FIRST EDITION

# Survey of mathematics


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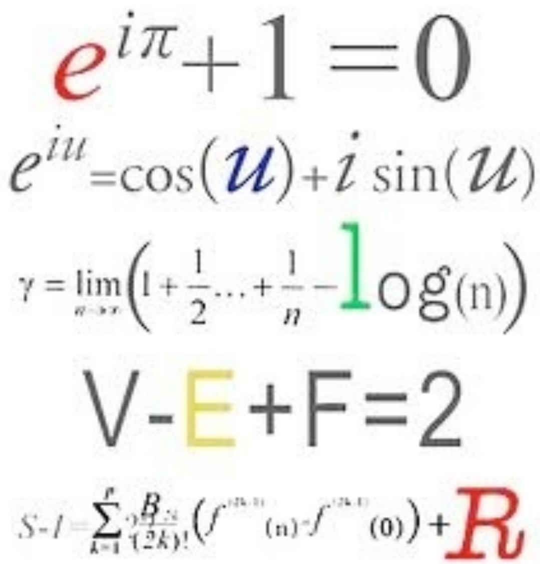
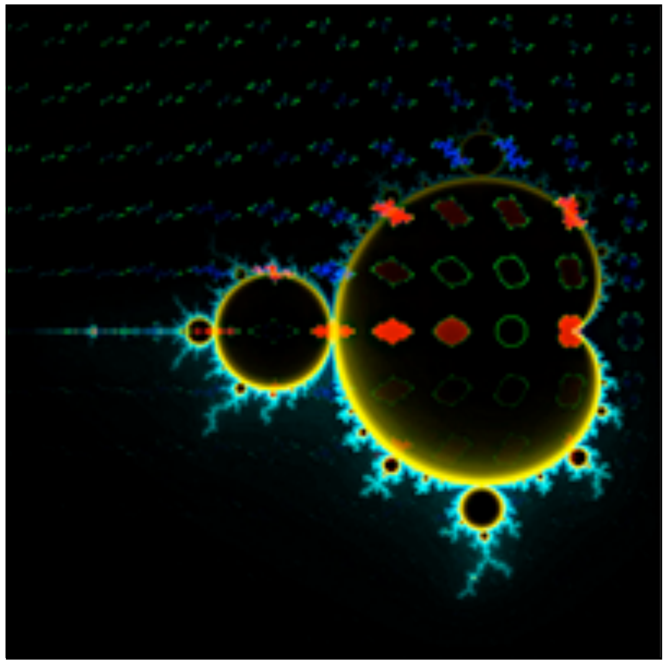
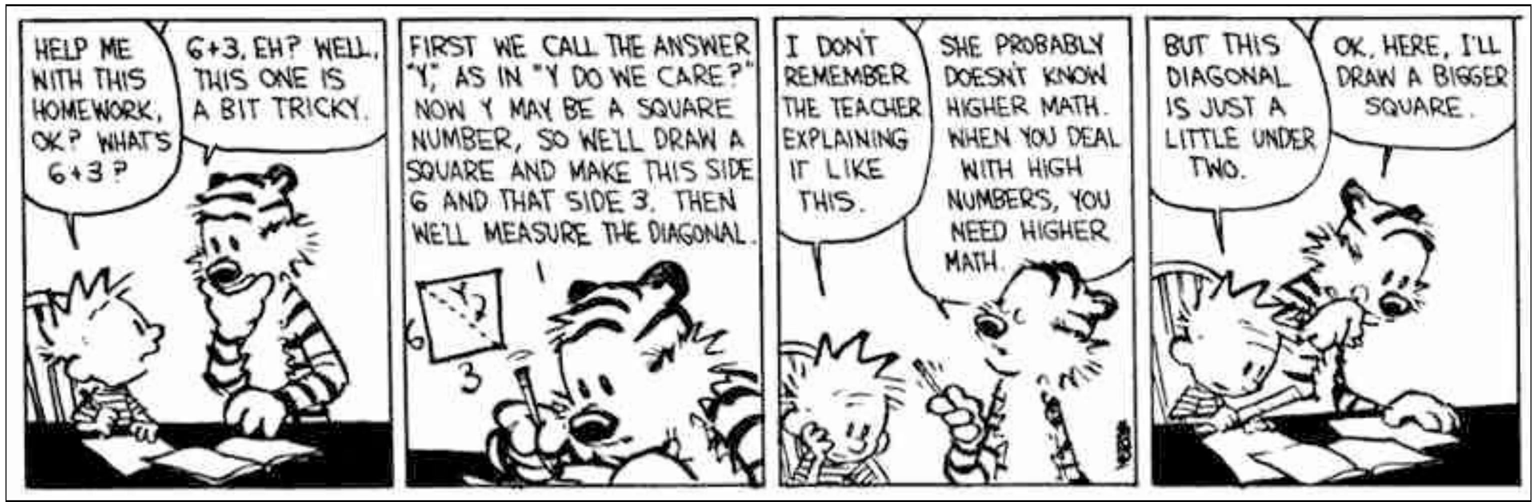


