 24$:

  I. $(4, 0)$ is an x-intercept of the graph.
 II. $(-4, 0)$ is an x-intercept of the graph.
III. $(0, -24)$ is the only y-intercept of the graph.

Which of the three statements are true?

(A) Only I   (B) Only II   (C) Only III

(D) Only I and II (E) Only I and III

What is the distance between the points $(-1, \, -3)$ and $(-4, \, 2)$?

(A) 2.5   (B) 7   (C) $\sqrt{10}$

(D) $\sqrt{34}$ (E) $\sqrt{26}$

Which of the following is the correct result of completing the square on the expression $x^2 + 5x - 2$?

(A) $(x+5)^2 - 27$ (B) $(x+5)^2 - 2$ (C) $(x+\frac{5}{2})^2 - \frac{33}{4}$

(D) $(x+\frac{5}{2})^2 - 27$ (E) None of these are correct

Shown here are three correspondences between domain sets A and range sets B. Which of these correspondences are functions?

I.
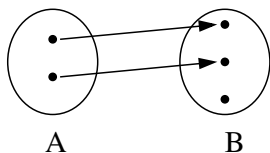
II.
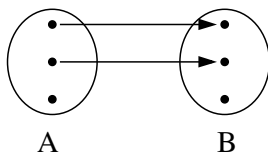
III. $A =$ the set of letters in the alphabet and $B =$ the set of words in the dictionary. A letter in A corresponds to a word in B if the word begins with that letter.

(A) None of them   (B) Only I   (C) Only II

(D) Only III   (E) Only I and III

Shown here are the graphs of three relations in x and y. For which of these relations is y a function of x?

I.

II.

III.

(A) Only I   (B) Only II   (C) Only III

(D) Only I and II   (E) Only I and III

An experiment is being conducted to study a population of penguins. The population P is a linear function of time t. Three years after the start of the experiment the population was 200, and six years after the start of the experiment the population was 380. What will the population be nine years after the start of the experiment?

(A) 460 (B) 500 (C) 510

(D) 550 (E) 560

Find the x-values at which relative maxima occur in the graph of $f(x)$ shown here.

(A) There are none

(B) $-2, 1$

(C) -2

(D) $-2, 0$

(E) $-3, -1$
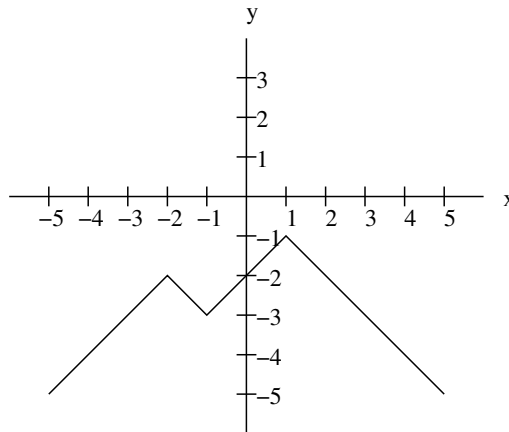
Define the piecewise function $h(x)$ as:

$$h(x) = \begin{cases} 2(x-1)^2 + 3 & \text{for } x \leq 1 \\ \sqrt{x} + 4 & \text{for } 1 < x < 4 \\ 3 - 2x & \text{for } x \geq 4 \end{cases}$$

What is $h(4)$?

(A) $h(4) = 6$ (B) $h(4) = 4$ (C) $h(4) = -5$

(D) $h(4) = 3$ (E) $h(4) = 21$

Solve the inequality $-\frac{2}{3}x - 7 < 3$.

(A) $x < -15$ (B) $x > -15$ (C) $x > \frac{20}{3}$

(D) $x < -\frac{20}{3}$ (E) $x > 10$

Math 135   Question 17 of 39   Sample C

Let $f(x) = x^2 - 4$ and $g(x) = \dfrac{1}{x-5}$. What is the domain of

$(g \circ f)(x) = \dfrac{1}{x^2 - 9}$?

(A) $x \neq 0$ (B) $x \neq \pm 3$ (C) $(-\infty, \infty)$

(D) $(0, \infty)$ (E) $(-3, 3)$

Math 135   Question 18 of 39   Sample C

Let $f(x) = 3x + 1$ and $g(x) = 4 - \sqrt{x}$. What is $f(g(x))$?

(A) $3 - \sqrt{3x}$ (B) $5 - \sqrt{3x}$ (C) $11 - 3\sqrt{x}$

(D) $13 - 3\sqrt{x}$ (E) $4 - \sqrt{3x - 1}$

Math 135   Question 19 of 39   Sample C

Shown here is the graph of

$$y = \frac{(x+2)(x-1)}{3x}$$

Solve the inequality:

$$\frac{(x+2)(x-1)}{3x} \le 0$$
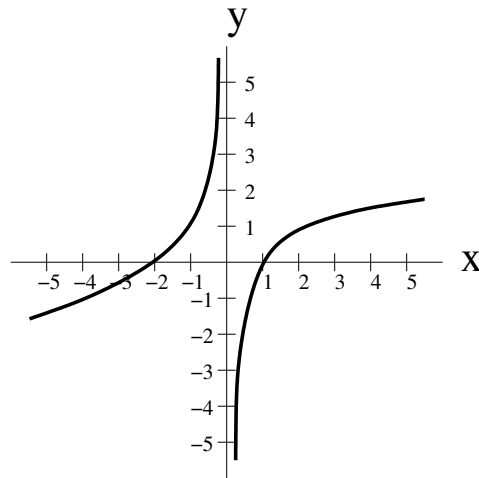
(A) $x \le -2$ or $0 < x \le 1$ (B) $x > 0$ (C) $x < 0$

(D) $x \le -2$ or $x > 0$ (E) $-2 \le x \le 1$

Simplify: $\dfrac{x^{-1/4}\,\sqrt[3]{y^2}}{\sqrt{x}\,y}$

(A) $x^{-1/2}\,y^{1/3}$ (B) $x^{-9/4}\,y^2$ (C) $x^{-3/4}\,y^{-1/3}$

(D) $x^{-3/4}\,y^{-2/3}$ (E) $x^{1/2}\,y^{1/3}$

Shown here are two numbers, A and B, on a number line where each mark represents 1 unit.

B　　　　　　　　　　　A

What is $|A - B|$?

(A) 5　　　　　(B) 6　　　　　(C) -5

(D) -6　　　　　(E) It cannot be determined

−7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7
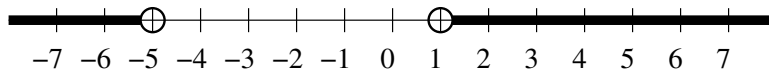
The intervals shown on this number line can be expressed as:

(A) $|x + 2| < 3$　　　　　(B) $|x + 2| > 3$　　　　　(C) $|x - 2| < 3$

(D) $|x - 2| > 3$　　　　　(E) $|x - 5| > 6$

The graph of $g(x)$ is shown here. What are the zeros of $g(x)$?

(A) 3

(B) $1, 3$

(C) $-2, 2, 3$

(D) $2, 3$

(E) $-2, 3$

Shown here is the graph of $f(x) = 3 - x$. What are the coordinates of point P?

(A) $(3, -1)$

(B) $(-1, 3)$

(C) $(-1, 4)$

(D) $(-1, 2)$

(E) $(3, 1)$
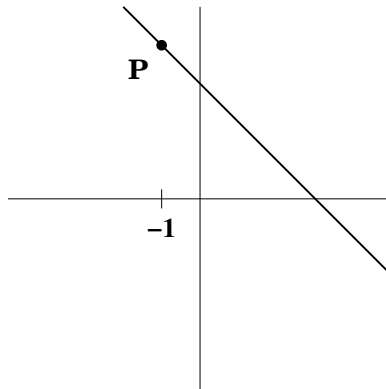
Solve the inequality $\frac{x-5}{(x+1)(x-4)} \geq 0$.

(A) $x > -1$ (B) $x > 4$

(C) $x < -1$ or $4 < x \leq 5$ (D) $x \leq 5$

(E) $-1 < x < 4$ or $x \geq 5$

Find the midpoint of the line segment between the points $(1, -9)$ and $(-3, 7)$.

(A) $(1, -8)$ (B) $(-2, -1)$ (C) $(-1, -1)$

(D) $(-1, 8)$ (E) $(-4, -2)$

Find functions f and g so that $(f \circ g)(x) = (\sqrt{x} - x)^3 + 10$.

(A) $f(x) = \sqrt{x} - x$ and $g(x) = x^3 + 10$

(B) $f(x) = x^3 + 10$ and $g(x) = \sqrt{x} - x$

(C) $f(x) = x^3$ and $g(x) = \sqrt{x} - x + 10$

(D) $f(x) = \sqrt{x} - x + 10$ and $g(x) = x^3$

(E) $f(x) = x + 10$ and $g(x) = (\sqrt{x} - x)^3$

Find all real numbers that solve the equation
$9x^2 - 12x = -4$.

(A) $\dfrac{2}{3}$

(B) $\dfrac{2 + \sqrt{2}}{3}$ and $\dfrac{2 - \sqrt{2}}{3}$

(C) $\dfrac{-12 + \sqrt{128}}{18}$ and $\dfrac{-12 - \sqrt{128}}{18}$

(D) $-\dfrac{2}{3}$ and $\dfrac{2}{3}$

(E) There are no real number solutions

Compute the difference quotient $\dfrac{f(x + h) - f(x)}{h}$ for

$f(x) = \frac{1}{x}$, $x = \frac{1}{2}$, and $h = \frac{1}{2}$.

(A) -2　　　　(B) 1　　　　(C) $\frac{1}{2}$

(D) 2　　　　(E) $\frac{1}{3}$

Find the coordinates of the center and the radius of the circle $x^2 + y^2 - 2x - 12y = 12$.

(A) Center $(2, -12)$ Radius 12

(B) Center $(1, 6)$ Radius 12

(C) Center $(-1, 6)$ Radius 7

(D) Center $(2, -12)$ Radius $\sqrt{12}$

(E) Center $(1, 6)$ Radius 7

How many x-intercepts does the graph of the parabola $y = -x^2 - 2x + 1$ have?

(A) 0   (B) 1   (C) 2

(D) 3   (E) 4

Simplify $\dfrac{x - 16}{\sqrt{x} + 4}$.

(A) $x - 4$ (B) $\sqrt{x} - 4$ (C) $\sqrt{x}$

(D) $\sqrt{x} + 4$ (E) $\dfrac{(x - 16)(\sqrt{x} - 4)}{x + 16}$

Find all values of x that solve the compound inequality:

$$-5 \leq 3 - 2x \leq 3$$

(A) There are no solutions

(B) $0 \leq x \leq -4$

(C) $0 \leq x \leq 4$

(D) $4 \leq x \leq 0$

(E) $-4 \leq x \leq 0$

Find all real number solutions to the equation
$x - 5\sqrt{x} + 4 = 0$.

(A) $x = \sqrt{\frac{1}{2}}$ and $x = \sqrt{2}$

(B) $x = 4$

(C) $x = 1$ and $x = 4$

(D) $x = 1$ and $x = 16$

(E) There are no real number solutions

Find the equation of the line that passes through the points $(4, -7)$ and $(-3, -2)$.

(A) $y = -\frac{5}{7}x - \frac{29}{7}$

(B) $y = -\frac{7}{5}x - \frac{31}{5}$

(C) $y = -\frac{5}{7}x + \frac{31}{7}$

(D) $y = -\frac{7}{5}x + \frac{41}{5}$

(E) $y = -x - 3$

What is the domain of the function $\sqrt{2x} - \dfrac{2}{x-2}$?

(A) $[0, \infty)$

(B) $[0, 2)$

(C) $[0, 2) \cup (2, \infty)$

(D) $(-\infty, \infty)$

(E) $(-\infty, 2) \cup (2, \infty)$

Perform the addition: $\dfrac{2}{x} + \dfrac{2x-4}{x+6}$

(A) $\dfrac{2x^2 - 2x + 12}{x^2 + 6x}$ (B) $\dfrac{2x-8}{x^2+6x}$ (C) $\dfrac{x-2}{2x+6}$

(D) $\dfrac{x^2 + 5x - 1}{x^2 + 3x}$ (E) $\dfrac{x^2 - 10x + 4}{x^2 + 6x}$

Simplify: $\dfrac{\dfrac{y}{x^2} + \dfrac{1}{y}}{\dfrac{1}{x} - \dfrac{x}{y^2}}$

(A) $\dfrac{y^3 + yx^2}{y^2 x - x^3}$ (B) $\dfrac{y^4 - x^2}{x^2 y^3}$

(C) $\dfrac{(y^2 + x)(y^2 - x)}{x^2 y^3}$ (D) $\dfrac{(x + y^2)(x - y^2)}{xy}$

(E) $\dfrac{x + y^2}{x^2 + y}$

Name:

UH ID:

Directions: Print your name and ID number. Clearly mark your answers. You may write on the exam, but will only be graded on the answer you mark. If you need to change answers, please completely erase or cross out the old answer and write and circle your final choice.

The passing score is 26.

Answers:

1.C	2.A	3.C	4.C	5.D	6.D	7.B	8.C	9.C	10.D
11.C	12.D	13.E	14.B	15.D	16.B	17.B	18.C	19.B	20.D
21.D	22.C	23.C	24.C	25.D	26.D	27.D	28.D	29.E	30.C
31.B	32.A	33.E	34.B	35.B	36.D	37.E	38.C	39.C	

What is an equation for the line graphed here?
(Points with integer coordinates have been emphasized with a dot.)

(A) $y = 2x - 4$

(B) $y = 2x + 4$

(C) $y = \frac{1}{2}x - 2$

(D) $y = 2x - 2$

(E) $y = \frac{1}{2}x - 4$
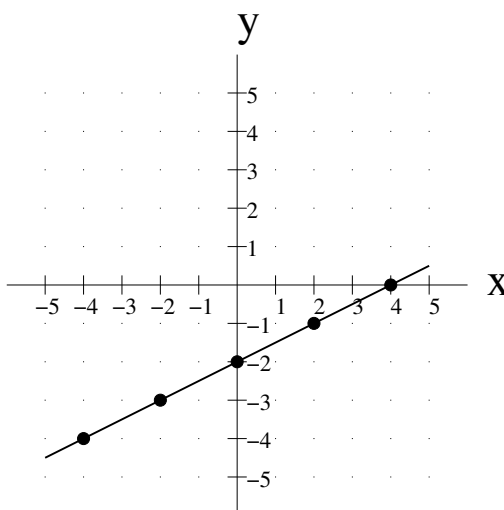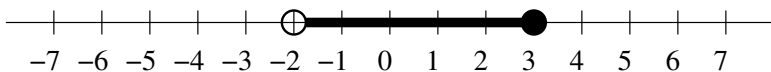
The interval shown on this number line can be expressed in interval notation as:

(A) $(-\infty, 3] \cap (-2, \infty)$

(B) $(-\infty, 3] \cup (-2, \infty)$

(C) $(-\infty, 3) \cap [-2, \infty)$

(D) $(-\infty, 3) \cup [-2, \infty)$

(E) $(-\infty, \infty] \cup (-2, 3)$

Here are three statements about the circle which is
the graph of the equation $(x - 1)^2 + (y + 1)^2 = 4$.

  I. The circle has center $(-1, -1)$.
 II. The circle has radius 4.
III. The point $(1, 1)$ is on the circle.

Which of the three statements are true?

(A) Only I   (B) Only II   (C) Only III

(D) Only II and III(E) All three are true

Which of the following is a correct equation for the line that
is perpendicular to $y = 4x - 4$ and passes through the point $(1, -4)$?
(A) $y - 4 = 4(x + 1)$

(B) $y - 4 = 4(x - 1)$

(C) $y + 4 = -\frac{1}{4}(x - 1)$

(D) $y + 4 = 4(x - 1)$

(E) $y + 4 = \frac{1}{4}(x - 1)$

What is the domain of the function $f(x) = 4\sqrt{4 - x} + 7$?

(A) $x \geq 7$ (B) $x < 4$ (C) $x < -4$

(D) $x \leq 4$ (E) $x \leq -4$

Which of the these three equations have a graph that passes through the point $(3, -2)$?

I. $y = -(x-3)^2 + 7$

II. $\dfrac{4}{y} - \dfrac{x}{2} = -\dfrac{1}{2} - x$

III. $(x-6)^2 + (y+2)^2 = 9$

(A) Only I　　　　　(B) Only II　　　　　(C) Only III

(D) Only II and III　(E) All three

The graph of the function $f(x)$ is shown here. What is its domain?

(A) $[-2, 1) \cup [2, 3]$

(B) $[-2, 3]$

(C) $[-2, 2) \cup (2, 3]$

(D) $(-2, 2) \cup (2, 3)$

(E) $[2, 3]$
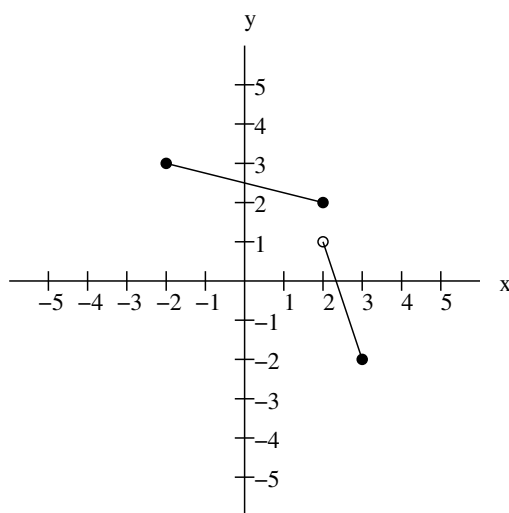
How many distinct, real-number solutions does
the equation $3x^2 - x + 3 = x + 5$ have?

