Why Do We Study Calculus?

or,

a brief look at some of the history of mathematics

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The question I am asked most often is, "why do we study this?" (or its variant, "will this be on the exam?"). Indeed, it's not immediately obvious how some of the stuff we're studying will be of any use to the students. Though some of them will eventually *use* calculus in their work in physics, chemistry, or economics, almost none of those people will ever need *prove* anything about calculus. They're willing to trust the pure mathematicians whose job it is to certify the reliability of the theorems. Why, then, do we study epsilons and deltas, and all these other abstract concepts of proofs?

Well, calculus is not a just vocational training course. In part, students should study calculus for the same reasons that they study Darwin, Marx, Voltaire, or Dostoyevsky: These ideas are a basic part of our culture; these ideas have shaped how we perceive the world and how we perceive our place in the world. To understand how that is true of calculus, we must put calculus into a historical perspective; we must contrast the world before calculus with the world after calculus. (Probably we should put more history into our calculus courses. Indeed, there is a growing movement among mathematics teachers to do precisely that.)

The earliest mathematics was perhaps the arithmetic of commerce: If 1 cow is worth 3 goats, how much does 4 cows cost? Geometry grew from the surveying of real estate. And so on; math was useful and it grew.

The ancient Greeks did a great deal of clever thinking, but very few experiments; this led to some errors. For instance, Aristotle observed that a rock falls faster than a feather, and concluded that heavier objects fall faster than lighter objects. Aristotle's views persisted for centuries, until the discovery of air resistance.

The most dramatic part of the story of calculus comes with astronomy. People studied and tried to predict things that were out of human reach and apparently beyond human control. "Fear not this dark and cold," they would say; "warm times will come again. The seasons are a cycle. The time from the beginning of one planting season to the beginning of the next planting season is almost 13 cycles of the moon -- almost 13 cycles of the blood of fertility." The gods who lived in the heavens were cruel and arbitrary -- too much rain or too little rain could mean famine.

The earth was the center of the universe. Each day, the sun rose in the east and set in the west. Each night, the constellations of stars rose in the east and set in the west. The stars were fixed in position, relative to each other, except for a handful of "wanderers," or "planets". The motions of these planets were extremely erratic and complicated. Astrologers kept careful records of the motions of the planets, so as to predict their future motions and (hopefully) their effects on humans.

In 1543 Copernicus published his observations that the motions of the planets could be explained more simply by assuming that the planets move around the sun, rather than around the earth -- and that the earth moves around the sun too; it is just another planet. This makes the planets' orbits approximately circular. The church did not like this idea, which made earth less important and detracted from the idea of humans as God's central creation.

During the years 1580-1597, Brahe and his assistant Kepler made many accurate observations of the planets. Based on these observations, in 1596 Kepler published his refinement of Copernicus's ideas. Kepler showed that the movements of the planets are described more accurately by ellipses, rather than circles. Kepler gave three "laws" that described, very simply and accurately, many aspects of planetary motion:

1. the orbits are ellipses, with the sun at one focus
2. the velocity of a planet varies in such a way that the area swept out by the line between planet and sun is increasing at a constant rate
3. the square of the orbital period of a planet is proportional to the cube of the planet's average distance from the sun.

The few people who understood geometry could see that Kepler had uncovered some very basic truths. This bore out an earlier statement of Plato: "God eternally geometrizes."

In 1609 Galileo took a "spyglass" -- a popular toy of the time -- and used it as a telescope to observe the heavens. He discovered many celestial bodies that could not be seen with the naked eye. The moons of Jupiter clearly went around Jupiter; this gave very clear and simple evidence supporting Copernicus's idea that not everything goes around the earth. The church punished Galileo, but his ideas, once released to the world, could not be halted.