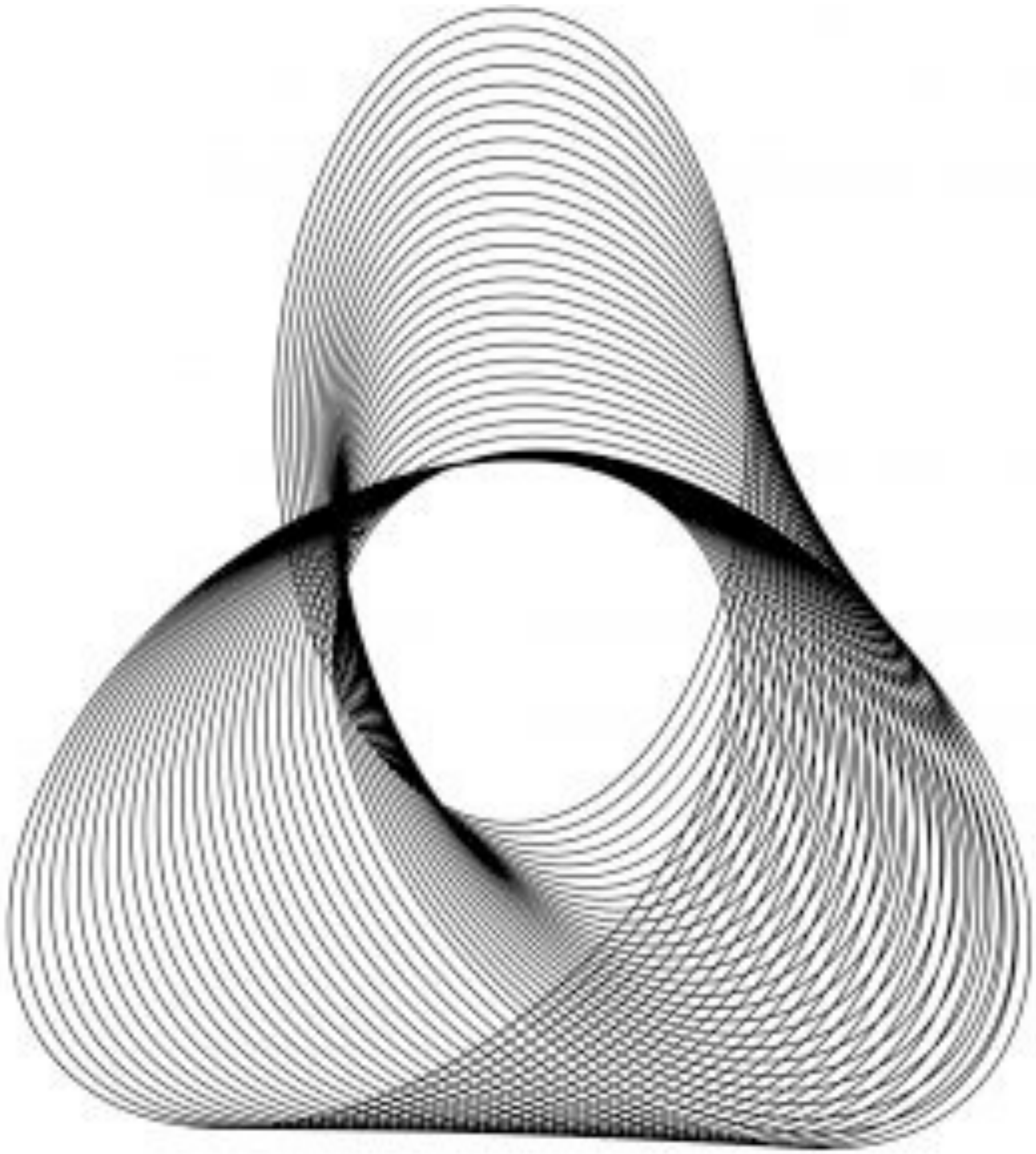





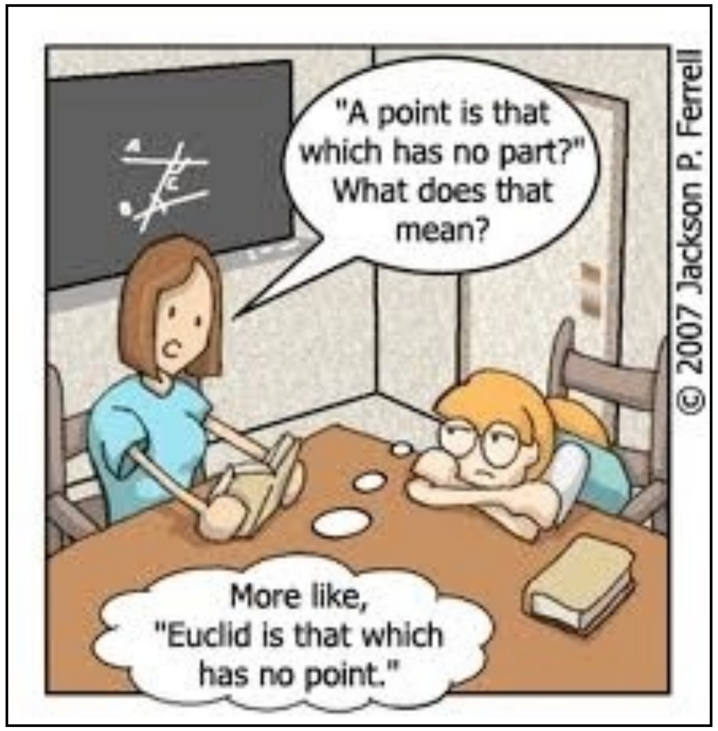
CHAPTER 1

# Geometry

*At age eleven, I [Bertrand Russell] began Euclid, with my brother as my tutor. This was one of the greatest events of my life, as dazzling as first love. I had not imagined that there was anything as delicious in the world.*
(Bertrand Russell, quoted from K.Hoechsmann, Editorial, $\pi$ in the Sky, Issue 9, Dec. 2005. A few paragraphs later K.H. adjoined: An innocent look at a page of contemporary theorems is no doubt less likely to evoke feelings of first love)
*La Géométrie d'Euclide a certainement de très grands avantages, elle accoutume l'esprit à la rigueur, à l'élégance des démonstrations et à l'enchainement mtéhodique des idées...* (J.-V. Poncelet 1822, p. xxv)
*There never has been, and till we see it we never shall believe that there can be, a system of geometry worthy of the name, which has any material departures ... from the plan laid down by Euclid.* (De Morgan 1848; copied from the Preface of Heath 1926.)