(A) 0   (B) 1   (C) 2

(D) 3   (E) 4

Here are three statements about the graph of
the equation $y = -x^2 - 10x + 25$:

  I. $(-5, 0)$ is an x-intercept of the graph.
 II. $(5, 0)$ is an x-intercept of the graph.
III. $(0, 25)$ is the only y-intercept of the graph.

Which of the three statements are true?

(A) Only I   (B) Only II   (C) Only III

(D) Only I and II   (E) Only I and III

What is the distance between the points $(10, -2)$ and $(7, -5)$?

(A) 5   (B) 7.5   (C) $\sqrt{13}$

(D) $\sqrt{18}$ (E) 11.5

Which of the following is the correct result of completing the square on the expression $x^2 + 6x - 3$?

(A) $(x+3)^2$　　　　(B) $(x+3)^2 - 3$　　　　(C) $(x+3)^2 - 12$

(D) $(x+3)^2 - 6$　　(E) None of these are correct

Shown here are three correspondences between domain sets A and range sets B. Which of these correspondences are functions?

I.　　　　　　　　　　　**II.**

　　A　　　　B　　　　　　　A　　　　B
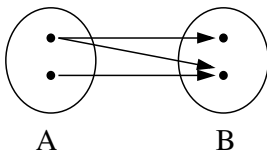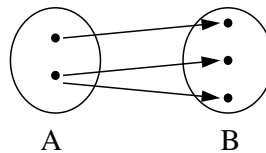
III. $A =$ the set of real numbers and $B = [0, \infty)$. A number x in A corresponds to a number y in B if $y = x^2$.

(A) None of them　　　　(B) Only I　　　　(C) Only II

(D) Only III　　　　(E) Only II and III

Shown here are the graphs of three relations in x and y. For which of these relations is y a function of x?

I.

II.

III.
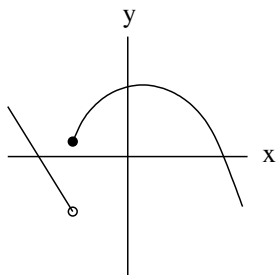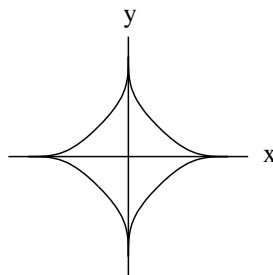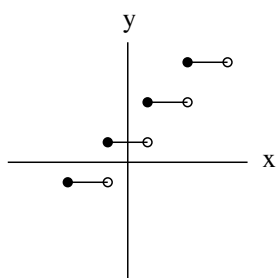
(A) Only I   (B) Only II   (C) Only III

(D) Only I and II   (E) Only I and III

An experiment is being conducted to study a population of penguins. The population P is a linear function of time t. The initial population was 1000 penguins. Three years after the start of the experiment the population was 1600. What will the population be seven years after the start of the experiment?

(A) 2600   (B) 2400   (C) 2000

(D) 2100   (E) 2800

Find the x-values at which relative maxima occur in the graph of $f(x)$ shown here.

(A) There are none

(B) -2 and 0

(C) -2

(D) 0

(E) -2 and 2
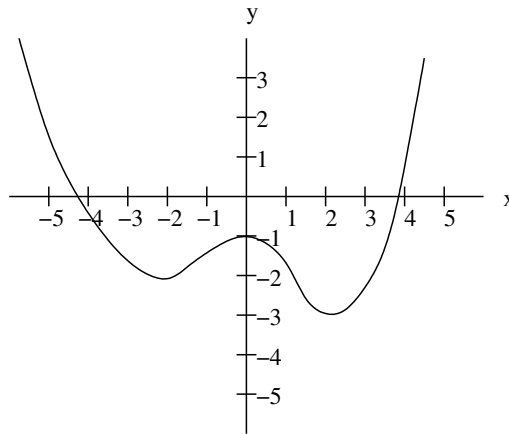
Define the piecewise function $h(x)$ as:

$$h(x) = \begin{cases} -x + 7 & \text{for } x < 0 \\ x^2 + 1 & \text{for } 0 \leq x \leq 4 \\ \dfrac{x}{x-4} & \text{for } x > 4 \end{cases}$$

What is $h(0)$?

(A) $h(0) = 7$ (B) $h(0) = 1$ (C) $h(0) = 0$

(D) $h(0) = -\frac{1}{4}$ (E) $h(4)$ is undefined

Solve the inequality $-17x - 40 \leq 30x + 7$.

(A) $x \leq \dfrac{1}{2}$ (B) $x \geq -1$ (C) $x \leq \dfrac{33}{47}$

(D) $x \leq -\dfrac{33}{47}$ (E) $x \leq -1$

Let $f(x) = x^2 + 5$ and $g(x) = \sqrt{-x}$. What is the domain of
$(f \circ g)(x) = (\sqrt{-x})^2 + 5$?

(A) $[5, \infty)$ (B) $x \neq 0$ (C) $(-\infty, 0]$

(D) $[0, \infty)$ (E) $(0, \infty)$

Let $f(x) = 5 - 2x$ and $g(x) = 8 - \sqrt{-x + 5}$. What is $g(f(x))$?

(A) $40 + 2x^{3/2}$ (B) $8 - \sqrt{2x}$ (C) $8 - \sqrt{5 - 2x}$

(D) $2\sqrt{x} - 11$ (E) $40 + 2\sqrt{x}$

Shown here is the graph of

$$y = \frac{1}{5}x^4 + \frac{1}{5}x^3 - \frac{6}{5}x^2$$

Solve the inequality:

$$\frac{1}{5}x^4 + \frac{1}{5}x^3 - \frac{6}{5}x^2 > 0$$
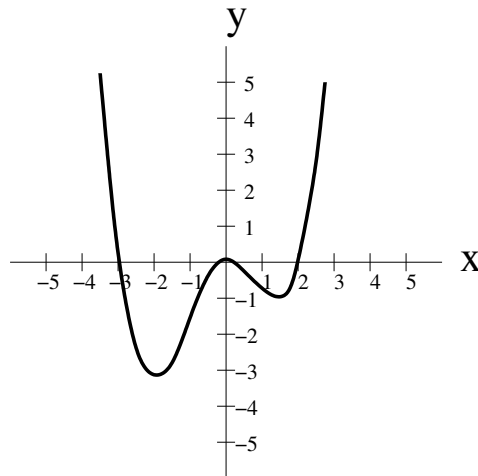
(A) $-3 \le x \le 2$ 　　　　(B) $x \le -3$ 　　　　(C) $x < 0$

(D) $x < -3$ or $x > 2$ 　　(E) $x \le -3$ or $x \ge 2$ or $x = 0$

Simplify: $\dfrac{(x^2\, y^{1/3})^6}{x^{-3}\, y^{-2}}$

(A) $x^{-15}\, y^4$ 　　　　(B) $x^{-6}\, y^{14}$ 　　　　(C) $x^{30}\, y^{-10}$

(D) $x^{15}\, y^4$ 　　　　(E) x^{15}

Shown here are two numbers, A and B, on a number line where each mark represents 1 unit.

B   A
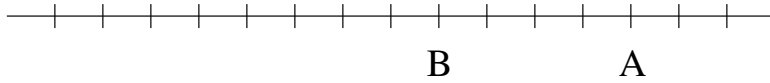
What is $|A - B|$?

(A) -4   (B) 9   (C) 4

(D) 3   (E) -3

−7 −6 −5 −4 −3 −2 −1   0   1   2   3   4   5   6   7
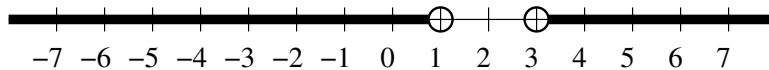
The intervals shown on this number line can be expressed as:

(A) $|x + 1| > 2$ (B) $|x + 2| < 1$ (C) $|x - 2| > 1$

(D) $|x - 2| < 1$ (E) $|x - 1| > 2$

The graph of $g(x)$ is shown here. What are the zeros of $g(x)$?

(A) $-4, -3, -2$

(B) $-2, 0, 1$

(C) -1

(D) $-1, 1, 2$

(E) $-4, -1, 0, 1$
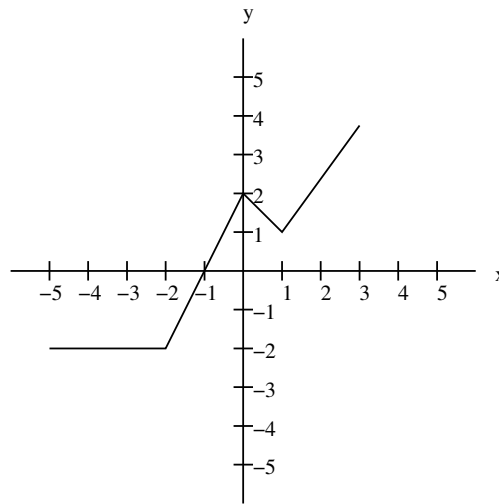
Shown here is the graph of $f(x) = 3 - x$. What are the coordinates of point P?

(A) $(-1, 4)$

(B) $(4, 4)$

(C) $(4, -4)$

(D) $(4, -1)$

(E) $(-1, -4)$
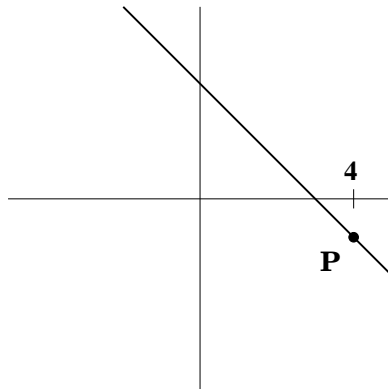
Solve the inequality $\frac{(x-3)^2}{x(x-7)} < 0$.

(A) $x > 3$　　　　　　　(B) $0 < x < 7$

(C) $x > 7$　　　　　　　(D) $0 < x < 3$ or $3 < x < 7$

(E) $x < 0$ or $x > 7$

Find the midpoint of the line segment between the
points $(4,\ 5)$ and $(1,\ 8)$.

(A) $(-3,\ 13)$　　　　　(B) $(2.5,\ 1.5)$　　　　　(C) $(-1.5,\ 6.5)$

(D) $(2.5,\ 6.5)$　　　　(E) $(-2.5,\ -1.5)$

Find functions f and g so that $(f \circ g)(x) = \sqrt{\sqrt{x+1} - (x+1)^2} + 4$.

(A) $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x+1} - x^2 + 4$

(B) $f(x) = \sqrt{x+1} - x^2 + 4$ and $g(x) = \sqrt{x}$

(C) $f(x) = \sqrt{x} + 4$ and $g(x) = \sqrt{x+1} - x^2$

(D) $f(x) = \sqrt{\sqrt{x} - x^2} + 4$ and $g(x) = x+1$

(E) $f(x) = \sqrt{x} - x^2$ and $g(x) = \sqrt{x+1} + 4$

Find all real numbers that solve the equation
$-2x^2 + 6x - 5 = 0$.

(A) $\dfrac{-3 + \sqrt{15}}{2}$ and $\dfrac{-3 - \sqrt{15}}{2}$

(B) $\dfrac{-3 + \sqrt{2}}{2}$ and $\dfrac{-3 - \sqrt{2}}{2}$

(C) $\dfrac{3 + \sqrt{15}}{2}$ and $\dfrac{3 - \sqrt{15}}{2}$

(D) $-3 + \sqrt{6}$ and $-3 - \sqrt{6}$

(E) There are no real number solutions

Compute the difference quotient $\dfrac{f(x + h) - f(x)}{h}$ for

$f(x) = \sqrt{x}$, $x = 81$, and $h = 19$.

(A) 10   (B) $-\frac{1}{11}$ (C) $\frac{1}{19}$

(D) 1   (E) $\sqrt{19}$

Find the coordinates of the center and the radius of the circle $x^2 + y^2 - 2x - 4y = 0$.

(A) Center $(1, 2)$ Radius $\sqrt{14}$

(B) Center $(1, 2)$ Radius $\sqrt{5}$

(C) Center $(2, 4)$ Radius 9

(D) Center $(2, 4)$ Radius 3

(E) Center $(-1, -2)$ Radius 5

How many x-intercepts does the graph of the parabola $y = -2x^2 - 2x - 10$ have?

(A) 0   (B) 1   (C) 2

(D) 3   (E) 4

Simplify $\dfrac{x + 9}{\sqrt{x} - 3}$.

(A) $x - 3$ (B) $\sqrt{x} - 3$ (C) $x + 3$

(D) $\sqrt{x} + 3$ (E) $\dfrac{(x + 9)(\sqrt{x} + 3)}{x - 9}$

Find all values of x that solve the compound inequality:

$$1 < -5 - 3x < 10$$

(A) $-2 < x < -5$ (B) $-5 < x < -2$

(C) $2 < x < 5$ (D) $5 < x < 2$

(E) There are no solutions

Find all real number solutions to the equation
$x - 3\sqrt{x} = 4$.

(A) $x = \sqrt{\frac{1}{2}}$ and $x = \sqrt{1}$

(B) $x = 16$

(C) $x = 1$

(D) $x = 4$ and $x = 16$

(E) There are no real number solutions

Find the equation of the line that passes through the points $(-8, -1)$ and $(4, 5)$.

(A) $y = \frac{1}{3}x + \frac{11}{3}$

(B) $y = -x - 9$

(C) $y = \frac{1}{2}x + \frac{3}{2}$

(D) $y = \frac{1}{2}x + 3$

(E) $y = -x + 9$

What is the domain of the function $\dfrac{(x-4)^2}{x^2 + 4}$?

(A) $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

(B) $(-\infty, -2) \cup (-2, 2) \cup (2, 4) \cup (4, \infty)$

(C) $(-\infty, -2) \cup (-2, 4) \cup (4, \infty)$

(D) $(2, \infty)$

(E) $(-\infty, \infty)$

Perform the addition: $\dfrac{2}{x} + \dfrac{x+2}{x-3}$

(A) $\dfrac{x^2 + 3x - 4}{x^2 - 6x}$ (B) $\dfrac{x^2 - 5x - 7}{x^2 - 8x}$ (C) $\dfrac{x^2 + 4x - 6}{x^2 - 3x}$

(D) $\dfrac{x^2 + 6x - 12}{x^2 - 3x}$ (E) $\dfrac{x + 6}{2x - 3}$

Simplify: $\dfrac{\dfrac{x^2}{y} - \dfrac{1}{y^2}}{\dfrac{1}{x} + \dfrac{1}{y^2}}$

(A) $\dfrac{(x + y^2)(x^2 - 1)}{-2y^2 + 2y}$ (B) $\dfrac{(x^2 y - 1)(x + y^2)}{xy^4}$

(C) $\dfrac{x^3 y - x}{y^2 + x}$ (D) $\dfrac{(x^2 y - 1)(x + y^2)}{y^2}$

(E) $\dfrac{x^2 y + 1}{xy^2}$

1. Solve quadratic inequalities and equations involving absolute value.

2. Solve a rational equation.

3. Solve an equation with square roots.

4. Perform polynomial long division.

5. Apply the factor theorem.

6. Determine the leading term, leading coefficient, and end behavior of a polynomial function.

7. Determine the equation of a polynomial function from its graph.

8. Determine the vertex and shape of a parabola from its equation.

9. Determine the equation of a parabola from its graph.

10. Determine the graph of a rational function from its equation.

11. Graph translations of functions.

12. Graph reflections of graphs of logarithmic and exponential functions.

13. Classify rational, polynomial, linear, quadratic, cubic, exponential, and logarithmic functions.

14. Given a graph of a function graph the inverse function.

15. Specify the relationship between the domain and range of inverse functions.

16. Given a graph of the function or a list of the points on its graph, determine the value of the inverse function at a specific point.

17. Compute the inverse of a rational, logarithmic, exponential, or a cubic function.

18. Find the range of a quadratic function.

19. Determine the behavior of the graph of a rational function with respect to vertical, horizontal, and oblique asymptotes.

20. Find the equations of the vertical and horizontal asymptotes of a rational function.

21. Evaluate a logarithm.

22. Express a logarithm in terms of a logarithm in a different base.

23. Determine the key values of logarithmic and exponential functions.

24. Solve logarithmic and exponential equations.

25. Simplify an expression with logarithmic or exponential functions

26. Expand a logarithm into a sum or difference of simpler logarithms.

27. Combine an expression involving logarithms into a single logarithm.

28. Express the volume or area - a two variable function - as a function of one variable.

29. Optimize the area of a figure with a constraint on the perimeter.

1. The diagram represents the schematic for an open box. The box is made by cutting out squares of length x from the corners of a cardboard rectangle measuring 20 by 26 inches. Express the volume V of the open box (in inches^3) as a function of x.
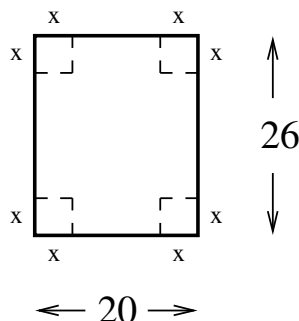
2. If $|x - 5| \geq 3$ and $|x + 1| = 7$, what is x?

