Math 100 Exam 3

1. Determine the number of possible ways to mark an answer sheet on a four-question true-or-false test.

\[
\begin{array}{cccc}
T/F & T/F & T/F & \ldots \\
2 & 2 & 2 & 16
\end{array}
\]

2. Phone numbers are 10 digits. How many phone numbers begin with ‘808’ as the first three digits (assume that any following numbers are allowed)?

\[
\begin{array}{cccc}
0 & 10 & 10 & 10 \ldots \\
1 & 1 & 0 & 16 \\
0 & 1 & 0 & 10 \ldots \\
8 & 0 & 0 & 1
\end{array}
\]

3. How many different ways could first- and second-place finishers occur in a race with five runner competing?

\[
\begin{array}{c}
5 \times 4 \\
= 20
\end{array}
\]

4. How many different three-member committees can be chosen from eight people?

\[
8C_3 = \frac{8!}{3!5!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56
\]

5. Two fair coins are tossed. What is the probability of getting both heads?

\[
\begin{array}{c}
\text{4 total outcomes} \\
\text{1 both heads} \\
\text{1} \\
\frac{1}{4}
\end{array}
\]
6. For the experiment of rolling a single fair die, find the probability of each event.

(a) odd or prime 
\[ \frac{3}{6} + \frac{3}{6} - \frac{2}{6} = \frac{4}{6} \]

(b) not less than 3 
\[ = 1 - P(less \ than \ 3) = 1 - P(1 \ or \ 2) = 1 - \frac{2}{6} = \frac{4}{6} \]

(c) not a 7 
\[ 1 - P(\text{a 7}) = 1 - 0 = 1 \]

7. A convenience store has five soft drinks: two in a red can and three in a blue can. If Astrid and Angmar, in that order, each select one drink at random without replacement, what is the probability that

(a) both select a red can 
\[ \frac{2}{5} \times \frac{1}{4} = \frac{2}{20} = \frac{1}{10} \]

(b) at least one of them selects a blue can 
\[ \text{at least one blue} = \text{not both red} \]
\[ \text{so } P(\text{at least one blue}) = 1 - P(\text{both red}) = 1 - 10 = \frac{9}{10} \]

8. If three fair coins are tossed, find the probability of each number of heads.

(a) 0 
\[ \frac{3C_0}{8} = \frac{1}{8} \]

(b) 2 
\[ \frac{3C_2}{8} = \frac{3}{8} \]

(c) 4 
\[ \frac{3C_4}{8} = \frac{0}{8} \] (or notice that it’s impossible so has prob. 0)
9. Write the first five rows of Pascal’s triangle. Circle the position on Pascal’s triangle that corresponds to \( \binom{4}{2} \), and put a square around the position that corresponds to \( \binom{2}{0} \).

\[
\begin{array}{cccccc}
& & & & 1 & \\
& & & 1 & 2 & 1 \\
& & 1 & 3 & 3 & 1 \\
& 1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]

10. What is \( nP_0 \)? Explain the meaning of this.

\[
nP_0 = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1
\]

There is only one way to select 0 items out of \( n \) total.

11. Why is \( nC_r = nC_{n-r} \)? Explain the meaning of this.

\[
nC_r = \frac{n!}{r!(n-r)!} = nC_{n-r}
\]

since multiplication is commutative, i.e., \( r!(n-r)! = (n-r)!r! \)

Selecting \( r \) items out of \( n \) total is the same as selecting \( n-r \) items if order doesn't matter.

(picking 5 things out of 15 to keep is the same as picking 10 to throw away)
12. How many ways can the following poker hands be dealt from a standard deck of cards? As in class, give the ways of getting exactly this hand. You can leave your answer as an expression in terms of \( nC_r \), \(nP_r \), and \( n! \) if desired. Briefly explain each expression.

(a) Pair.

\[
(\binom{13}{1})(\binom{4}{2}) \frac{48 \cdot 44 \cdot 40}{3!} \quad \text{choose 3 more cards that won't pair up with anything} \]

\[
\quad \text{choose a suit} \quad \text{choose two suits for this value} \quad \text{order of these doesn't matter} \]

(b) Full house.

\[
(\binom{13}{1})(\binom{4}{2})(\binom{12}{1})(\binom{4}{3}) \rightarrow \text{choose three suits for second value} \]

\[
\quad \text{choose first value} \quad \text{choose two suits for first value} \quad \text{choose second value} \]

(c) Straight flush.

\[
(\binom{4}{1})(\binom{9}{1}) \quad \text{choose a value for the low card of the straight} \]

\[
\quad \text{choose a suit} \quad \text{choose a value for the low card of the straight} \quad \text{(can be A, 2, 3, 4, \ldots, 8, 9 — a 10 would make a royal flush, and J, Q, K can't be lowest card in a straight)}
\]
Bonus: suppose a trickster put a 2 of squares (some fifth suit) in a standard deck of cards.

What is the probability of being dealt a pair of 2s with this deck?

\[
\frac{\binom{1}{1} \binom{5}{2}}{\binom{53}{5}} \cdot \frac{48}{52} \cdot \frac{44}{51} \cdot \frac{40}{50}
\]

\[\binom{53}{5}\]

This is normal part of a pair's non-matching cards.