*Die Lehrart, die man schon in dem ¨altesten auf unsere Zeit gekommenen Lehrbuche der Mathematik (den Elementen des Euklides) antrifft, hat einen so hohen Grad der Vollkommenheit, dass sie von jeher ein Gegenstand der Bewunderung ....* (B.Bolzano, Grössenlehre, p. 18r, 1848)

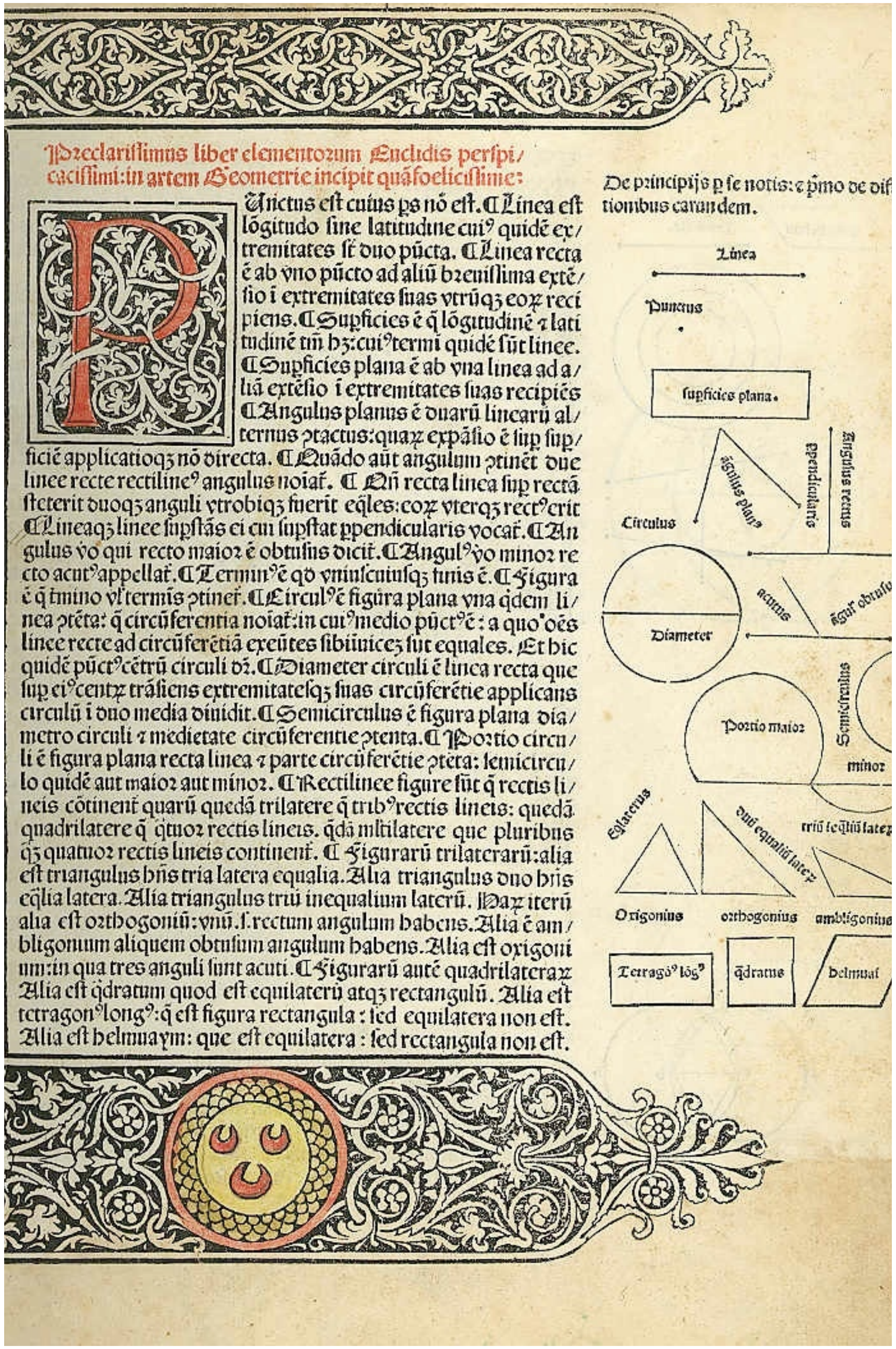


Euclid's Elements are considered by far the most famous mathematical oeuvre. Comprising about 500 pages which are organized in 13 books, they were written about 300 B.C. All mathematical knowledge of the epoch is collected there and presented in a rigour being unmatched in the following two thousand years. Over the years, the Elements have been copied, recopied, modified, commented and interpreted perpetually. Only the painstaking comparison of all available sources allowed Heiberg in 1888 to largely reconstruct the original version. The most important source (M.S. 190; this manuscript dates from the 10th century) was discovered in the treasury 1 of the Vatican, when Napoleon's troops invaded Rome in 1809. Heiberg's text has been translated into all scientific languages. The English translation by Sir Thomas L.Heath from 1908 (second enlarged edition 1926) is by far the most richly commented.