3. Consider the polynomial $p(x) = 7x^{10} + 300x^4 + 7x$. Which of the following statements are true?

    (a) $p(x)$ is a degree 4 polynomial.
    (b) The leading coefficient of $p(x)$ is 300.
    (c) The leading term of $p(x) = 7x^{10}$.
    (d) The graph of $p(x)$ behaves as pictured below:
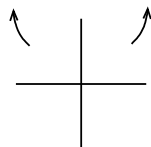
4. Consider the graph of a polynomial $p(x)$ pictured below. What is a possible formula for $p(x)$? Why is $-(x + 1)(x - 3)$ incorrect?
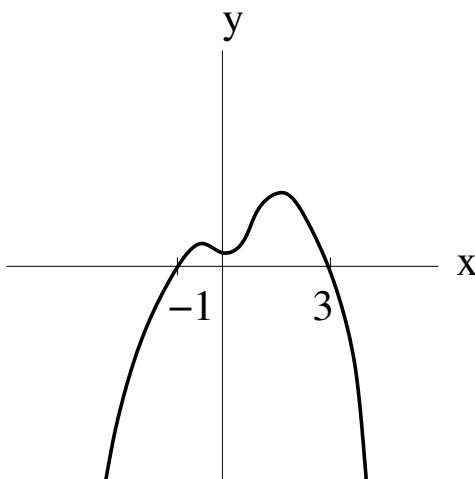
5. Consider the graph of some function $f(x)$. What is the equation of a function $g(x)$ which has the same graph as $f(x)$ but shifted 5 units to the left and one unit up?

6. Divide. List the quotient $q(x)$ and the remainder $r(x)$.

$$x^3 + 3 \,\overline{)\, 3x^4 - 7x^3 + x^2 + 5}$$

7. Which of these functions are rational? For those that are not, explain why.

   (a) $\frac{x-1}{x+1}$

   (b) $\sqrt{x-1} + 3$

   (c) $\frac{3x^3 + x^2 + e^x}{7x - 1}$

   (d) $x^{\frac{3}{2}} - x + 12$

8. Here are some functions. Classify each one as either logarithmic, exponential, polynomial, linear, quadratics, cubic, rational, or neither.

   (a) $x^3 + 3^x$

   (b) $x + 1$

   (c) $7x^3$

   (d) $x^2 + 2 + \ln 6$

   (e) $x^3 + 11x + \log_3 10$

   (f) $\log_2 x + 7x + 1$

   (g) $\frac{3x + \log_{10} 11}{x^2 + x + 1}$

   (h) $\frac{3x + \log_{10} 11}{x^2 + \sqrt{x} + 1}$

9. Compute $\log_{16} 32$

10. The graph of $f(x)$ is shown in blue. Which is the correct graph of the function $f^{-1}(x)$ shown in red?

A

B

C

D

E
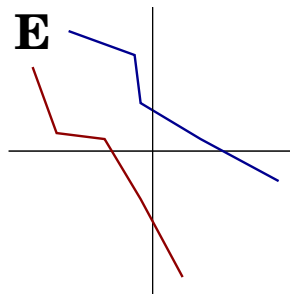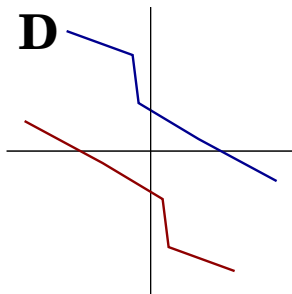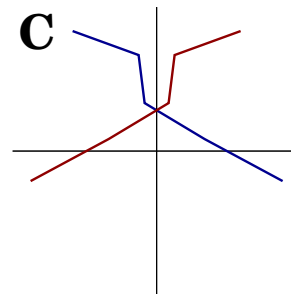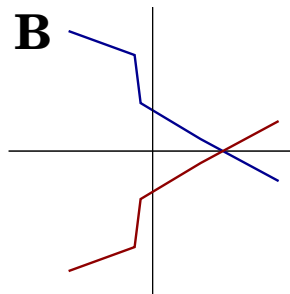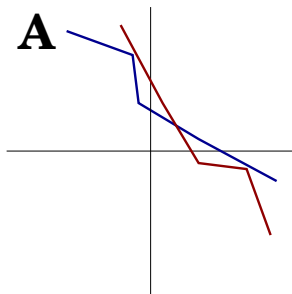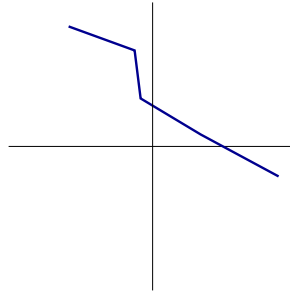
11. Shown here is the graph of $f(x)$. What is the value of $f^{-1}(-1)$?
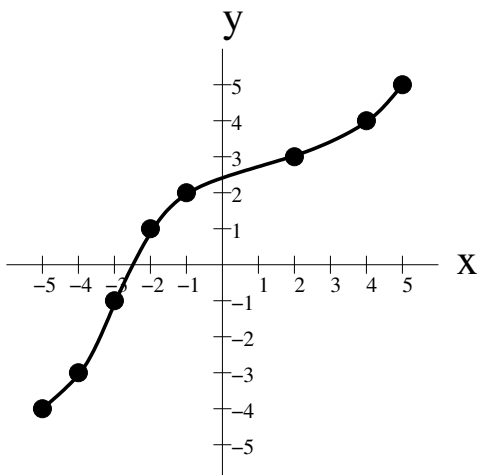
12. Which of the following is an exponential function with base 3?

    (a) $x^3 + 1$

    (b) a^{3x}

    (c) $\log_3 x$

    (d) 3^{x+1}

    (e) 3^e

13. Which of the following is a natural logarithmic function?

    (a) $\log_3 e$

    (b) $\log_3 3x^e$

    (c) $\log_e 5^x$

    (d) $\log_e x^5$

    (e) $\log_e$

14. Solve: $2 = 2\log_3(x-1) + 6$

15. Which is the graph of $\log_{\frac{3}{2}} x$ and which is the graph of $\left(\frac{3}{2}\right)^x$?
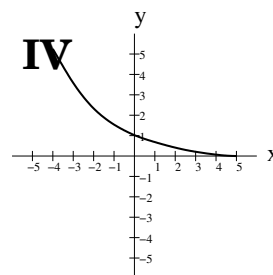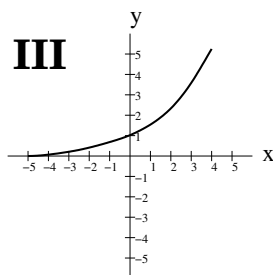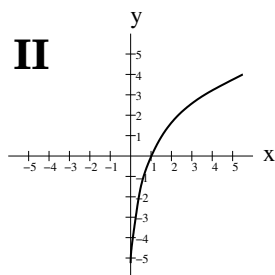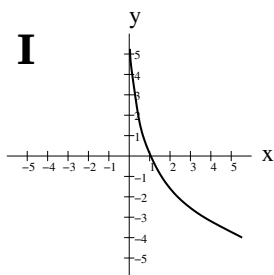
16. Express $\log_9 11$ using base 4 logarithms.

17. Find the inverse of $f(x) = e^{4x+1} - 2$

18. What is the range of the function $G(x) = -(x-2)^2 + 14$?

19. Find all vertical and horizontal asymptotes of the rational function

$$f(x) = \frac{5x^2 + 11x - 36}{x^3 - x^2 - 6x} = \frac{(5x-9)(x+4)}{x(x+2)(x-2)}$$

20. What is the remainder when we divide $x - 1$ into $x^{10} - 4x^3 + 2x + 5$?

21. Order from least to greatest: $3, \ln 3, e^3$.

22. Which of the following statements about the graph of polynomial rational functions are true?

    (a) The graph of a rational function may cross a vertical asymptote.

    (b) The graph of a rational function may cross a horizontal asymptote.

    (c) There graph of a polynomial function may have more than one horizontal asymptote.

    (d) If a graph of a rational function has a oblique asymptote, then it will also have a horizontal asymptote.

    (e) The graph of a degree 4 polynomial has at most 3 vertical asymptotes.

23. Solve the inequality $x^2 + 8x + 16 < 0$.

24. Simplify $\log_2(2^{3x})$.

25. Which of the graphs below is a best fit for the equation

$$f(x) = \frac{x+1}{(x-2)(x+3)}$$
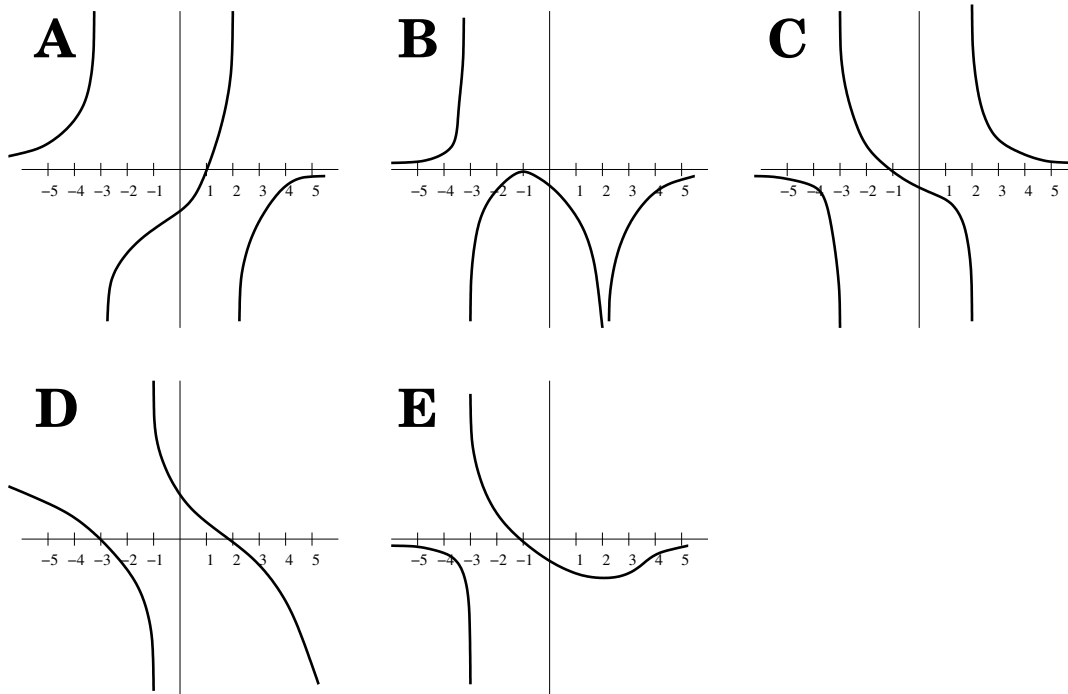
26. Let $f(x)$ be a one-to-one function with range $(-5, 11)$ and domain $(-\infty, 3]$. What is the domain of $f^{-1}(x)$?

27. Express $2 \ln a - \ln(b - c)$ as a single logarithm.

28. Which of these statements about the graph of of a parabola given by the equation $f(x) = \frac{2}{3}(x + 5)^2 - 1$ are true?

    (a) The parabola opens downwards.

    (b) The vertex of the parabola is the point $(5, 2)$.

    (c) The parabola is wider that $y = x^2$.

    (d) The line $y = -1$ is a horizontal asymptote for $f(x)$.

29. Find all values of t which solve the equation $\frac{3t-4}{t} + \frac{t}{2} = t$.

30. Find all values of x which solve the equation $\sqrt{x + 5} = x - 1$.

31. An office cubicle is being constructed against an existing wall. Two walls of length x and one wall of length y are required. If 60 ft of material are available, what should be the dimensions of the cubicle so that the enclosed area is as large as possible?

32. What is the equation of this parabola?
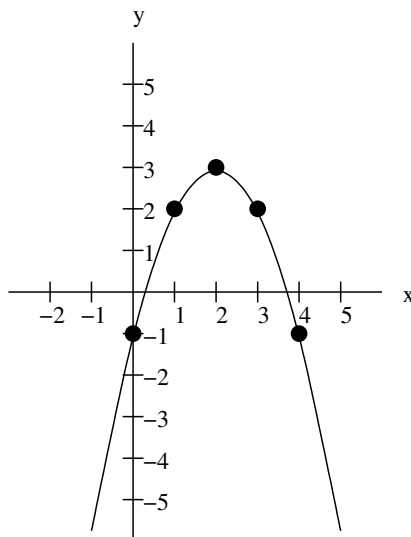
33. Which of the following is the graph of the polynomial function

$$-x^4 + 2x^3 + 11x^2 - 12x - 36 = -(x-3)^2(x+2)^2$$

(The $y-axis$ is not drawn to scale.)
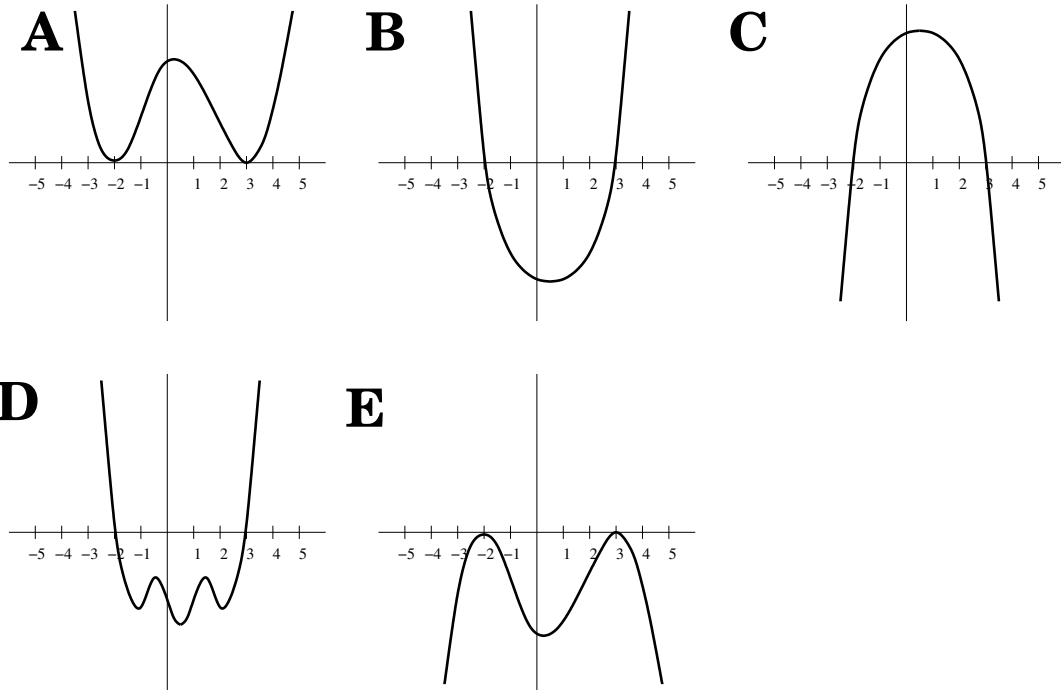
34. Find all values of x which solve the equation $\ln x + \ln(x+5) = \ln 6$.

Name:

UH ID:

Directions: Print your name and ID number. Clearly mark your answers. You may write on the exam, but will only be graded on the answer you mark. If you need to change answers, please completely erase or cross out the old answer and write and circle your final choice.

The passing score is 26.

Answers:
1.D	2.C	3.E	4.B	5.E	6.C	7.C	8.C	9.C	10.E
11.A	12.B	13.D	14.C	15.D	16.D	17.D	18.E	19.A	20.B
21.D	22.C	23.A	24.D	25.E	26.A	27.D	28.E	29.B	30.D
31.C	32.A	33.E	34.B	35.D	36.C	37.C	38.E	39.E	40.E

An open box is formed from a piece of cardboard measuring 20 inches by 26 inches by cutting out squares of length x from the four corners, as shown here.
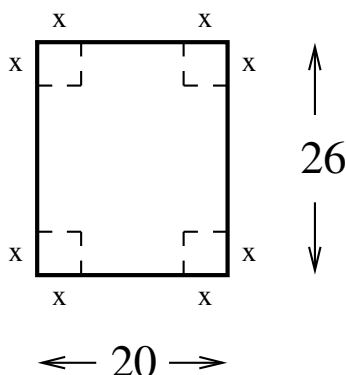
Which of the following is a correct formula for the volume V of the box (in inches^3), as a function of x?

(A) $V = 20 \cdot 26 \cdot x$

(B) $V = (20 - x) \cdot (26 - x)$

(C) $V = (20 - x) \cdot (26 - x) \cdot x$

(D) $V = (20 - 2x) \cdot (26 - 2x) \cdot x$

(E) $V = (20 - 2x) \cdot (26 - 2x) \cdot 2x$

If $|x - 4| < 3$ and $|x - 6| = 2$, what is x?