Galileo also began experiments to measure the effects of gravity; his ideas on this subject would later influence astronomy too. He realized that Aristotle was wrong -- that heavier objects do not fall faster than light ones. He established this by making careful measurements of the times that it took balls of different sizes to roll down ramps. There is a story that Galileo dropped objects of different sizes off the Leaning Tower of Pisa, but it is not clear that this really happened. However, we can easily run a "thought-experiment" to see what would happen in such a drop. If we describe things in the right way, we can figure out the results:

> Drop 3 identical 10-pound weights off the tower; all three will hit the ground simultaneously. Now try it again, but first connect two of the three weights with a short piece of thread; this has no effect, and the three weights still hit the ground simultaneously. Now try it again, but instead of thread, use superglue; the three weights will still hit the ground simultaneously. But if the superglue has dried, we see that we no longer have three 10-pound weights; rather, we have a 10-pound weight and a 20-pound weight.

Some of the most rudimentary ideas of calculus had been around for centuries, but it took Newton and Leibniz to put the ideas together. Independently of each other, around the same time, those two men discovered the Fundamental Theorem of Calculus, which states that integrals (areas) are the same thing as antiderivatives. Though Newton and Leibniz generally share credit for "inventing" calculus, Newton went much further in its applications. A derivative is a rate of change, and everything in the world changes as time passes, so derivatives can be very useful. In 1687 Newton published his "three laws of motion," now known as "Newtonian mechanics"; these laws became the basis of physics.

1. If no forces (not even gravity or friction) are acting on an object, it will continue to move with constant velocity -- i.e., constant speed and direction. (In particular, if it is sitting still, it will remain so.)
2. The force acting on an object is equal to its mass times its acceleration.
3. The forces that two objects exert on each other must be equal in magnitude and opposite in direction.

To explain planetary motion, Newton's basic laws must be combined with his law of gravitation:

- the gravitational attraction between two bodies is directly proportional to the product of the masses of the two bodies and inversely proportional to the square of the distance between them.

Newton's laws were simpler and more intuitive as Kepler's, but they yielded Kepler's laws as corollaries, i.e., as logical consequences.

Newton's universe is sometimes described as a "clockwork universe," predictable and perhaps even deterministic. We can predict how billiard balls will move after a collision. In principle we can predict everything else in the same fashion; a planet acts a little like a billiard ball.

(Our everyday experiences are less predictable, because they involve trillions of trillions of tiny little billiard balls that we call "atoms". But all the atoms in a planet stay near each other due to gravity, and combine to act much like one big billiard ball; thus the planets are more predictable.)

Suddenly the complicated movements of the heavens were revealed as consequences of very simple mathematical principles. This gave humans new confidence in their ability to understand -- and ultimately, to control -- the world around them. No longer were they mere subjects of incomprehensible forces. The works of Kepler and Newton changed not just astronomy, but the way that people viewed their relation to the universe. A new age began, commonly known as the "Age of Enlightenment"; philosophers such as Voltaire and Rousseau wrote about the power of reason and the dignity of humans. Surely this new viewpoint contributed to

- portable accurate timepieces, developed over the next couple of centuries, increasing the feasibility of overseas navigation and hence overseas commerce
- the steam engine, developed over the next century, making possible the industrial revolution
- the overthrow of "divine-right" monarchies, in America (1776) and France (1789).

Perhaps Newton's greatest discovery, however, was this fact about knowledge in general, which is mentioned less often: The fact that a partial explanation can be useful and meaningful. Newton's laws of motion did not fully explain gravity. Newton described *how much* gravity there is, with mathematical preciseness, but he did not explain *what causes* gravity. Are there some sort of "invisible wires" connecting each two objects in the universe and pulling them toward each other? Apparently not. How gravity works is understood a little better nowadays, but Newton had no understanding of it whatsoever. So when Newton formulated his law of gravity, he was also implicitly formulating a new principle of epistemology (i.e., of how we know things): we do not need to have a complete explanation of something, in order to have useful (predictive) information about it. That principle revolutionized science and technology.