## SECTION 1

# Definitions, Axioms and Postulates



Preclarissimus liber elementorum Euclidis perspi/ cacissimi: in artem Geometrie incipit quãfoelicissime:

Punctus est cuius ps nõ est. ¶ Linea est lõgitudo sine latitudine cui⁹ quidẽ ex/ tremitates sũt duo pũcta. ¶ Linea recta ẽ ab vno pũcto ad aliũ breuissima extẽ/ sio i extremitates suas vtrũqʒ eorꝫ reci piens. ¶ Supficies ẽ q̃ lõgitudinẽ ⁊ lati tudinẽ tñ hꝫ: cui⁹ termi quidẽ sũt linee. ¶ Supficies plana ẽ ab vna linea ad a/ liã extẽsio i extremitates suas recipiẽs ¶ Angulus planus ẽ duarũ linearũ al/ ternus ɔtactus: quarꝫ expãsio ẽ sup sup/ ficiẽ applicatioqʒ nõ directa. ¶ Quãdo aũt angulum ɔtinẽt due linee recte rectiline⁹ angulus noiaẽ. ¶ Qñ recta linea sup rectã steterit duoqʒ anguli vtrobiqʒ fuerit eq̃les: eorꝫ vterqʒ rect⁹ erit ¶ Lineaqʒ linee supstãs ei cui supstat ppendicularis vocaẽ. ¶ An gulus vo qui recto maior ẽ obtusus diciẽ. ¶ Angul⁹ vo minor re cto acut⁹ appellaẽ. ¶ Termin⁹ ẽ qd vniuscuiusqʒ finis ẽ. ¶ Figura ẽ q̃ tmino vl terminis ɔtineẽ. ¶ Circul⁹ ẽ figura plana vna q̃dem li/ nea ɔtẽta: q̃ circũferentia noiaẽ: in cui⁹ medio pũct⁹ ẽ: a quo oẽs linee recte ad circũferẽtiã exeũtes sibiinicẽ sũt equales. Et hic quidẽ pũct⁹ cẽtrũ circuli dr. ¶ Diameter circuli ẽ linea recta que sup ei⁹ centꝫ trãsiens extremitatesqʒ suas circũferẽtie applicans circulũ i duo media diuidit. ¶ Semicirculus ẽ figura plana dia/ metro circuli ⁊ medietate circũferentie ɔtenta. ¶ Portio circu/ li ẽ figura plana recta linea ⁊ parte circũferẽtie ɔtẽta: semicircu/ lo quidẽ aut maior aut minor. ¶ Rectilinee figure sũt q̃ rectis li/ neis cõtinenẽ quarũ quedã trilatere q̃ trib⁹ rectis lineis: quedã quadrilatere q̃ q̃tuor rectis lineis. q̃dã mltilatere que pluribus q̃ʒ quatuor rectis lineis continenẽ. ¶ Figurarũ trilaterarũ: alia est triangulus hñs tria latera equalia. Alia triangulus duo hñs eq̃lia latera. Alia triangulus triũ inequalium laterũ. Harꝫ iterũ alia est orthogoniũ: vnũ .s. rectum angulum habens. Alia ẽ am/ bligonium aliquem obtusum angulum habens. Alia est oxigoni um: in qua tres anguli sunt acuti. ¶ Figurarũ autẽ quadrilaterarꝫ Alia est q̃dratum quod est equilaterũ atqʒ rectangulũ. Alia est tetragon⁹ long⁹: q̃ est figura rectangula: sed equilatera non est. Alia est helmuaym: que est equilatera: sed rectangula non est.

The Elements start with a long list of 23 definitions. Euclid's definitions avoid any figure; below we give an overview of the most interesting definitions in the form of pictures in Fig.1. Definition 1.1.

**Definition 1.1.**

**1.** *A **point** is that which has no part.*

2. *A **line** is breadth-less length.*

3. *The **extremities** of a line are points.*

4. *A **straight line** is a line which lies evenly with the points on itself.*

8. *A **plane angle** is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.*

10. *When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**, and the straight line standing on the other is called a **perpendicular** to that on which it stands.*

15. *A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.*

21. ***Rectilineal figures** are those which are contained by straight lines, trilateral figures being those contained by three, ...*

23. ***Parallel** straight lines are straight lines which being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.*