(A) -3 (B) 3 (C) 4

(D) 6 (E) 8

Here are three statements about the polynomial
$p(x) = -2x^5 + 7x^4 - 3x + 4$:

  I. The degree of $p(x)$ is 4.

  II. The leading coefficient of $p(x)$ is -2.

  III. The graph of $p(x)$ has the end behavior pictured here:
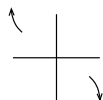
Which of the statements are true?

(A) Only I   (B) Only II   (C) Only III

(D) Only I and II   (E) Only II and III

Shown here is the graph of the polynomial $p(x)$. Which of the choices below is the only possible formula for $p(x)$?

(A) $p(x) = -(x+1)(x-3)$

(B) $p(x) = -(x+1)(x-3)(10x^2+1)$

(C) $p(x) = (x+1)(x-3)$

(D) $p(x) = (x+1)(x-3)(10x^2+1)$

(E) $p(x) = -(x+1)(10x^2+1)$
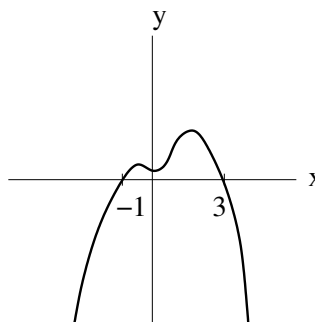
Which of these functions has a graph that is the same as the graph of $f(x)$, but shifted 7 units to the right?

(A) $f(x) - 7$　　　　(B) $7f(x)$　　　　(C) $f(x) + 7$

(D) $f(x + 7)$　　　　(E) $f(x - 7)$

Perform the polynomial division:

$$x^2 + 3 \overline{)\, x^4 + 2x^2 + 1}$$

Give both the quotient $q(x)$ and the remainder $r(x)$.

(A) $q(x) = x^2 - 4x + 19$ and $r(x) = -75$

(B) $q(x) = x^2 + x + 2$ and $r(x) = x - 3$

(C) $q(x) = x^2 - 1$ and $r(x) = 4$

(D) $q(x) = x^2 - 4x + 19$ and $r(x) = -75x + 1$

(E) $q(x) = x^2 - 1$ and $r(x) = 5$

Listed here are three functions. Which of these is a rational function?

I. $\dfrac{x + \sqrt{x} + 2}{3x^2 - x - 10}$ II. $\dfrac{x - 1}{x + 1}$

III. $\sqrt{x - 14} + 7$

(A) None of these   (B) Only I   (C) Only II

(D) Only III   (E) Only I and II

What type of function is $f(x) = -\frac{4}{x}$?

(A) exponential   (B) logarithmic

(C) rational   (D) linear

(E) quadratic

Compute $\log_{\frac{1}{6}} \frac{1}{36}$.

(A) -2 (B) $\frac{1}{2}$ (C) 2

(D) $-\frac{1}{2}$ (E) $\frac{1}{6}$

Shown here is the graph of the function $f(x)$. Which of the choices below shows the graph of $f^{-1}(x)$ in red?

A **B** **C**

D **E**
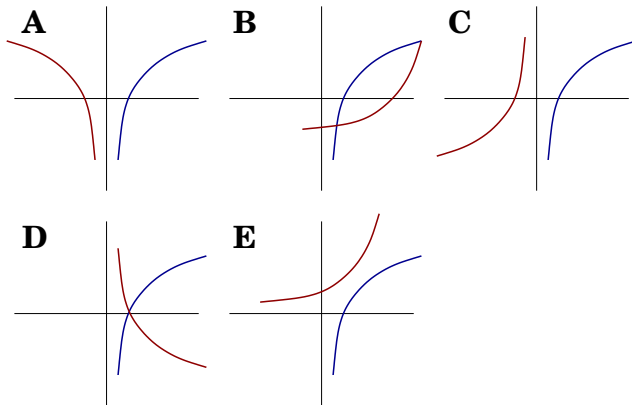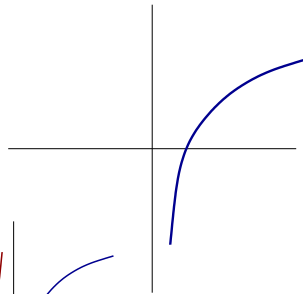
Shown here is a graph of $y = f(x)$. What is $f^{-1}(-1)$?

(A) 2

(B) 4

(C) -2

(D) $-\frac{1}{2}$

(E) $\frac{1}{2}$
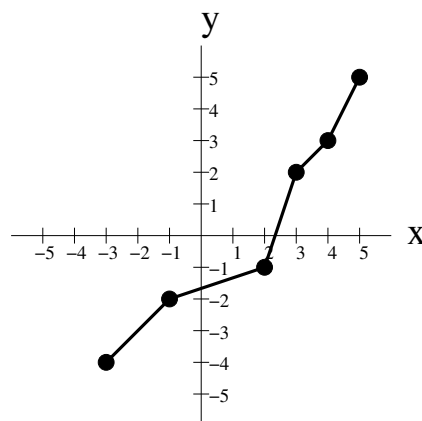
Which of the functions below is an exponential function with base 10?

(A) x^{10} (B) 10^x (C) $\log_{10} x$

(D) $\log_x 10$ (E) e^{10}

Which of the functions below is the natural logarithm function?

(A) $\log_{10} x$ (B) e^x (C) $\log 10$

(D) $\log_e x$ (E) $\log_{10} e$

Solve for x in the equation $3 = 3\log_2(x - 3)$.

(A) 11   (B) 9   (C) 5

(D) 515   (E) 64

Shown here are several graphs. Which is the graph of 2^x and which is the graph of $\left(\frac{1}{2}\right)^x$?
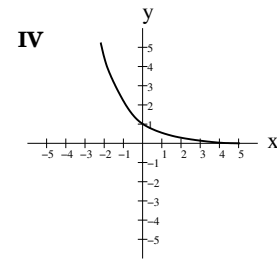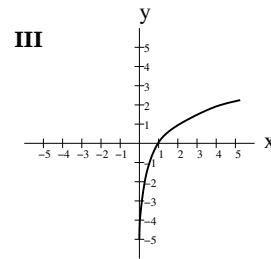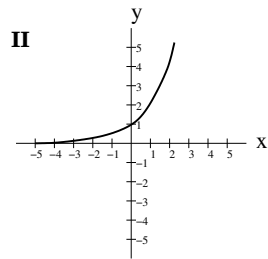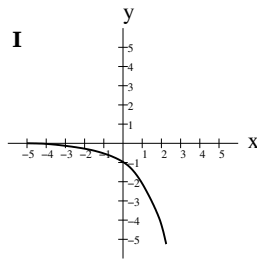
(A) I is 2^x and III is $\left(\frac{1}{2}\right)^x$ (B) II is 2^x and III is $\left(\frac{1}{2}\right)^x$

(C) II is 2^x and I is $\left(\frac{1}{2}\right)^x$ (D) II is 2^x and IV is $\left(\frac{1}{2}\right)^x$

(E) III is 2^x and IV is $\left(\frac{1}{2}\right)^x$

Express $\log_7 3$ in terms of base 10 logarithms.

(A) $\log_{10} 37$ (B) $\dfrac{\log_{10} 7}{\log_{10} 3}$ (C) $\log_{10} \dfrac{3}{7}$

(D) $\dfrac{\log_{10} 3}{\log_{10} 7}$ (E) $\log_{10} \dfrac{7}{3}$

Find a formula for the inverse of $f(x) = \dfrac{x+10}{x-3}$.

(A) $\dfrac{x-3}{3x+10}$ (B) $\dfrac{7}{x-1}$ (C) $\dfrac{-x-3}{3x+10}$

(D) $\dfrac{3x+10}{x-1}$ (E) $\dfrac{-3x+10}{x-1}$

Here is a list of values for the function $g(x)$:

x	1	2	3	4	5	6
$g(x)$	2	4	5	6	8	10

Compute $g^{-1}(2)$.

(A) 6   (B) $\dfrac{1}{10}$ (C) 4

(D) 3   (E) 1

Find the equation of the horizontal or oblique asymptote (if any)

of the function $f(x) = \dfrac{10x^2 - 6x + 1}{2x - 4}$.

(A) $y = 5x + 7$ (B) $y = 5$ (C) There are none

(D) $y = 0$ (E) $y = 2x + 2$

Listed here are three functions. Which of these is a polynomial?

I. $\frac{x^2+2\,x+1}{x+3}$ II. $\frac{1}{7}\,x^5 - \sqrt[3]{5}\,x^2 + 2\,x + \pi$

III. $2 + x + \frac{4}{3}\,x^2 + 3\,x^{2/3}$

(A) Only I   (B) Only II   (C) Only III

(D) Only I and II (E) Only II and III

Find all vertical and horizontal asymptotes of the rational function $f(x)$, which has been factored for your convenience.

$$f(x) = \frac{3x^2 - x - 4}{x^2 + 5x + 6} = \frac{(3x-4)(x+1)}{(x+2)(x+3)}$$

(A) $x = -2$, $x = -3$, $y = 0$

(B) $x = \frac{4}{3}$, $x = -1$, $y = 3$

(C) $x = 2$, $x = 3$, $y = 0$

(D) $x = -2$, $x = -3$, $y = 3$

(E) $x = \frac{4}{3}$, $x = -1$, $y = -\frac{2}{3}$

What is the remainder when we divide $x + 1$ into $x^{10} - 4x^3 + 2x + 5$?
(HINT: Use the Remainder Theorem).

(A) 1   (B) -3 (C) 8

(D) -2 (E) 4

Put these numbers in order from least to greatest:

$$0 \qquad \ln 2 \qquad \ln e$$

(A) $0 < \ln 2 < \ln e$

(B) $0 < \ln e < \ln 2$

(C) $\ln 2 < 0 < \ln e$

(D) $\ln 2 < \ln e < 0$

(E) $\ln e < \ln 2 < 0$

Here are three statements about graphs of polynomial and rational functions :

  I. The graph of a rational function may not cross a horizontal asymptote.

  II. The graph of a polynomial may not have a vertical asympote.

  III. The graph of a rational function may have several vertical asymptotes.

Which of the statements are true?

(A) Only I   (B) Only II   (C) Only III

(D) Only II and III(E) All three are true

Solve the inequality $x^2 + 3x - 28 \geq 0$.

(A) $x \leq -7$ (B) $-7 \leq x \leq 4$ (C) $x \leq 4$

(D) $x \geq 4$ (E) $x \leq -7$ or $x \geq 4$

Simplify the expression $e^{\ln(x^3)}$.

(A) x^3 (B) e^{3x} (C) $3e^x$

(D) $3x$ (E) 3

Which of the five graphs below is the graph of $y = \frac{(x-3)^2}{x-1}$?
(The y-scales have been intentionally omitted.)

A

B

C

D

E
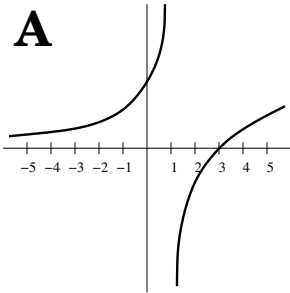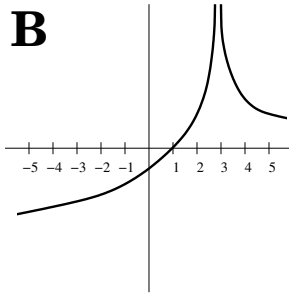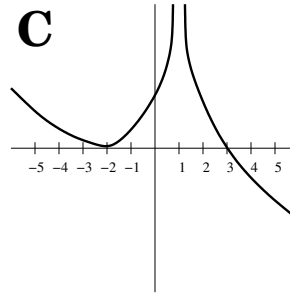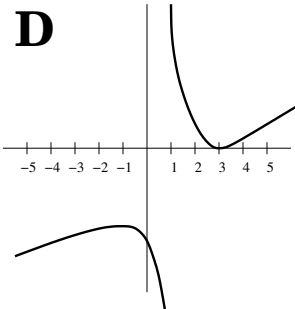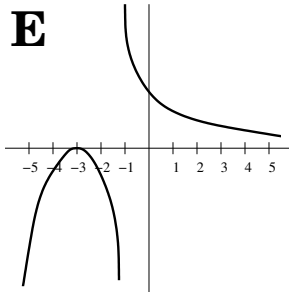
Let $f(x)$ be a one-to-one function with domain $(-3, 4)$ and range $(-3, 10)$.
What is the domain of $f^{-1}(x)$?

(A) $(-3, 4)$ (B) $(-4, 3)$ (C) $(1, 10)$

(D) $(-3, 1)$ (E) $(-3, 10)$

Express $\ln a - 2\ln b - \ln c$ as a single logarithm.

(A) $\ln\left(\frac{ac}{b^2}\right)$

(B) $\ln\left(\frac{a}{b^2c}\right)$

(C) $\ln\left(ab^2c\right)$

(D) $\ln\left(\frac{a-b^2}{c}\right)$

(E) $\ln\left(a - b^2 + c\right)$

Here are three statements about the parabola which is the graph of the equation $y = -3(x+1)^2 + 4$.

  I. The vertex of the parabola is at $(1, 4)$.
 II. The parabola is not wider than $y = x^2$.
III. The parabola opens downward.

Which of the three statements are true?

(A) Only I   (B) Only II   (C) Only III

(D) Only II and III (E) All three are true

The graph of the equation $y = x^2 - 3x + 4$ is a parabola.
What are the coordinates of its vertex?

(A) $(3, 4)$ (B) $\left(-\frac{3}{2}, \frac{7}{4}\right)$ (C) $\left(\frac{3}{2}, \frac{7}{4}\right)$

(D) $(3, -5)$ (E) $(-3, -5)$

Find all x-values that solve the equation:

$$\frac{5}{x-2} + \frac{x}{x+2} = \frac{14}{x^2-4}$$

(A) $x = -4$ and $x = 1$ (B) $x = -2$ and $x = 2$

(C) $x = \frac{7}{3}$ (D) $x = 0$ and $x = 1$

(E) $x = \frac{4}{3}$

Find all values of x that solve the equation:

$$\sqrt{x-31} + 1 = x$$

(A) $x = 32$ (B) $x = 35$ and $x = 39$

(C) $x = -4$ and $x = 2$ (D) $x = -5$ and $x = 6$

(E) There are no solutions

An office cubicle is being constructed against an existing wall of the building. Two walls of length x and one wall of length y are required, as shown here. If 20 feet of material are available, what is the largest area that can be enclosed by the cubicle?
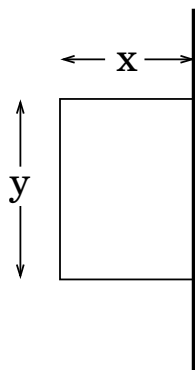
(A) 20 ft^2 (B) 50 ft^2 (C) 25 ft^2

(D) 100 ft^2 (E) 64 ft^2

What is the equation of the parabola graphed here?

(A) $y = (x+2)^2 + 3$

(B) $y = -(x+2)^2 + 3$

(C) $y = (x-2)^2 + 3$

(D) $y = -(x-2)^2 + 3$

(E) $y = -(x+2)^2 - 3$
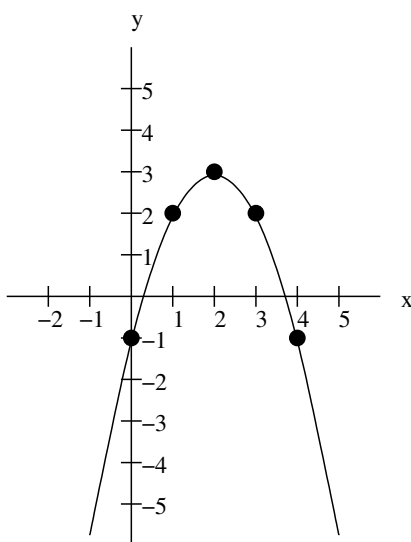
Which of the five graphs below is the graph of the polynomial

$$y = x^3 - 12x + 16 = (x - 2)^2(x + 4)$$

(The y-scales have been intentionally omitted.)