That principle can be seen in the calculus itself. Newton and Leibniz knew how to correctly give the derivatives of most common functions, but they did not have a precise definition of "derivative"; they could not actually prove the theorems that they were using. Their descriptions were not explanations. They explained a derivative as a quotient of two infinitesimals (i.e., infinitely small but nonzero numbers). This explanation didn't really make much sense to mathematicians of that time; but it was clear that the computational methods of Newton and Leibniz were getting the right answers, regardless of their explanations. Over the next couple of hundred years, other mathematicians -- particularly Weierstrass and Cauchy -- provided better explanations (epsilons and deltas) for those same computational methods.

It may be interesting to note that, in 1960, logician Abraham Robinson finally found a way to make sense of infinitesimals. This led to a new branch of mathematics, called *nonstandard analysis*. Its devotees claim that it gives better intuition for calculus, differential equations, and related subjects; it yields the same kinds of insights that Newton and Leibniz originally had in mind. Ultimately, the biggest difference between the infinitesimal approach and the epsilon-delta approach is in what kind of language you use to hide the quantifiers:

- The numbers epsilon and delta are "ordinary-sized", in the sense that they are not infinitely small. They are moderately small, e.g., numbers like one billionth. We look at what happens when we vary these numbers and make them smaller. In effect, these numbers are changing, so there is motion or action in our description. We can make these numbers smaller than any ordinary positive number that has been chosen in advance.
- The approach of Newton, Leibniz, and Robinson involves numbers that do not need to change, because the numbers are infinitesimals -- i.e., they are already smaller than any ordinary positive number. But one of the modern ways to represent an infinitesimal is with a sequence of ordinary numbers that keep getting smaller and smaller as we go farther out in the sequence.

To a large extent, mathematics -- or any kind of abstract reasoning -- works by selectively suppressing information. We choose a notation or terminology that hides the information we're not currently concerned with, and focuses our attention on the aspects that we currently want to vary and study. The epsilon-delta approach and the infinitesimal approach differ only slightly in how they carry out this suppression.

A college calculus book based on the infintesimal approach was published by Keisler in 1986. However, it did not catch on. I suspect the reason it didn't catch on was simply because the ideas in it were too unfamiliar to most of the teachers of calculus. Actually, most of the unfamiliar ideas were relegated to an appendix; the new material that was really central to the book was quite small.

Yet another chapter is still unfolding in the interplay between mathematics and astronomy: We are working out what is the shape of the universe. To understand that question, let us first consider the shape of the planet. On its surface, the earth looks mostly flat, with a few local variations such as mountains. But if you went off in one direction, traveling in what seemed a straight line, sometimes by foot and sometimes by boat, you'd eventually arrive back where you started, because the earth is round. Magellan confirmed this by sailing around the world, and astronauts confirmed this with photographs in the 1960's. But the radius of the earth is large (4000 miles), and so the curvature of the two-dimensional surface is too slight to be evident to a casual observer.

In an analogous fashion, our entire universe, which we perceive as three-dimensional, may have a slight curvature; this question was raised a couple of hundred years ago when Gauss and Riemann came to understand non-Euclidean geometries. If you take off in a rocketship and travel in what seems a straight line, will you eventually return to where you began? The curvature of the physical universe is too slight to be detected by any instruments we have yet devised. Astronomers hope to detect it, and deduce the shape of the universe, with more powerful telescopes that are being built even now.

Human understanding of the universe has gradually increased over the centuries. One of the most dramatic events was in the late 19th century, when Georg Cantor "tamed" infinity and took it away from the theologians, making it a secular concept with its own arithmetic. We may still have a use for theologians, since we do not yet fully understand the human spirit; but infinity is no longer a good metaphor for that which transcends our everyday experience.