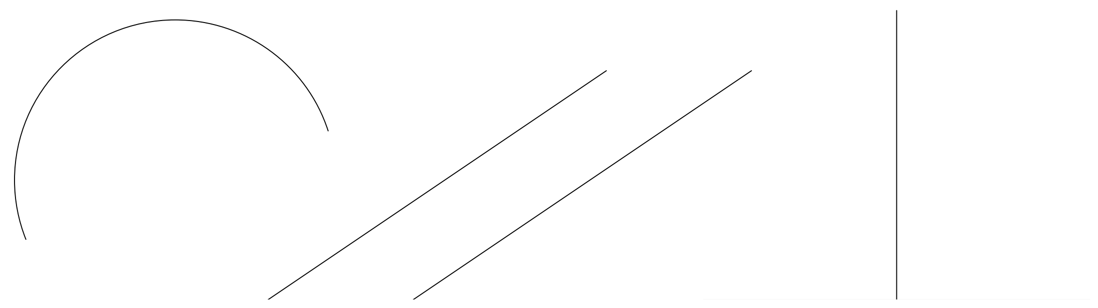


Figure 1

The definitions describe some objects of geometry. When we discuss a modern axiom system for Euclidean geometry, we will see that certain fundamental concepts must remain undefined. The first of these is the **point**. We assume that the Euclidean plane is an abstract set **E** whose elements are called "points", whatever they may be. We go along with Euclid to the extend of illustrating points as chalk marks on the blackboard. Notice that Euclid calls any bent or straight curve a "line" and that lines and straight lines all have end points. We will use the term **line** to denote a second kind of undefined objects which are certain **subsets** of **E** and correspond to Euclid's "straight lines produced indefinitely in either direction". So line and straight line is the same for us, and lines have no endpoints. What is a straight line for Euclid is a **line segment** AB for us. We must settle for the moment for our intuitive pictorial idea of a line segment. If we pretend that we know the concept of a **half ray** emanating from an **initial point**, then we can define "angle" rigorously (see below). If A,B are distinct points, then they determine a unique half ray, denoted $\overrightarrow{AB}$, which has A as initial point and contains B. In Definition 10, Euclid talks without explanation about "equal" angles and similarly he takes for granted a concept of “equal” line segments and triangles. As a matter of fact, we must accept an undefined relation between line segments AB and A′B′, called **congruence** and we write AB ≡ A′B′ if the relation holds. Similarly, we assume an undefined relation between angles ∠A,∠A′ called **angle congruence** or simply **congruence** and write ∠A ≡∠A′ if the angles are in fact congruent. We now list some modern definitions in order to clarify subsequent discussions.

**Definition 1.2.**

1. *A set of points is **collinear** if the set is contained in some straight line.*

2. *A triangle ΔABC is any set {A,B,C} of non–collinear points. The points A,B,C are the **vertices** of the triangle. The line segments AB, BC, CA are called the **sides** of the triangle.*

3. *An **angle** is a set of two half–rays h,k with common initial point not both contained in the same line. We write ∠(h,k) = {h,k}.*

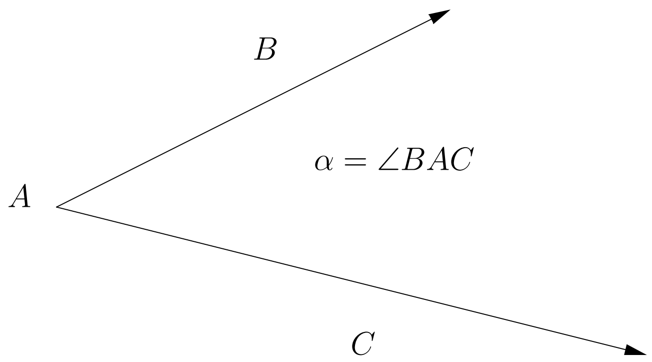



Figure 2

4. *Let ΔABC be a triangle. The angles* $\angle A = \angle(\overrightarrow{AB},\overrightarrow{AC})$*,* $\angle B = \angle(\overrightarrow{BC}, \overrightarrow{BA})$*, and* $\angle C = \angle(\overrightarrow{CA}, \overrightarrow{CB})$ *are the **angles** of the triangle.*

5. *Two triangles are **congruent** if their vertices can be matched in such a way that that all the corresponding sides are congruent and all the corresponding angles are congruent. If the vertices of the one triangles are labeled A,B,C and the corresponding vertices of the other are labeled A′,B′,C′, then we write ΔABC ≡ ΔA′B′C′ and we have AB ≡ A′B′, BC ≡ B′C′,CA ≡ C′A′, ∠A ≡∠A′, ∠B ≡∠B′, and ∠C ≡∠C′.*

6. *Let C be a point, and AB a line segment. The **circle** with **center** C and **radius** AB is the set of all points P such that PC ≡ AB.*

7. *Two (different) lines are **parallel** if they do not intersect (in the sense of set theory). We also agree, for technical reasons, that a line is parallel to itself.*