A

B

C

D

E
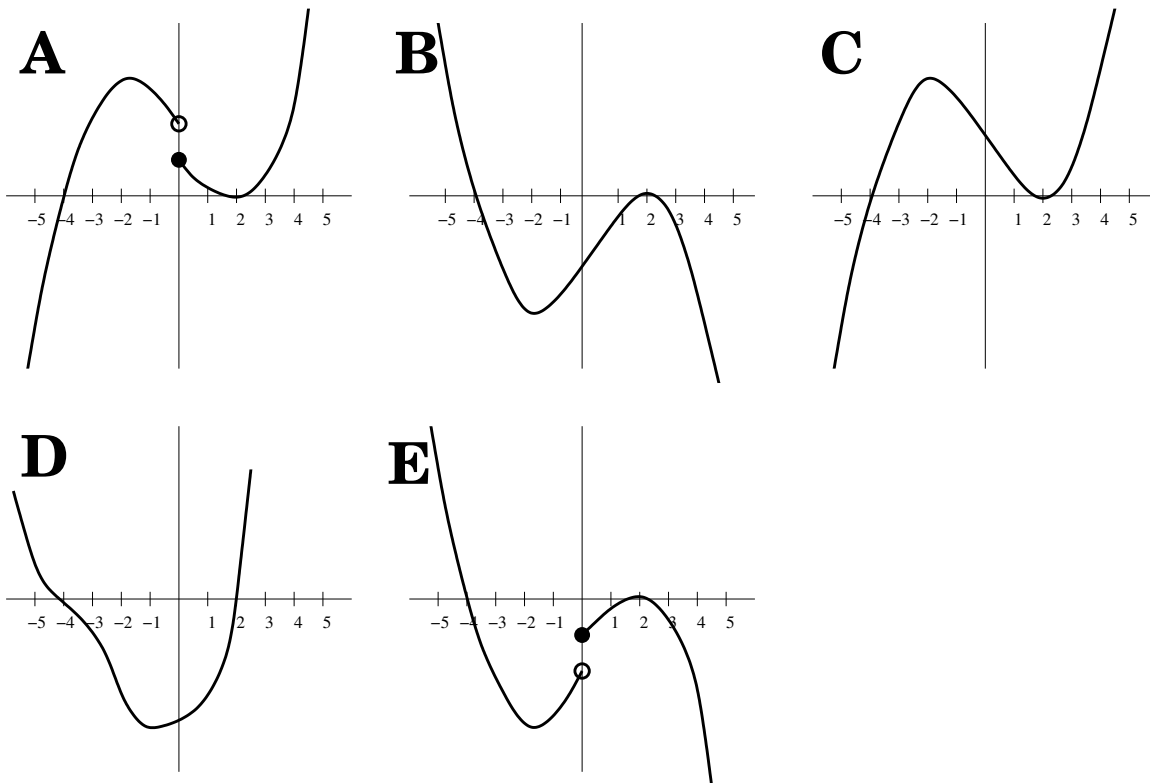
Solve for x in the equation: $\log_{10} x = 2$.

(A) 3   (B) 10   (C) 100

(D) 1000   (E) $\frac{3}{10}$

Solve for x in the equation: $5^{4x-1} = 25$.

(A) $x = -4$ (B) $x = 4$ (C) $x = -1$

(D) $x = 1$ (E) $x = \frac{3}{4}$

Find all values of x that satisfy the equation:

$$\log_2 x - \log_2(x + 3) = 2$$

(A) $x = -4$ (B) $x = 1$

(C) $x = -4$ and $x = 1$ (D) $x = 2$ and $x = -3$

(E) No Solution

Expand the logarithmic expression: $\ln \frac{(x+4)^2(x-1)}{x-7}$.

(A) $\frac{2\ln(x+4)+\ln(x-1)}{x-7}$ (B) $2\ln(x+4) + 2\ln(x-1) - \ln(x-7)$

(C) $2(\ln x + \ln 4) - \ln(-7)$ (D) $\frac{2\ln(x+4)+\ln(x-1)}{\ln(x-7)}$

(E) $2\ln(x+4) + \ln(x-1) - \ln(x-7)$

Name:

UH ID:

Directions: Print your name and ID number. Clearly mark your answers. You may write on the exam, but will only be graded on the answer you mark. If you need to change answers, please completely erase or cross out the old answer and write and circle your final choice.

The passing score is 26.

Answers:
1.C   2.D   3.C   4.A   5.A   6.B   7.D   8.A   9.A   10.A
11.A   12.D   13.B   14.B   15.C   16.A   17.B   18.B   19.D   20.E
21.E   22.B   23.C   24.E   25.D   26.B   27.C   28.B   29.E   30.C
31.B   32.E   33.E   34.D   35.B   36.B   37.D   38.C   39.E   40.D

A Norman window is a rectangle with a semicircle at the top, as shown here. Let the radius of the semicircle be r inches and the height of the rectangular part be y inches. If $y = 3r$ for a particular window, find a formula for the area A of this window (in inches^2), as a function of r only.
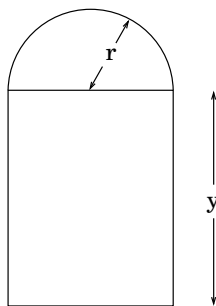
(A) $A = \pi r^2 + 3r$

(B) $A = \dfrac{\pi r^2}{2} + 3r^2$

(C) $A = \dfrac{\pi r^2}{2} + 6r^2$

(D) $A = \pi r + 5r$

(E) $A = \pi r^2 + 9r^2$

If $|x + 2| \geq 4$ and $|x + 1| = 3$, what is x?

(A) 6

(B) -6

(C) 4

(D) 2

(E) -8

Here are three statements about the polynomial
$p(x) = 22x^3 - x^4 + 4$:

   I. The leading coefficient of $p(x)$ is 22.

  II. The degree of $p(x)$ is 4.

 III. The graph of $p(x)$ has the end behavior pictured here:

Which of the statements are true?

(A) None of these　　　　　(B) Only I　　　　(C) Only II

(D) Only III　　　　　　　(E) All of these

Shown here is the graph of the polynomial $p(x)$. Which of the choices below is the only possible formula for $p(x)$?

(A) $p(x) = (x - 2)^2(x + 3)$

(B) $p(x) = -(x - 2)(x + 3)$

(C) $p(x) = x(x - 2)(x + 3)$

(D) $p(x) = -(x - 2)^2(x + 3)^2$

(E) $p(x) = (x + 2)(x + 3)$
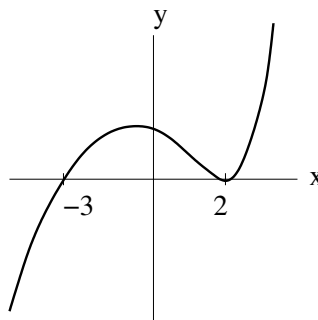
Which of these functions has a graph that is the same as the graph of $f(x)$, but shifted 4 units down?

(A) $f(x) - 4$ (B) $f(x + 4)$ (C) $4f(x)$

(D) $f(x - 4)$ (E) $f(x) + 4$

Perform the polynomial division:

$$x - 5 \,\overline{)\, x^3 - 3x^2 - 8x - 7}$$

Give both the quotient $q(x)$ and the remainder $r(x)$.

(A) $q(x) = x^2 - 4x - 1$ and $r(x) = -2$

(B) $q(x) = x^2 + 2x + 2$ and $r(x) = 3$

(C) $q(x) = x^2 + 5x - 3$ and $r(x) = 8$

(D) $q(x) = x^2 + 3x + 5$ and $r(x) = 18$

(E) $q(x) = x^2 - 7x - 45$ and $r(x) = -233$

Listed here are three functions. Which of these is a rational function?

I. $\dfrac{-1}{x}$　　　　　　II. $\dfrac{9x^7 - 4x^2 + 2}{3x + 4}$

III. $\dfrac{3x^2 - x - 1}{\sqrt{x} + x^2 + 5x^3}$

(A) Only I　　　　(B) Only II　　　　(C) Only III

(D) Only I and II　(E) All of them

What type of function is $f(x) = 8 - 2^x$?

(A) exponential　　　　　　(B) logarithmic

(C) rational　　　　　　　　(D) linear

(E) quadratic

Compute $\log_{16} 2$.

(A) $\frac{1}{4}$　　　　(B) $\frac{1}{2}$　　　　(C) 2

(D) $-\frac{1}{2}$　　　(E) -2

Shown here is the graph of the function $f(x)$. Which of the choices below shows the graph of $f^{-1}(x)$ in red?

A   **B**   **C**

D   **E**
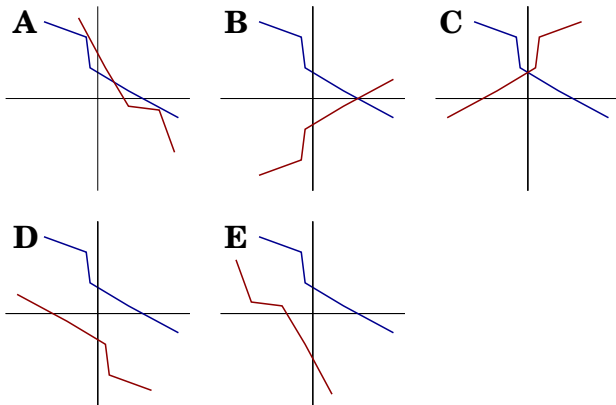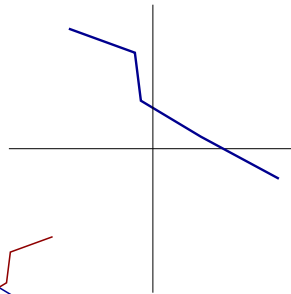
Shown here is a graph of $y = f(x)$. What is $f^{-1}(-2)$?

(A) 1   (B) -1

(C) -2 (D) 5

(E) -5
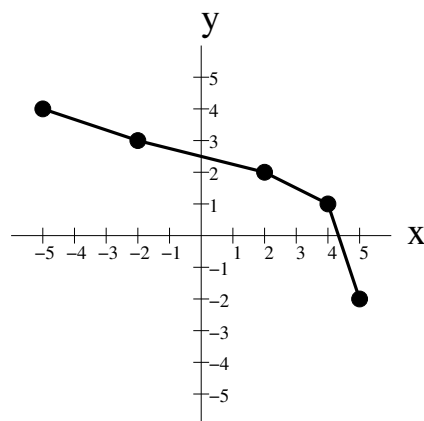
Which of the functions below is an exponential function with base 3?

(A) $\log_3 x$ 　　　(B) x^3 　　　(C) $\log_x 3$

(D) 3^x 　　　(E) e^3

Which of the functions below is a logarithmic function with base 4?

(A) 4^x 　　　(B) $\log_4 x$ 　　　(C) $\ln 4$

(D) $\log 4$ 　　　(E) $\ln x$

Solve for x in the equation $10 = 5\log_6(8x + 4)$.

(A) 30 　　　(B) 4 　　　(C) 16

(D) $\dfrac{1}{72}$ 　　　(E) $\dfrac{1}{36}$

Shown here are several graphs. Which is the graph of e^x and which is the graph of $\ln x$?

I

II

III

IV
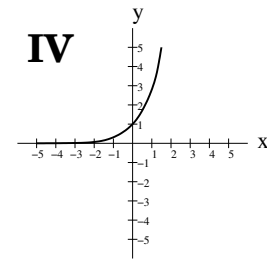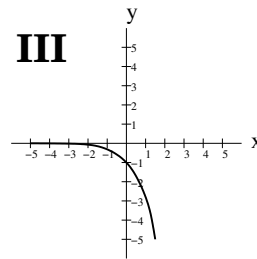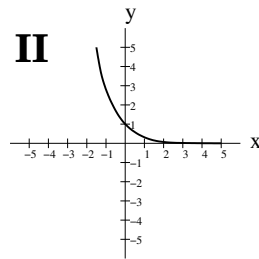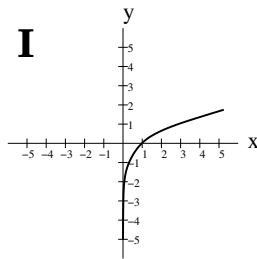
(A) IV is e^x and II is $\ln x$

(B) II is e^x and III is $\ln x$

(C) IV is e^x and I is $\ln x$

(D) III is e^x and IV is $\ln x$

(E) II is e^x and I is $\ln x$

Express $\log_{10} 6$ in terms of base e logarithms.

(A) $\dfrac{\ln 6}{\ln 10}$ (B) $\ln \dfrac{10}{6}$ (C) $\dfrac{\ln 10}{\ln 6}$

(D) $10 \ln 6$ (E) $\ln \dfrac{6}{10}$

Find a formula for the inverse of $f(x) = \dfrac{2x+3}{5x+4}$.

(A) $\dfrac{x-3}{2}$ (B) $\dfrac{4x-3}{2-5x}$ (C) $\dfrac{3x+2}{2x+4}$

(D) $\dfrac{4x+3}{5x-2}$ (E) $\dfrac{7}{5x-2}$

Here is a list of values for the function $h(x)$:

x	-7	-4	-1	0	3	5
$h(x)$	2	3	5	8	11	16

Compute $h^{-1}(16)$.

(A) -1 (B) 5 (C) $\dfrac{1}{11}$

(D) 11 (E) -4

Find the equation of the horizontal or oblique asymptote (if any)

of the function $f(x) = \dfrac{x+8}{2x^2-1}$.

(A) $y = \dfrac{1}{2}$ (B) $y = 2x - 8$ (C) There are none

(D) $y = 0$ (E) $y = 2x - 9$

Listed here are three functions. Which of these is a polynomial?

I. $\sqrt{x^2 - 9}$ II. $\frac{1}{x} + 2 + x - 5x^2$

III. $x^3 - 2x^2 + 5x - 3e^x$

Find all vertical and horizontal asymptotes of the rational function $f(x)$, which has been factored for your convenience.

$$f(x) = \frac{x^3 - 2x^2 - 5x + 6}{2x^2 - 7x - 4} = \frac{(x-1)(x+2)(x-3)}{(2x+1)(x-4)}$$

(A) $x = -\frac{1}{2}$, $x = 4$, $y = \frac{1}{2}$

(B) $x = 1$, $x = -2$, $x = 3$

(C) $x = 1$, $x = -2$, $x = 3$, $y = \frac{1}{2}$

(D) $x = \frac{1}{2}$, $y = -\frac{3}{2}$

(E) $x = -\frac{1}{2}$, $x = 4$

What is the remainder when we divide $x + 1$ into $x^8 + 3x^5 - x + 2$?
(HINT: Use the Remainder Theorem).

(A) -1 (B) 1 (C) 5

(D) 2 (E) 4

Put these numbers in order from least to greatest:

$$3 \qquad \ln 3 \qquad e^3$$

(A) $3 < \ln 3 < e^3$

(B) $3 < e^3 < \ln 3$

(C) $\ln 3 < 3 < e^3$

(D) $\ln 3 < e^3 < 3$

(E) $e^3 < \ln 3 < 3$

Here are three statements about graphs of polynomial and rational functions :

  I. The graph of a rational function may have both horizontal and vertical asymptotes.

 II. The graph of a rational function may not cross a vertical asymptote.

III. The graph of a degree 2 polynomial may have a horizontal asympote.

Which of the statements are true?

(A) None of these   (B) Only I   (C) Only II

(D) Only III   (E) Only I and II

Solve the inequality $-x^2 + 10x - 24 > 0$.

(A) $x < -6$ or $x > -4$ (B) $-6 < x < -4$

(C) $x < 4$ or $x > 6$ (D) $4 < x < 6$

(E) There are no solutions

Simplify the expression $7^{\log_7(2x)}$.

(A) 49 (B) $2x$ (C) $2\log_7 x$

(D) 7^x (E) x^2

Which of the five graphs below is the graph of

$$y = \frac{x+1}{(x-2)(x+3)} \quad ?$$

(The y-scales have been intentionally omitted.)
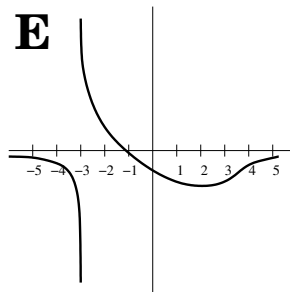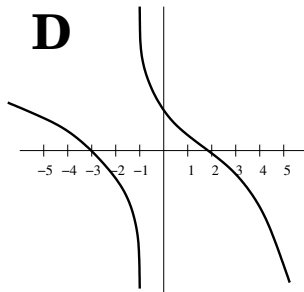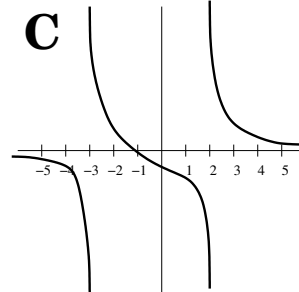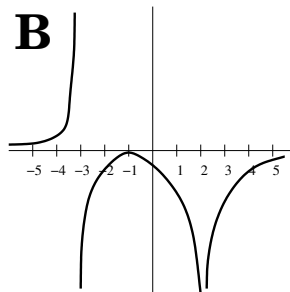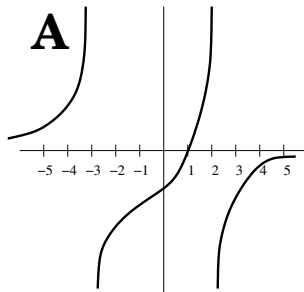
Let $f(x)$ be a one-to-one function with domain $(-2, 8)$ and range $(-1, 7)$. What is the domain of $f^{-1}(x)$?

(A) $(-2, 1)$ (B) $(-1, 7)$ (C) $(-8, 2)$

(D) $(-2, 8)$ (E) $(-7, 1)$

Express $3 \ln (a - 2b) - \ln c$ as a single logarithm.

(A) $\ln (3a - 6b + c)$ (B) $\ln (a^3 - 8b^3 + c)$

(C) $\ln ((a - 2b)^3 c)$ (D) $\ln (a^3 c - 8b^3 c)$

(E) $\ln \left(\frac{(a - 2b)^3}{c} \right)$

Here are three statements about the parabola which is
the graph of the equation $y = -\frac{1}{2}(x - 2)^2 - 7$.

  I. The vertex of the parabola is at $(2, 7)$.
 II. The parabola is not wider than $y = x^2$.
III. The parabola opens downward.

Which of the three statements are true?