Cantor was studying the convergence of Fourier series and was led to consider the relative sizes of certain infinite subsets of the real line. Earlier mathematicians had been bewildered by the fact that an infinite set could have "the same number of elements" as some proper subset. For instance, there is a one-to-one correspondence between the natural numbers

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad \ldots$$

and the even natural numbers

$$2, \quad 4, \quad 6, \quad 8, \quad 10, \quad \ldots$$

But this did not stop Cantor. He said that two sets "have the same cardinality" if there exists a one-to-one correspondence between them; for instance, the two sets above have the same cardinality. He showed that it is possible to arrange the rational numbers into a table (for simplicity, we'll consider just the *positive* rational numbers):

1/1	1/2	1/3	1/4	**...**
2/1	2/2	2/3	2/4	**...**
3/1	3/2	3/3	3/4	**...**
4/1	4/2	4/3	4/4	**...**
...	**...**	**...**	**...**	**...**

Following along successive diagonals, we obtain a *list*:

1/1, 1/2, 2/1, 1/3, 2/2, 3/1, 1/4, 2/3, 3/2, 4/1, 1/5, **...**

This shows that the set of all ordered pairs of positive integers is *countable* -- i.e., it can be arranged into a list; it has the same *cardinality* as the set of positive integers. Now, run through the list, crossing out any fraction that is a repetition of a previous fraction (e.g., 2/2 is a repetition of 1/1). This leaves a slightly "shorter" (but still infinite) list

1/1, 1/2, 2/1, 1/3, 3/1, 1/4, 2/3, 3/2, 4/1, 1/5, **...**

containing each positive rational number exactly once. Thus the set of positive rational numbers is countable. A similar argument with a slightly more complicated diagram shows that the set of *all* rational numbers is also countable. However, by a different argument (not given here), Cantor showed that the real numbers cannot be put into a list -- thus the real numbers are *uncountable*. Cantor showed that there are even bigger sets (e.g., the set of all subsets of the reals); in fact, there are infinitely many different infinities.

As proof techniques improved, gradually mathematics became more rigorous, more reliable, more certain. Today our standards of rigor are extremely high, and we perceive mathematics as a collection of "immortal truths," arrived at by pure reason, not even dependent on physical observations. We have developed a mathematical language which permits us to formulate each step in our reasoning with complete certainty; then the conclusion is certain as well. However, it must be admitted that modern mathematics has become detached from the physical world. As Einstein said,

> As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.

For instance, use a pencil to draw a line segment on a piece of paper, perhaps an inch long. Label one end of it "0" and the other end of it "1," and label a few more points in between. The line segment represents the interval [0,1], which (at least, in our minds) has uncountably many members. But in what sense does that uncountable set *exist*? There are only finitely many graphite molecules marking the paper, and there are only finitely many (or perhaps countably many) atoms in the entire physical universe in which we live. An uncountable set of points is easy to imagine mathematically, but it does not exist anywhere in the physical universe. Is it merely a figment of our imagination?

It may be our imagination, but "merely" is not the right word. Our purely mental number system has proved useful for practical purposes in the real world. It has provided our best explanation so far for numerical quantities. That explanation has made possible radio, television, and many other technological achievements --- even a journey from the earth to the moon and back again. Evidently we are doing something right; mathematics cannot be dismissed as a mere dream.

The "Age of Enlightenment" may have reached its greatest heights in the early 20th century, when Hilbert tried to put all of mathematics on a firm and formal foundation. That age may have ended in the 1930's, when Gödel showed that Hilbert's program cannot be carried out; Gödel discovered that even the language of mathematics has certain inherent limitations. Gödel proved that, in a sense, some things cannot be proved. Even a mathematician must accept some things on faith or learn to live with uncertainty.

Some of the ideas developed in this essay are based on the book *Mathematics: The Loss of Certainty*, by Morris Kline. I enjoyed reading that book very much, but I should mention that I disagreed with its ending. Kline suggests that Gödel's discovery has led to a general disillusionment with mathematics, a disillusionment that has spread through our culture (just as Newton's successes spread earlier). I disagree with Kline's pessimism. Mathematics may have some limitations, but in our human experience we seldom bump into those limitations. Gödel's theorem in no way invalidates Newton, Cantor, or the moon trip. Mathematics remains a miraculous device for seeing the world more clearly.