### 1.3 Euclid’s Postulates

1.*To draw a straight line from any point to any point.*

Figure 3

2. *To produce a finite straight line continuously in a straight line.*

3. *To describe a circle with any center and distance.*

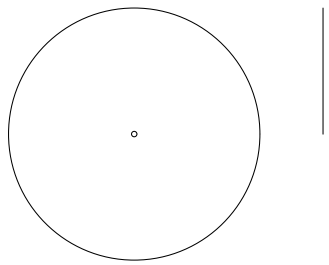


Figure 4

4.*That all right angles are equal to one another.*

5. *That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

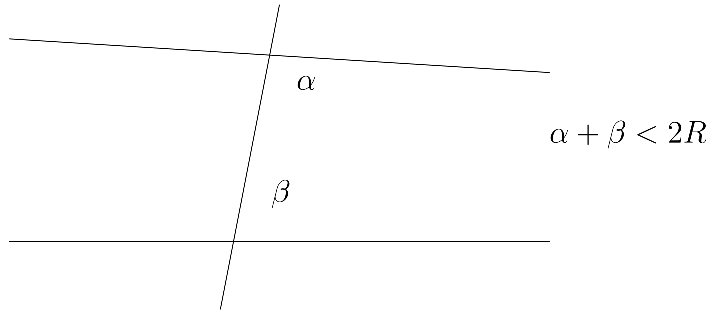


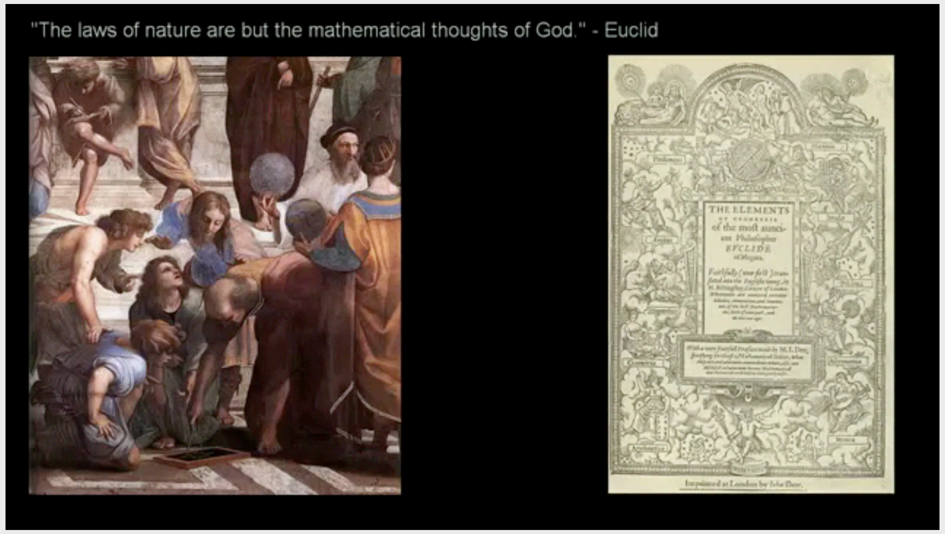

Figure 5

**Remark 1.4.** *The first three postulates are apparently motivated by the usual constructions with ruler (Post. 1 and 2) and compass (Post. 3). The fourth postulate defines the right angle as a universal measure for angles; the fifth postulate, finally, constitutes the celebrated parallel postulate. Over the centuries, it gave rise to endless discussions. The postulates are followed by common notions (also called axioms in some translations) which comprise the usual rules for equations and inequalities.*

### 1.5. Euclid's Common Notions or Axioms

1. *Things which are equal to the same thing are also equal to one another.*
2. *If equals be added to equals, the wholes are equal.*
3. *If equals be subtracted from equals, the remainders are equal.*
4. *Things which coincide with one another are equal to one another.*
5. *The whole is greater than the part.*

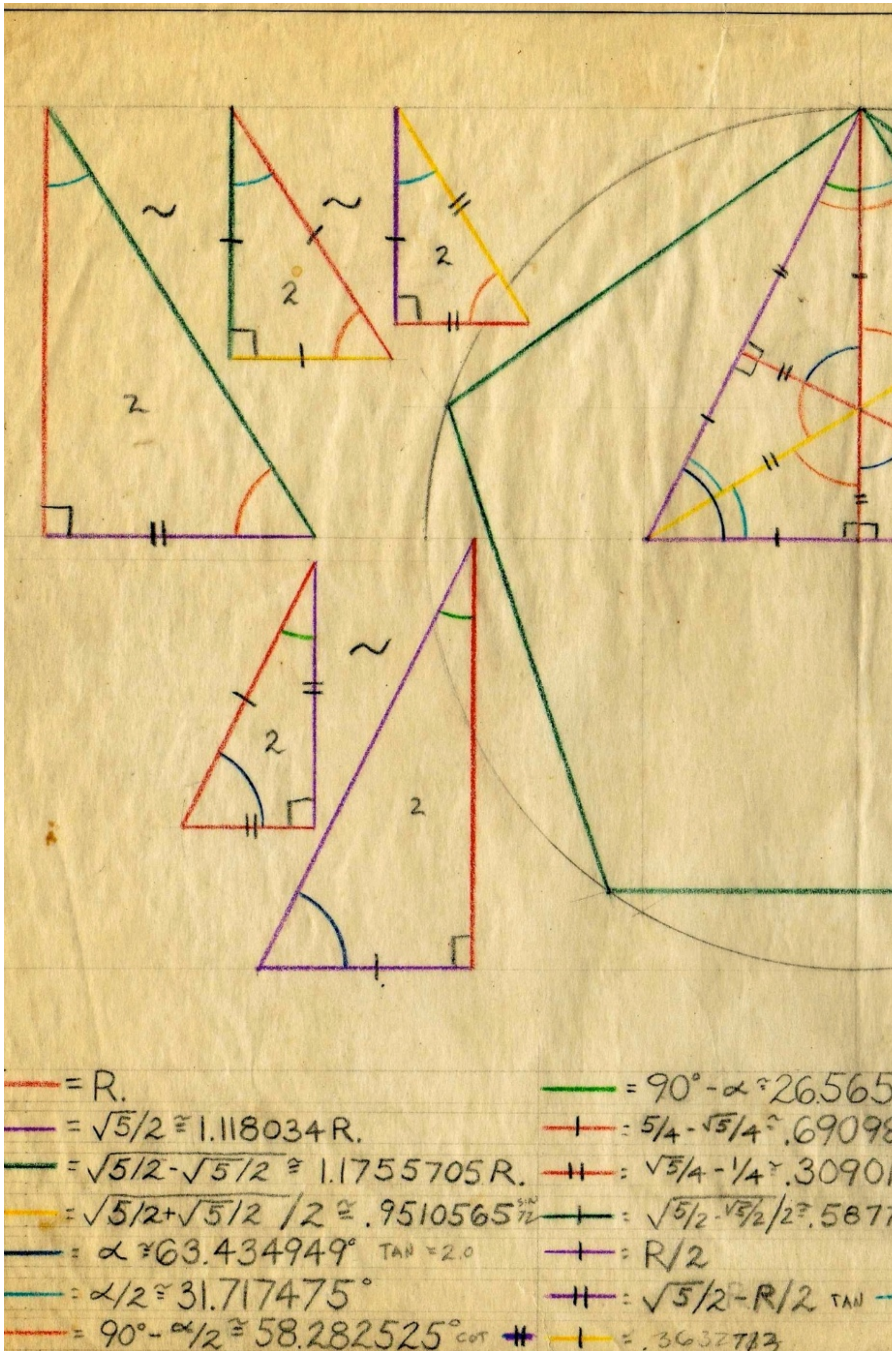
MOVIE 1.1 Euclid, the father of Geometry




Thank you Euclid!!