(A) Only I   (B) Only II   (C) Only III

(D) Only II and III(E) All three are true

The graph of the equation $y = x^2 + 5x - 1$ is a parabola.
What are the coordinates of its vertex?

(A) $(5, 1)$ (B) $\left(-\frac{5}{2}, -\frac{29}{4}\right)$ (C) $\left(-\frac{5}{2}, -1\right)$

(D) $\left(\frac{5}{2}, -1\right)$ (E) $(-5, 1)$

Find all x-values that solve the equation:

$$\frac{1}{x} - \frac{3}{4} = \frac{5}{2}$$

(A) $x = \frac{7}{4}$ (B) $x = -\frac{4}{3}$ (C) $x = -\frac{1}{3}$

(D) $x = 0$ (E) $x = \frac{4}{13}$

Find all values of x that solve the equation:

$$\sqrt{x+2} - 4 = x$$

(A) $x = -1$ and $x = 2$ (B) $x = -2$

(C) $x = 2$ and $x = 7$ (D) $x = 7$

(E) There are no solutions

An office cubicle is being constructed against an existing wall of the building. Two walls of length x and one wall of length y are required, as shown here. If 16 feet of material are available, what is the largest area that can be enclosed by the cubicle?
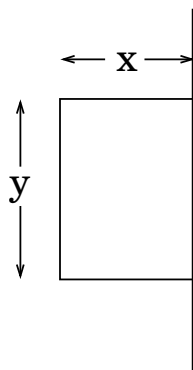
(A) 36 ft^2 (B) 256 ft^2 (C) 16 ft^2

(D) 32 ft^2 (E) 64 ft^2

What is the equation of the parabola graphed here?

(A) $y = -\frac{1}{4}(x-2)^2 - 1$

(B) $y = \frac{1}{4}(x+2)^2 - 1$

(C) $y = 4(x+2)^2 - 1$

(D) $y = 4(x-2)^2 - 1$

(E) $y = \frac{1}{4}(x+2)^2 + 1$
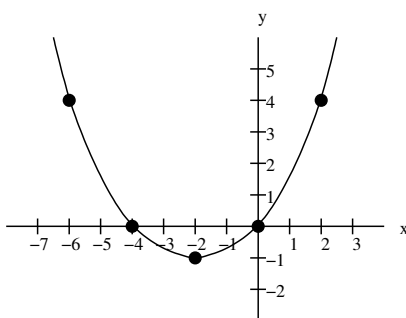
Which of the five graphs below is the graph of the polynomial

$$y = x^4 - 6x^3 - 3x^2 + 56x - 48 = (x + 3)(x - 1)(x - 4)^2$$

(The y-scales have been intentionally omitted.)

A

B

C

D

E
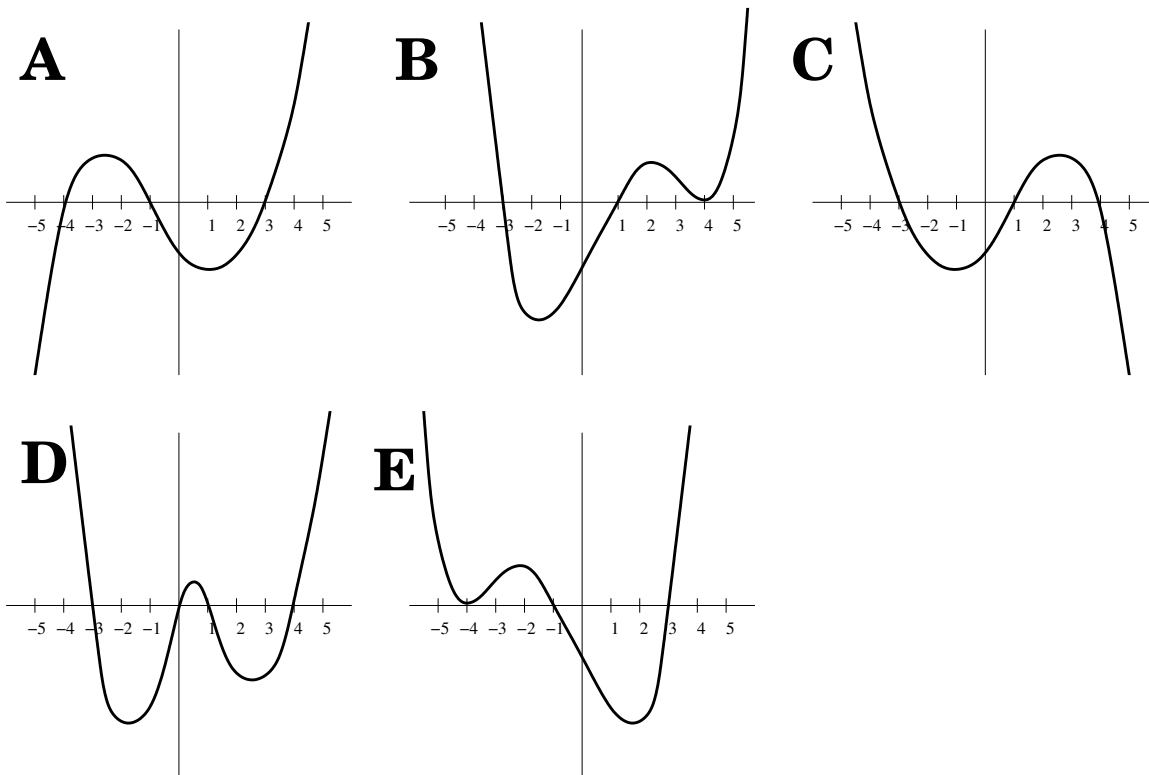
Solve for x in the equation: $\log_4 x = -\frac{1}{2}$.

(A) -2 (B) 2 (C) $\frac{1}{8}$

(D) $\frac{1}{2}$ (E) $\frac{1}{16}$

Solve for x in the equation: $3^{8-2x} = \frac{1}{27}$.

(A) $x = -2$ (B) $x = 4$ (C) $x = \frac{11}{2}$

(D) $x = \frac{1}{8}$ (E) $x = 5$

Find all values of x that satisfy the equation:

$$\ln x + \ln(x + 5) = \ln 6$$

(A) $x = e$ (B) $x = -6$ and $x = 1$

(C) $x = \frac{11}{6}$ (D) $x = -\frac{11}{6}$

(E) $x = 1$

Expand the logarithmic expression: $\ln \sqrt{\frac{a^2 b d^5}{c^3}}$.

(A) $\ln a + \ln b - 3\ln c + 5\ln d$ (B) $\ln a + \frac{1}{2}\ln b - \frac{3}{2}\ln c - \frac{5}{2}\ln d$

(C) $\ln a + \ln b - 3\ln c - 5\ln d$ (D) $\ln a + \frac{1}{2}\ln b - \frac{3}{2}\ln c + \frac{5}{2}\ln d$

(E) $\frac{2\ln a + \ln b}{6\ln c + 10\ln d}$

Name:

UH ID:

Directions: Print your name and ID number. Clearly mark your answers. You may write on the exam, but will only be graded on the answer you mark. If you need to change answers, please completely erase or cross out the old answer and write and circle your final choice.

The passing score is 26.

Answers:

1.A	2.D	3.B	4.C	5.D	6.A	7.A	8.B	9.B	10.B
11.A	12.A	13.C	14.E	15.B	16.B	17.E	18.A	19.C	20.D
21.A	22.C	23.C	24.A	25.E	26.B	27.B	28.A	29.E	30.D
31.A	32.C	33.B	34.E	35.D	36.D	37.C	38.A	39.C	40.B

A gutter is constructed by bending a rectangular sheet of metal into right angles at the dotted lines as shown here. The sheet of metal measures 6 inches by 100 inches and the bends are made at measurements of x inches as shown. Give a formula for the volume V of the resulting gutter (in inches^3), as a function of x.
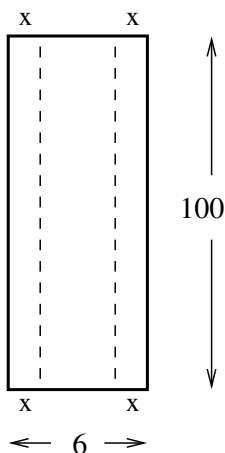
(A) $V = 100x\,(6 - 2x)$

(B) $V = (100 - 2x)\,(6 - 2x)\,x$

(C) $V = 600 - 2x$

(D) $V = 100x\,(6 - 4x)$

(E) $V = 6x\,(100 - 2x)$

If $|x + 3| < 4$ and $|x - 5| = 5$, what is x?

(A) 10 (B) -1 (C) 2

(D) 0 (E) 8

Here are three statements about the polynomial
$p(x) = 9 - x + x^2 - x^3$:

  I. The degree of $p(x)$ is 3.

  II. The leading coefficient of $p(x)$ is 1.

  III. The graph of $p(x)$ has the end behavior pictured here:

Which of the statements are true?

(A) None of these   (B) Only I   (C) Only II

(D) Only III   (E) Only I and II

Shown here is the graph of the polynomial $p(x)$. Which of the choices below is the only possible formula for $p(x)$?

(A) $p(x) = (x + 2)(x - 3)(x - 5)$

(B) $p(x) = -(x + 2)(x - 3)^2(x - 5)^2$

(C) $p(x) = (x + 2)(x - 3)(x - 5)(8x^2 + 1)$

(D) $p(x) = -(x + 2)(x - 3)(x - 5)(8x^2 + 1)$

(E) $p(x) = (x + 2)(x - 3)^2(x - 5)^2$
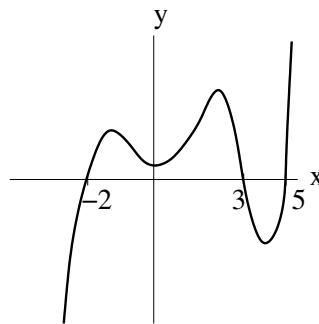
Which of these functions has a graph that is the same as the graph of $f(x)$, but shifted 6 units up?

(A) $f(x) - 6$ (B) $f(x - 6)$ (C) $6f(x)$

(D) $f(x) + 6$ (E) $f(x + 6)$

Perform the polynomial division:

$$x^2 + x - 2 \,\overline{\smash{\big)}\, x^3 - 2x^2 - 8x + 10}.$$

Give both the quotient $q(x)$ and the remainder $r(x)$.

(A) $q(x) = x - 3$ and $r(x) = -3x + 4$

(B) $q(x) = x - 3$ and $r(x) = -2x + 4$

(C) $q(x) = x - 4$ and $r(x) = -3$

(D) $q(x) = x - 4$ and $r(x) = 2x + 1$

(E) $q(x) = x - 5$ and $r(x) = 0$

Listed here are three functions.　Which of these is a rational function?

I.　$\dfrac{\sqrt{2}\ x^2+4}{2x-5}$　　　　　II.　$\dfrac{3+x+e^x}{4x+3x^2}$

III.　$\sqrt{16-x^2}$

(A) Only I　　　　(B) Only II　　　(C) Only III

(D) Only I and II　(E) None of them

What type of function is $f(x) = \log_3 x + 5$?

(A) exponential　　　　　　(B) logarithmic

(C) rational　　　　　　　　(D) linear

(E) cubic

Compute $\log_8 \sqrt{2}$.

(A) $\frac{1}{4}$　　　　(B) $\frac{1}{6}$　　　　(C) 2

(D) $\frac{1}{3}$　　　　(E) 4

Shown here is the graph of the function $f(x)$. Which of the choices below shows the graph of $f^{-1}(x)$ in red?

A

B

C

D

E
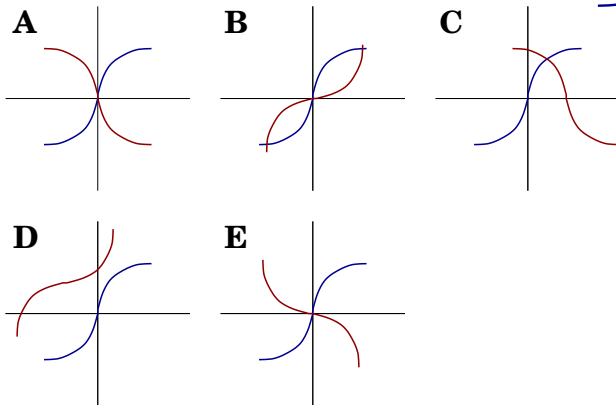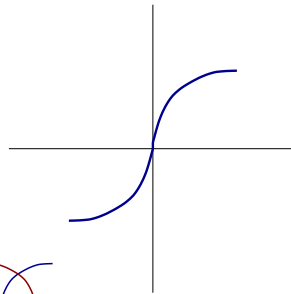
Shown here is a graph of $y = f(x)$. What is $f^{-1}(-3)$?

(A) $-\frac{1}{4}$ (B) $\frac{1}{4}$

(C) 5 (D) 4

(E) -5
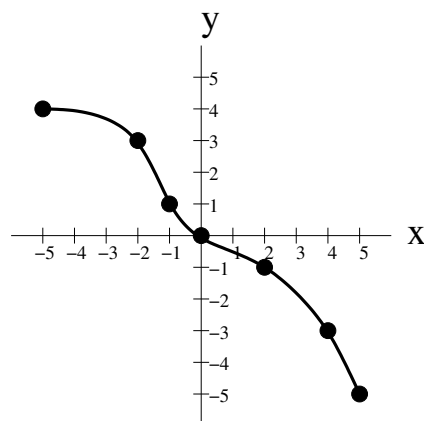
Which of the functions below is an exponential function with base e?

(A) e^x　　　　(B) x^e　　　　(C) $\ln x$

(D) $\log_x e$　　(E) e^2

Which of the functions below is a logarithmic function with base 10?

(A) $\log_x 10$　　　　(B) 10^x　　　　(C) $\log_{10} x$

(D) $\ln 10$　　　　(E) $\ln x$

Solve for x in the equation $2 = 2\log_3 (x - 1) - 2$.

(A) $\dfrac{10}{9}$　　　　(B) -2　　　　(C) $\dfrac{244}{243}$

(D) 5　　　　(E) 10

Shown here are several graphs. Which is the graph of $\log_{\frac{3}{2}} x$ and which is the graph of $\left(\frac{3}{2}\right)^x$?
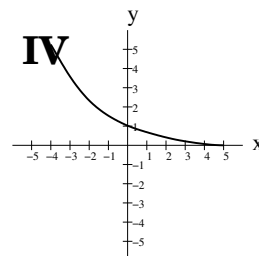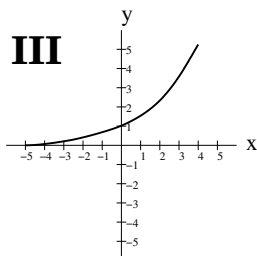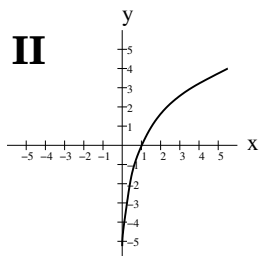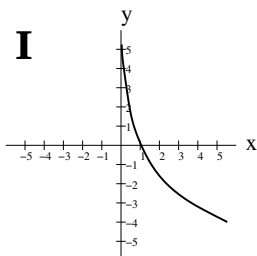
(A) I is $\log_{\frac{3}{2}} x$ and IV is $\left(\frac{3}{2}\right)^x$ (B) II is $\log_{\frac{3}{2}} x$ and III is $\left(\frac{3}{2}\right)^x$

(C) II is $\log_{\frac{3}{2}} x$ and IV is $\left(\frac{3}{2}\right)^x$ (D) I is $\log_{\frac{3}{2}} x$ and III is $\left(\frac{3}{2}\right)^x$

(E) IV is $\log_{\frac{3}{2}} x$ and I is $\left(\frac{3}{2}\right)^x$

Express $\ln 11$ in terms of base 3 logarithms.

(A) $\log_3 \dfrac{e}{11}$ (B) $\dfrac{\log_3 11}{\log_3 e}$ (C) $\log_3 11$

(D) $\dfrac{\log_3 e}{\log_3 11}$ (E) $11 \log_3 e$

Find a formula for the inverse of $f(x) = (x+4)^3 - 2$.

(A) $(x+4)^{-3} - 2$ (B) $(x+4)^{-1/3} + 2$

(C) $\sqrt[3]{x} - 2$ (D) $\sqrt[3]{x} + 2$

(E) $\sqrt[3]{x+2} - 4$

Here is a list of values for the function $k(x)$:

x	-1	0	1	2	3	4
$k(x)$	-3	-2	-1	0	1	2

Compute $k^{-1}(-2)$.

(A) 0 (B) 4 (C) -2

(D) $-\dfrac{1}{2}$ (E) It is undefined

Find the equation of the horizontal or oblique asymptote (if any)

of the function $f(x) = \dfrac{12x^3 - 1}{3x + 3}$.

(A) $y = 4x - 4$ (B) $y = 4$ (C) There are none

(D) $y = -1$ (E) $y = 0$

Listed here are three functions. Which of these is a polynomial?

  I. $x^{2/3} + 5x^{5/3} + 7$ II. $\dfrac{x - 1}{x + 1}$

III. $3 + 4x^2 - \frac{2}{3}x^3$

(A) None of them   (B) Only I   (C) Only II

(D) Only III   (E) Only I and III

Find all vertical and horizontal asymptotes of the rational function $f(x)$, which has been factored for your convenience.

$$f(x) = \frac{5x^2 + 11x - 36}{x^3 - x^2 - 6x} = \frac{(5x - 9)(x + 4)}{x(x + 2)(x - 3)}$$

(A) $x = 0$, $x = -2$, $x = 3$, $y = 0$

(B) $x = \frac{9}{5}$, $x = -4$

(C) $x = \frac{9}{5}$, $x = -4$, $y = 0$

(D) $x = 0$, $x = -2$, $x = 3$, $y = 5$

(E) $x = 5$, $y = 0$, $y = -2$, $y = 3$

What is the remainder when we divide $x - 1$ into $x^9 - x^2 + 3$?
(HINT: Use the Remainder Theorem).

(A) 1　　　　　(B) -1　　　　　(C) 3

(D) 2　　　　　(E) -4

Put these numbers in order from least to greatest:

$$e^{-2} \qquad \ln \frac{1}{e} \qquad e^0$$

(A) $e^{-2} < \ln \frac{1}{e} < e^0$

(B) $e^{-2} < e^0 < \ln \frac{1}{e}$

(C) $\ln \frac{1}{e} < e^{-2} < e^0$

(D) $\ln \frac{1}{e} < e^0 < e^{-2}$

(E) $e^0 < e^{-2} < \ln \frac{1}{e}$

Here are three statements about graphs of polynomial and rational functions :

  I. The graph of a degree 3 polynomial may have up to 3 vertical asympotes.

 II. The graph of a rational function may have both a horizontal and an oblique asymptote.

III. The graph of a rational function may cross a vertical asymptote.

Which of the statements are true?

(A) None of these   (B) Only I   (C) Only II

(D) Only III   (E) Only II and III

Solve the inequality $x^2 + 8x + 16 < 0$.

(A) $x < 4$ (B) $x > 4$ (C) $x < -4$

(D) $x > -4$ (E) There are no solutions

Simplify the expression $\log_4 \left(4^{x-3} \right)$.

(A) $4x - 12$ (B) $x - 3$ (C) $4^x - 64$

(D) 64 (E) $\log_4 (x - 3)$

Which of the five graphs below is the graph of

$$y = \frac{x+1}{(x+2)^2} \quad ?$$

(The y-scales have been intentionally omitted.)

A

B

C

D

E
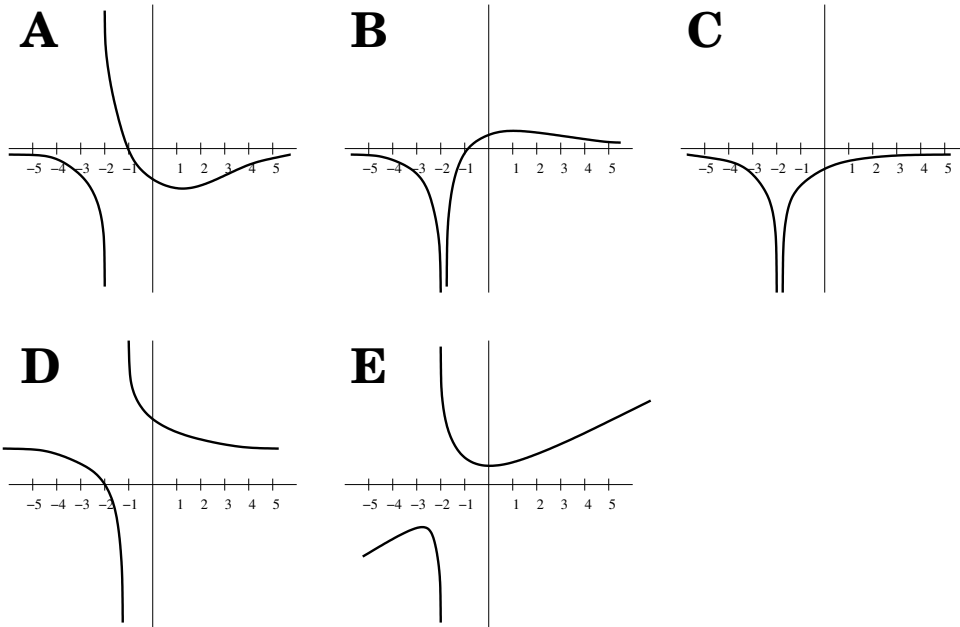
Let $f(x)$ be a one-to-one function with domain $\left(-\frac{1}{2}, \frac{1}{4}\right)$ and range $(3, 12)$. What is the range of $f^{-1}(x)$?

(A) $\left(-\frac{1}{2}, \frac{1}{4}\right)$ (B) $\left(\frac{1}{3}, \frac{1}{12}\right)$ (C) $\left(-\frac{1}{3}, -\frac{1}{12}\right)$

(D) $(-2, 4)$ (E) $(3, 12)$

Express $2 \ln a - \ln (b - c)$ as a single logarithm.

(A) $\ln (a^2 - b - c)$ 　　　　(B) $\ln (2a - b - c)$

(C) $\ln \left(\frac{2a}{b-c} \right)$ 　　　　(D) $\ln (ab - ac)$

(E) $\ln \left(\frac{a^2}{b-c} \right)$

Here are three statements about the parabola which is the graph of the equation $y = \frac{2}{3}(x + 5)^2 - 1$.

　I. The vertex of the parabola is at $(5, -1)$.
　II. The parabola is wider than $y = x^2$.
　III. The parabola opens upward.

Which of the three statements are true?

(A) Only I 　　　(B) Only II 　　　(C) Only III

(D) Only II and III(E) All three are true

The graph of the equation $y = 2x^2 + 8x + 3$ is a parabola.
What are the coordinates of its vertex?

(A) $(-2, -5)$ 　　　(B) $(-4, -13)$ 　　　(C) $(-2, -1)$

(D) $(-4, -1)$ 　　　(E) $(4, -13)$

Find all values of t that solve the equation:

$$\frac{3t - 4}{t} + \frac{t}{2} = t$$

(A) $t = \frac{4}{3}$ (B) $t = 1$ and $t = 5$

(C) $t = 2$ and $t = 4$ (D) $t = -1$ and $t = \frac{2}{3}$

(E) There are no solutions

Find all values of x that solve the equation:

$$\sqrt{3x + 1} + 3 = x$$

(A) $x = 2$ and $x = 5$ (B) $x = 8$

(C) $x = 5$ (D) $x = 1$ and $x = 8$

(E) There are no solutions

An office cubicle is being constructed against an existing wall of the building. Two walls of length x and one wall of length y are required, as shown here. If 36 feet of material are available, what dimensions should the cubicle have so that it encloses the largest possible area?
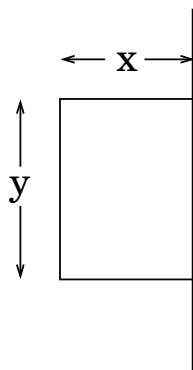
(A) $x = 15$ ft and $y = 6$ ft

(B) $x = 18$ ft and $y = 14$ ft

(C) $x = 10$ ft and $y = 16$ ft

(D) $x = 12$ ft and $y = 12$ ft

(E) $x = 9$ ft and $y = 18$ ft

What is the equation of the parabola graphed here?
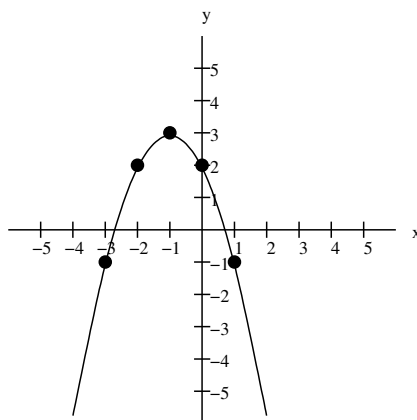
(A) $y = (x - 3)^2 - 1$

(B) $y = (x - 1)^2 + 3$

(C) $y = -(x - 1)^2 + 3$

(D) $y = -(x + 1)^2 + 3$

(E) $y = (x - 1)^2 - 3$

Which of the five graphs below is the graph of the polynomial

$$y = -x^3 - 3x^2 + x + 3 = -(x-1)(x+1)(x+3)$$

(The y-scales have been intentionally omitted.)

A

B

C

D

E
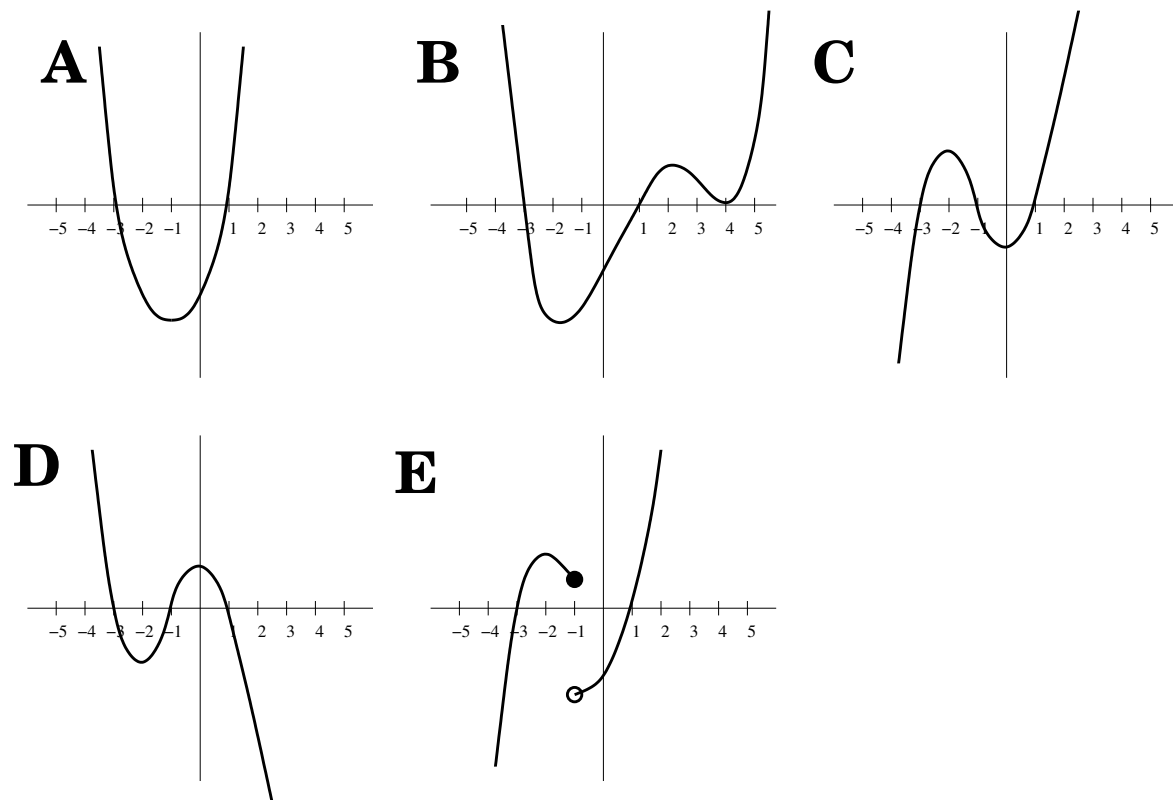
Solve for x in the equation: $\log_9 x = -2$.

(A) -3 (B) 3 (C) $\frac{1}{81}$

(D) $\frac{1}{3}$ (E) There is no solution

Solve for x in the equation: $\left(\frac{1}{4}\right)^{3-5x} = 16$.

(A) $x = 1$ (B) $x = -3$ (C) $x = \frac{4}{5}$

(D) $x = \frac{1}{5}$ (E) $x = 3$

Find all values of x that satisfy the equation:

$$\log_7(1 - x) - \log_7 x = \log_7 5$$

(A) $x = \frac{1}{4}$ (B) $x = 1$

(C) $x = \frac{1}{6}$ (D) $x = 49$

(E) There are no solutions

Expand the logarithmic expression: $\ln \sqrt{\frac{\sqrt{m}}{p^3+w^5}}$.

(A) $\frac{1}{4}\ln m - \ln(p^3 + w^5)$ (B) $\frac{1}{4}\ln m - \frac{1}{2}\ln(p^3 + w^5)$

(C) $\frac{1}{4}\ln m - 3\ln p - 5\ln w$ (D) $\frac{1}{4}\ln m - 3\ln p + 5\ln w$

(E) $\frac{1}{4}\ln m - \frac{3}{2}\ln p - \frac{5}{2}\ln w$

Name:

UH ID:

Directions: Print your name and ID number. Clearly mark your answers. You may write on the exam, but will only be graded on the answer you mark. If you need to change answers, please completely erase or cross out the old answer and write and circle your final choice.

The passing score is 26.

Answers:
1.C	2.B	3.E	4.B	5.E	6.E	7.C	8.E	9.C	10.D
11.C	12.C	13.E	14.E	15.A	16.E	17.D	18.A	19.B	20.E
21.D	22.B	23.E	24.C	25.B	26.B	27.A	28.E	29.B	30.A
31.D	32.E	33.E	34.B	35.A	36.E	37.D	38.B	39.C	40.D

Two circles are drawn inside a larger circle so that each pair of circles intersect at precisely one point. The large circle has radius r as shown. Write a formula for the area A of the region that is inside the large circle, but outside of both smaller circles.
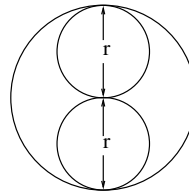
(A) $A = \frac{1}{4}\pi r^2$ (B) $A = \frac{1}{8}\pi r^2$ (C) $A = \frac{1}{2}\pi r^2$

(D) $A = \pi r$ (E) $A = \frac{1}{2}\pi r$

If $|x - 7| > 9$ and $|x - 6| = 9$, what is x?

(A) 3   (B) -3 (C) 4

(D) 8   (E) 10

Here are three statements about the polynomial
$p(x) = 7x^{10} + 300x^4 + 7x$:

   I. The degree of $p(x)$ is 7.
  II. The leading coefficient of $p(x)$ is 7.
 III. The graph of $p(x)$ has the end behavior pictured here:
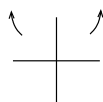
Which of the statements are true?

(A) None of these   (B) Only I   (C) Only II

(D) Only III   (E) Only II and III

Shown here is the graph of the polynomial $p(x)$. Which of the choices below is the only possible formula for $p(x)$?

(A) $= -x(x-1)(x+3)$

(B) $= -x(x-1)^2(x+3)^2$

(C) $= -x(x-1)(x+3)(x^2+1)$

(D) $= -(x-1)^2(x+3)^2$

(E) $= (x-1)(x+3)(x^2+1)$
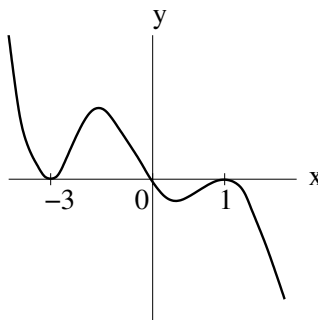
Which of these functions has a graph that is the same as the graph of $f(x)$, but shifted 2 units to the left?

(A) $f(2x)$ (B) $f(x) - 2$ (C) $f(x - 2)$

(D) $f(x) + 2$ (E) $f(x + 2)$

Perform the polynomial division:

$$x^3 + 2 \,\big)\, 3x^4 - 7x^3 + 2x^2 + 6.$$

Give both the quotient $q(x)$ and the remainder $r(x)$.

(A) $q(x) = 3x - 7$ and $r(x) = x^2 - 6x + 19$

(B) $q(x) = 3x + 1$ and $r(x) = x^2 + 3$

(C) $q(x) = 3x + 1$ and $r(x) = -x^2 + 5$

(D) $q(x) = 3x^3 + x^2 + 1$ and $r(x) = 4x - 7$

(E) $q(x) = 3x - 7$ and $r(x) = 2x^2 - 6x + 20$

Listed here are three functions. Which of these is a rational function?

  I. $x^{7/2} - 7x + 2$ II. $\frac{-2 + 5x + \sqrt[3]{x}}{-\sqrt{x} + 6x^2}$

III. $\frac{x^3 + x - 2}{11x + 105}$

(A) Only I   (B) Only II   (C) Only III

(D) Only I and II   (E) None of them

What type of function is $f(x) = x^2 - 2x^3$?

(A) exponential   (B) logarithmic

(C) linear   (D) quadratic

(E) cubic


Compute $\log_{\frac{1}{3}} \frac{1}{27}$.

(A) $\frac{1}{9}$ (B) $\frac{1}{3}$ (C) 3

(D) $-\frac{1}{3}$ (E) 9

Shown here is the graph of the function $f(x)$. Which of the choices below shows the graph of $f^{-1}(x)$ in red?