**SECTION 2**

# Book I. Propositions



After the definitions, postulates, and axioms, the **propositions** follow with proofs.

In the following some propositions are stated in the translation given in the Book I of *Euclid, The Thirteen Books of THE ELEMENTS*, Translated with introduction and commentary by Sir Thomas L. Hearth, Dover Publications 1956. Most propositions are translated into modern mathematical language and labeled by a decimal number indicating section number and item number. These results may be used and should be referred to in exercises.

Propositions 1 to 3 state that certain constructions are possible.

**2.1 Proposition 4** *If two triangles have two sides equal to two sides respectively, and have the enclosed angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.*

**Proof.** Superposition

**2.2 Proposition 4 bis** Let $\Delta ABC$ and $\Delta A'B'C'$ be triangles such that $AB \equiv A'B'$, $AC \equiv A'C'$ and $\angle A \equiv \angle A'$. Then $\Delta ABC \equiv \Delta A'B'C'$. (**sas**)

**2.3 Proposition 5** In $\Delta ABC$, if $AB \equiv AC$ then $\angle B \equiv \angle C$.

**Proof.** Let us see how Euclid has proved this proposition. One extends, see Fig.8, $CA$ and $CB$ (Post. 2) towards the points $F$ et $G$ with $AF = BG$. Thus the triangles $FCB$ and $GCA$ are equal, i.e., $\alpha + \delta = \beta + \varepsilon$, $\eta = \zeta$ and $FB = GA$. Now, the triangles $AFB$ and $BGA$ are equal and thus $\delta = \varepsilon$. Using the above identity, one has $\alpha = \beta$. This appears to be a brilliant proof, but is actually superfluous. This proposition is immediately followed by proposition 2.4, where the opposite implication is proved.

**2.4 Proposition 6** In $\Delta ABC$, if $\angle B \equiv \angle C$ then $AB \equiv AC$.

Proposition 7 is preparatory to Proposition 8.

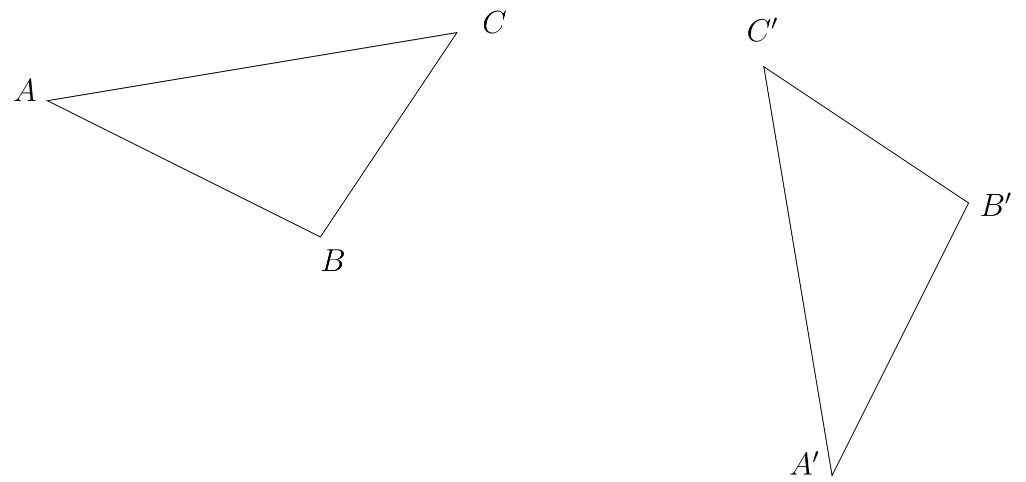



Figure 6

**2.5 Proposition 8** Let $\Delta ABC$ and $\Delta A'B'C'$ be triangles such that $AB \equiv A'B'$, $BC \equiv B'C'$ and $CA \equiv C'A'$. Then $\Delta ABC \equiv \Delta A'B'C'$. (sss)

The proof of Philo of Byzantium, which can be seen in Fig.9 is more elegant than that by Euclid.

Proposition 9 describes a method for bisecting an angle. Similarly, Proposition 10 tell how to bisect a line segment. Proposition 11 contains a construction of the perpendicular to a line at a point on the line.

**Exercise 2.6.**

1. *Describe a compass and straight-edge construction for the bisector of a given angle. Prove that the construction works.*

2. *Describe a compass and straight-edge construction for the perpendicular bisector of a given line segment. Prove that the construction works.*

3. *Describe a compass and straight-edge construction for the perpendicular to a given line at a given point on the line. Prove that the construction works.*

**2.7. Proposition 12** *To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line.*

**2.8. Proposition 12** *There is a compass and straight-edge construction for the perpendicular of a given line passing through a point not on the line.*

**Construction.** Choose a point D in the half plane of the given line l not containing the given point C. Draw the circle with center C and radius CD. It cuts l in points G,E. Let H be the midpoint of GE. Then HC is the desired perpendicular.