A

B

C

D

E
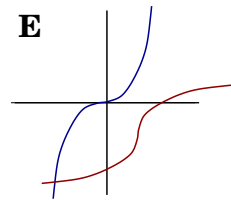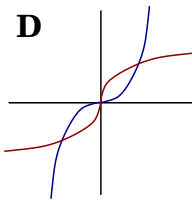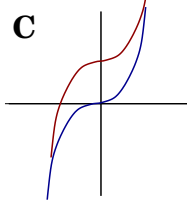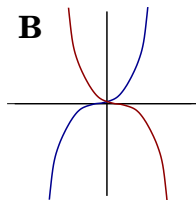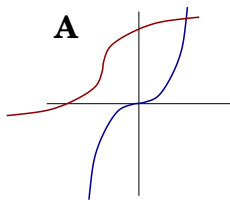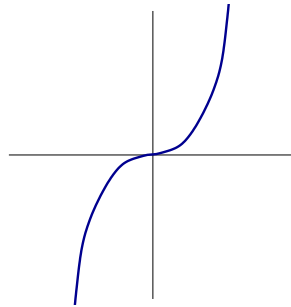
Shown here is a graph of $y = f(x)$. What is $f^{-1}(1)$?

(A) $-\frac{1}{2}$ (B) -3

(C) -2 (D) 1

(E) $\frac{1}{2}$
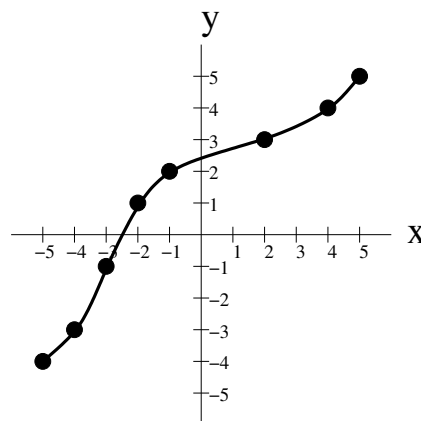
Which of the functions below is an exponential function with base 4?

(A) $\log_x 4$ (B) e^4 (C) 4^x

(D) $\log_4 x$ (E) x^4

Which of the functions below is a logarithmic function with base 5?

(A) $\ln 5$ (B) $\log_x 5$ (C) 5^x

(D) $\ln 5x$ (E) $\log_5 x$

Solve for x in the equation $6 = -8\log_{16}(4x) + 2$.

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{12}$ (C) $\dfrac{16}{3}$

(D) $\dfrac{1}{3}$ (E) $\dfrac{1}{16}$

Shown here are several graphs. Which is the graph of $\log_2 x$ and which is the graph of 2^x?
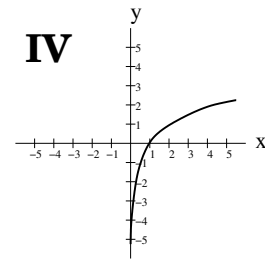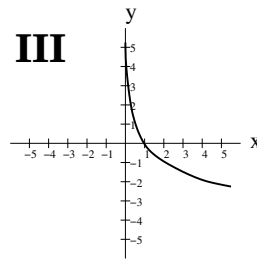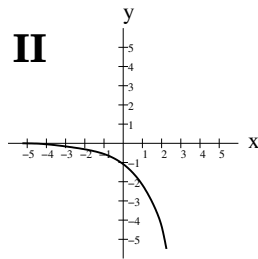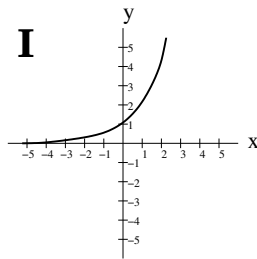
I   **II**   **III**   **IV**

(A) IV is $\log_2 x$ and I is 2^x

(B) III is $\log_2 x$ and I is 2^x

(C) IV is $\log_2 x$ and III is 2^x

(D) I is $\log_2 x$ and II is 2^x

(E) II is $\log_2 x$ and I is 2^x

Express $\log_{100} 13$ in terms of base 17 logarithms.

(A) $13 \log_{17} 100$

(B) $100 \log_{17} 13$

(C) $\dfrac{\log_{17} 100}{\log_{17} 13}$

(D) $\log_{17} \dfrac{13}{100}$

(E) $\dfrac{\log_{17} 13}{\log_{17} 100}$

Find a formula for the inverse of $f(x) = e^{3x+1} + 2$.

(A) $\ln(3x + 1) - 2$

(B) $\ln(3x + 1) + 2$

(C) $\dfrac{\ln(x + 2) - 1}{3}$

(D) $\dfrac{\ln(x - 2) - 1}{3}$

(E) $e^{-(3x+1)} - 2$

Here is a list of values for the function $R(x)$:

x	-2	-1	0	1	2	3
$R(x)$	$-1/4$	$-1/2$	1	2	4	8

Compute $R^{-1}(2)$.

(A) 1   (B) 2   (C) $\dfrac{1}{2}$

(D) 0   (E) -1

Find the equation of the horizontal or oblique asymptote (if any)

of the function $f(x) = \dfrac{x^3 + 5x^2 + 2x - 1}{-3x^3 - 2x + 1}$.

(A) $y = 0$ (B) $y = -\dfrac{1}{3}$ (C) $y = -3x + 2$

(D) $y = 3x + 1$ (E) There are none

Listed here are three functions. Which of these is a polynomial?

  I. $-\dfrac{8}{11} x^5 + \dfrac{2}{9} x^3 + 2x - 10$ II. $\dfrac{6}{7} x^{7/4} + x^5 - x^2$

III. $2 + \sqrt{2}\, x + 2x^2 + 4x^3$

(A) None of them   (B) Only I   (C) Only II

(D) Only III   (E) Only I and III

Find all vertical and horizontal asymptotes of the rational function $f(x)$, which has been factored for your convenience.

$$f(x) = \frac{x^2 + 3x - 28}{2x^2 + x - 3} = \frac{(x-4)(x+7)}{(2x+3)(x-1)}$$

(A) $x = 4$, $x = -7$, $y = 0$

(B) $x = \frac{1}{2}$, $y = -\frac{3}{2}$, $y = 1$

(C) $x = -\frac{3}{2}$, $x = 1$

(D) $x = -\frac{3}{2}$, $x = 1$, $y = \frac{1}{2}$

(E) $x = 4$, $x - 7$, $y = \frac{28}{3}$

What is the remainder when we divide $x - 1$ into $x^7 - 2x + 6$?
(HINT: Use the Remainder Theorem).

(A) 4   (B) 5   (C) 10

(D) -2 (E) 7

Put these numbers in order from least to greatest:

$$\ln e \qquad e^{-1} \qquad 0$$

(A) $\ln e < e^{-1} < 0$

(B) $\ln e < 0 < e^{-1}$

(C) $e^{-1} < \ln e < 0$

(D) $e^{-1} < 0 < \ln e$

(E) $0 < e^{-1} < \ln e$

Here are three statements about graphs of polynomial and rational functions :

  I. The graph of a rational function may not have more than one vertical asymptote.

 II. The graph of a rational function may cross an oblique asymptote.

III. The graph of a degree 4 polynomial may have up to 4 horizontal asympotes.

Which of the statements are true?

(A) None of these   (B) Only I   (C) Only II

(D) Only III   (E) Only I and III

Solve the inequality $x^2 + 10x + 16 \leq 0$.

(A) $x \leq -8$ or $x \geq -2$ (B) $-8 \leq x \leq -2$

(C) $x \geq 0$ (D) $x \geq -2$

(E) All real numbers are solutions

Simplify the expression $\log_2\left(2^{3x^3}\right)$.
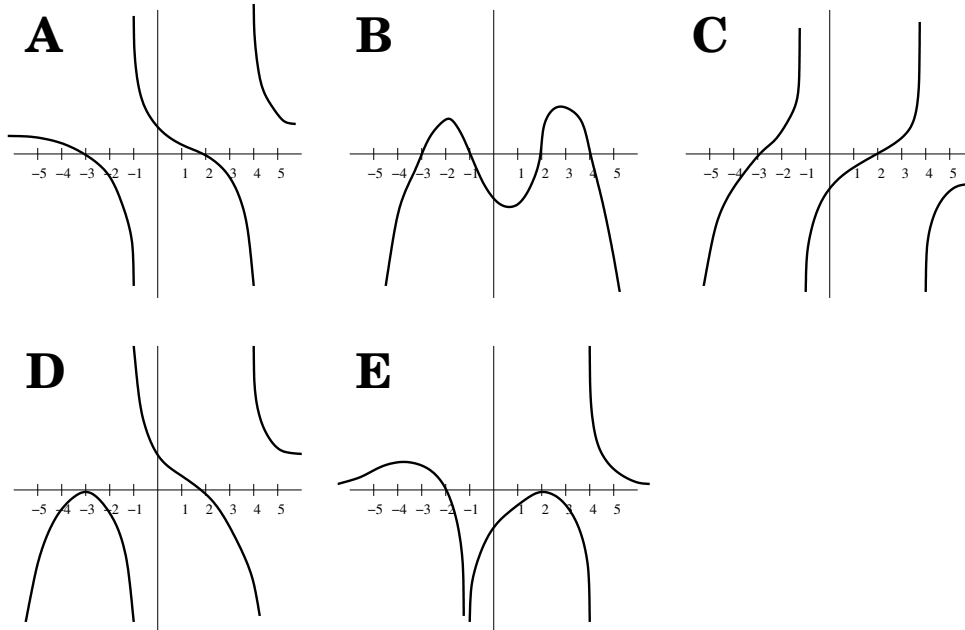
(A) $3x$ (B) $3x^3$ (C) 8

(D) $\log_2(3x)$ (E) $8x$

Which of the five graphs below is the graph of

$$y = \frac{(x+3)(x-2)}{(x+1)(x-4)} \quad ?$$

(The y-scales have been intentionally omitted.)

A

B

C

D

E

Let $f(x)$ be a one-to-one function with domain $(-6, -1)$ and range $(-4, -1)$. What is the range of $f^{-1}(x)$?

(A) $(-6, 3)$ (B) $\left(-\frac{1}{6}, \frac{1}{3}\right)$ (C) $\left(-\frac{1}{4}, -1\right)$

(D) $(-4, -1)$ (E) $(-6, -1)$

Express $\frac{1}{5} \ln a - \frac{2}{3} \ln b$ as a single logarithm.

(A) $\ln \left(\frac{a}{5} + \frac{2b}{3} \right)$ (B) $\ln \left(\dfrac{a^{1/5}}{b^{2/3}} \right)$

(C) $\ln \left(a^{2/15} b^{2/15} \right)$ (D) $\ln \left(a^{1/5} + b^{2/3} \right)$

(E) $\ln \left(\frac{2ab}{15} \right)$

Here are three statements about the parabola which is the graph of the equation $y = 2(x + 4)^2 + 10$.

  I. The vertex of the parabola is at $(4, 10)$.
 II. The parabola is wider than $y = x^2$.
III. The parabola opens upward.

Which of the three statements are true?

(A) Only III   (B) Only I and II   (C) Only I and III

(D) Only II and III   (E) All three are true

The graph of the equation $y = -x^2 + 6x - 2$ is a parabola.
What are the coordinates of its vertex?

(A) $(-3, -9)$ (B) $(3, -9)$ (C) $(3, -7)$

(D) $(3, 7)$ (E) $(6, -2)$

Find all values of y that solve the equation:

$$\frac{1}{y+2} - \frac{2}{y-3} = 1$$

(A) $y = 1$ and $y = -1$ (B) $y = -2$ and $y = 3$

(C) $y = -5$ and $y = 0$ (D) $y = \frac{2}{3}$ and $y = 6$

(E) There are no solutions

Math 135   Question 32 of 40   Sample D

Find all values of x that solve the equation:

$$\sqrt{x-5} = x + 1$$

(A) $x = 4$ (B) $x = -1$ and $x = 4$

(C) $x = -5$ (D) $x = -5$ and $x = 0$

(E) There are no solutions

Math 135   Question 33 of 40   Sample D

An office cubicle is being constructed against an existing wall of the building. Two walls of length x and one wall of length y are required, as shown here. If 60 feet of material are available, what dimensions should the cubicle have so that it encloses the largest possible area?
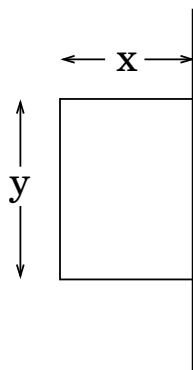
(A) $x = 10$ ft and $y = 40$ ft

(B) $x = 15$ ft and $y = 30$ ft

(C) $x = 30$ ft and $y = 12$ ft

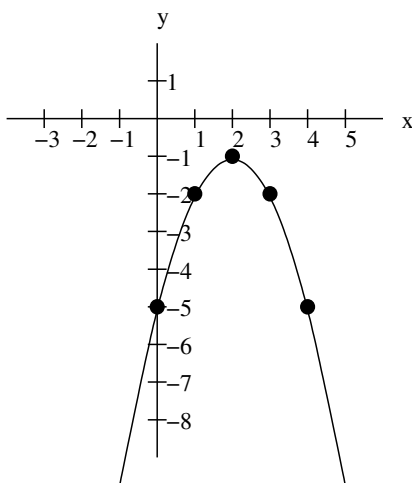(D) $x = 25$ ft and $y = 10$ ft

(E) $x = 20$ ft and $y = 20$ ft

What is the equation of the parabola graphed here?

(A) $y = -(x-2)^2 - 1$

(B) $y = -(x-1)^2 + 2$

(C) $y = -(x-1)^2 - 2$

(D) $y = -(x+2)^2 - 1$

(E) $y = -(x+1)^2 - 2$

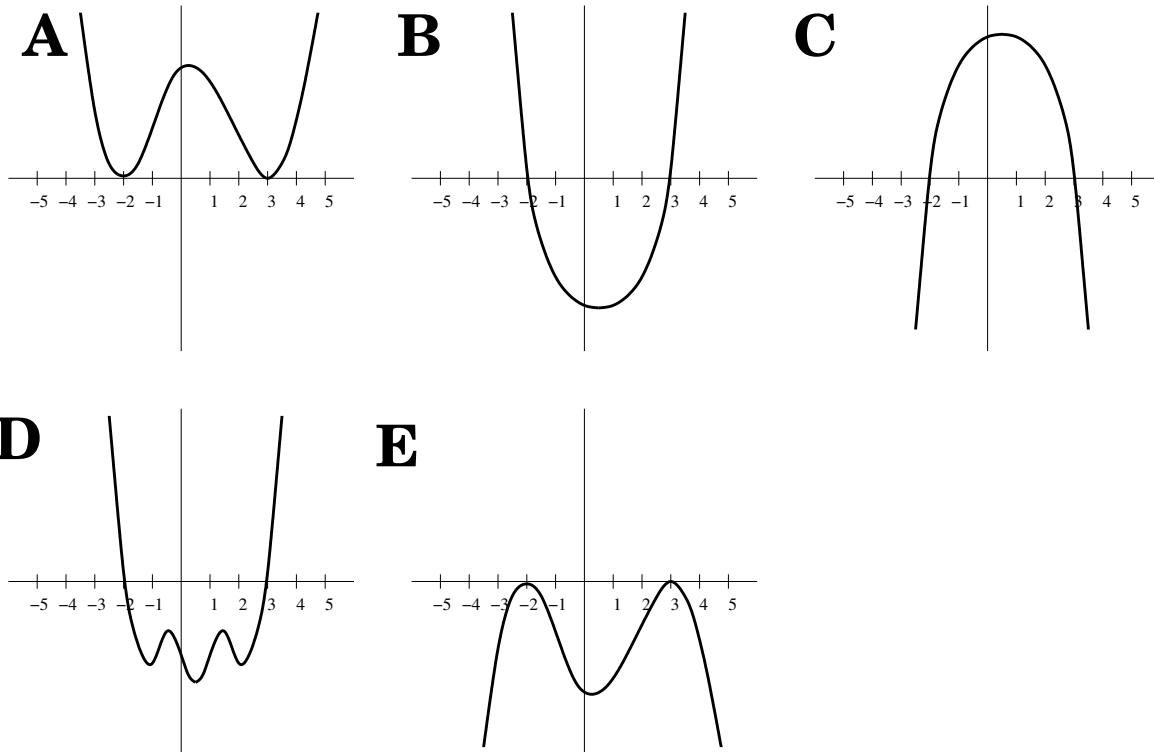Which of the five graphs below is the graph of the polynomial

$$y = -x^4 + 2x^3 + 11x^2 - 12x - 36 = -(x-3)^2(x+2)^2$$

(The y-scales have been intentionally omitted.)

A

B

C

D

E

Solve for x in the equation:　$\log_2 x = 1$.

(A)　0　　　　　(B)　1　　　　　(C)　-1

(D)　2　　　　　(E)　There is no solution

Solve for x in the equation: $125^{2x-1} = 5$.

(A) $x = \frac{3}{4}$ (B) $x = \frac{2}{3}$ (C) $x = -3$

(D) $x = -5$ (E) $x = -1$

Find all values of x that satisfy the equation:

$$\log_6(x-5) + \log_6 x = 1$$

(A) $x = -4$ and $x = 9$ (B) $x = -9$ and $x = 6$

(C) $x = 6$ (D) $x = 9$

(E) There are no solutions

Expand the logarithmic expression: $\ln\left(\frac{a^7}{\sqrt[3]{b^2+c}}\right)^3$.

(A) $10\ln a - 5\ln b - 4\ln c$ (B) $21\ln a - 3\ln(b^2+c)$

(C) $21\ln a - 6\ln b - 3\ln c$ (D) $21\ln a - \ln(b^2+c)$

(E) $21\ln a - 6\ln b + 3\ln c$