**2.9. Proposition 13** *Vertical angles are congruent*

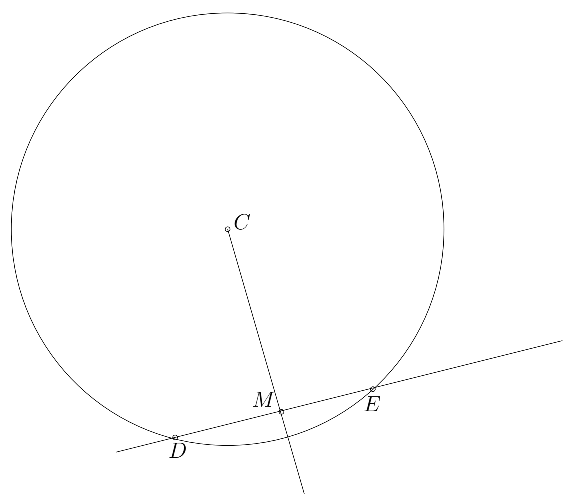



Figure 7

What follows now are "geometric inequalities". They are proved without the use of Postulate 5.

**2.10. Proposition 16** *In $\Delta ABC$, the exterior angle at C is larger than either $\angle A$ or $\angle B$.*

**2.11. Proposition 17** *In $\Delta ABC$, $\angle A + \angle B < 2R$.*

**2.12. Proposition 18** *In $\Delta ABC$, if $BC > AC$, then $\angle A > \angle B$.*

**2.13. Proposition 19** *In $\Delta ABC$, if $\angle A > \angle B$ then $BC > AC$.*

It is interesting that the Proposition 18 implies its converse, Proposition 19.

**2.14. Proposition 20 (Triangle Inequality)** *In any $\Delta ABC$, $AC + BC > AB$.*

**Exercise 2.15.**

1. *A farmer's house and his barn are on the same side of a straight river. The farmer has to walk from his house to the river and to fetch water and then to the barn to feed and water his horses. At which point on the river should he fetch water so that his path from the house via the river to the barn is as short as possible?*

2. *Prove Euclid's proposition 21.*

**2.16. Proposition 21** *Let $\Delta ABC$ be given, and let $C'$ be a point in the interior of $\Delta ABC$. Then $AC + BC > AC' + BC'$ and $\angle C' > \angle C$.*

**2.17. Proposition 22** If $a$, $b$, and $c$ are line segments such that $a + b > c$ then there is a triangle $\Delta ABC$ such that $AB \equiv c$, $BC \equiv a$, and $CA \equiv b$.

**2.18. Proposition 25** If two triangles have two sides equal to two sides respectively, but have the base greater than the base, they will also have the one of the angles contained by the equal straight lines greater than the other.

**2.19. Proposition 25** *Let $\Delta ABC$ and $\Delta A'B'C'$ be triangles such that $AB \equiv A'B'$ and $AC \equiv A'C'$ but $BC > B'C'$ then $\angle A > \angle A'$.*

**2.20. Proposition 26** *Let $\Delta ABC$ and $\Delta A'B'C'$ be triangles such that $AB \equiv A'B'$, $\angle A \equiv \angle A'$ and $\angle B \equiv \angle B'$ then $\Delta ABC \equiv \Delta A'B'C'$.* (**asa**)

The following congruence theorem does not appear in the Elements.

**2.21. Proposition** *Let $\Delta ABC$ and $\Delta A'B'C'$ be triangles such that $\angle A \equiv \angle A'=R$, $AB \equiv A'B'$, $BC \equiv B'C'$ and $BC > AB$ then $\Delta ABC \equiv \Delta A'B'C'$.* (**rss**)

**2.22. Proposition 27** *If a line cuts a pair of lines such that the alternating angles are congruent then the lines of the pair are parallel.*

**2.23. Proposition 28** *If a line cuts a pair of lines such that corresponding angles are congruent, then the lines of the pair are parallel.*

**Exercise 2.24.** Note that Proposition 2.8 says in particular that given a line l and a point P not on l, there exists a perpendicular from P to l. By Proposition 2.23 the perpendicular is unique.

1. *Let P be a point not on the line l and let $Q \in l$ be the foot of the perpendicular from P to l. Let X be any point of l, $X \neq Q$. Prove that $PX > PQ$. Hint: 2.10 and 2.13.*

*The line segment PQ in 1. is called the (**segment) distance of P from l**. The (segment) **distance of the point A from the point B i**s the line segment AB.*

2. *Let a,b be distinct lines intersecting in the point A. Prove: A point X has congruent segment distances from line a and line b if and only if X lies on the angle bisector of one of the four angles formed by the line a and b.*

3. *Let A,B be two distinct points. Prove that a point X has congruent segment distances from point A and point B if and only if X lies on the perpendicular bisector of AB.*

**Exercise 2.25.** Prove the following facts.

1. *The bisectors of the three angles of a triangle meet in a point.*

2. *The perpendicular bisectors of the three sides of a triangle meet in a single point.*

**Exercise 2.26**. *Let C be the circle with center A and radius AB. The interior of C is the set of all points X such that $AX < AB$; the exterior of C is the set of all points X such that $AX > AB$. Take for granted the fact that a line which contains an interior point of C intersects C in more than one point.*

1. *Let $P \in$ **C** and let t be the unique line containing P such that t is perpendicular to $\overleftrightarrow{AP}$. Prove that every point of t except P belongs to the exterior of **C**.*

2. *Let t be a line which intersects **C** in exactly one point T. Prove that $\overleftrightarrow{AP}$ is perpendicular to t.*

3. *Prove that a line intersects a circle in at most two points.*

Now, for the first time, Postulate 5 will be used.

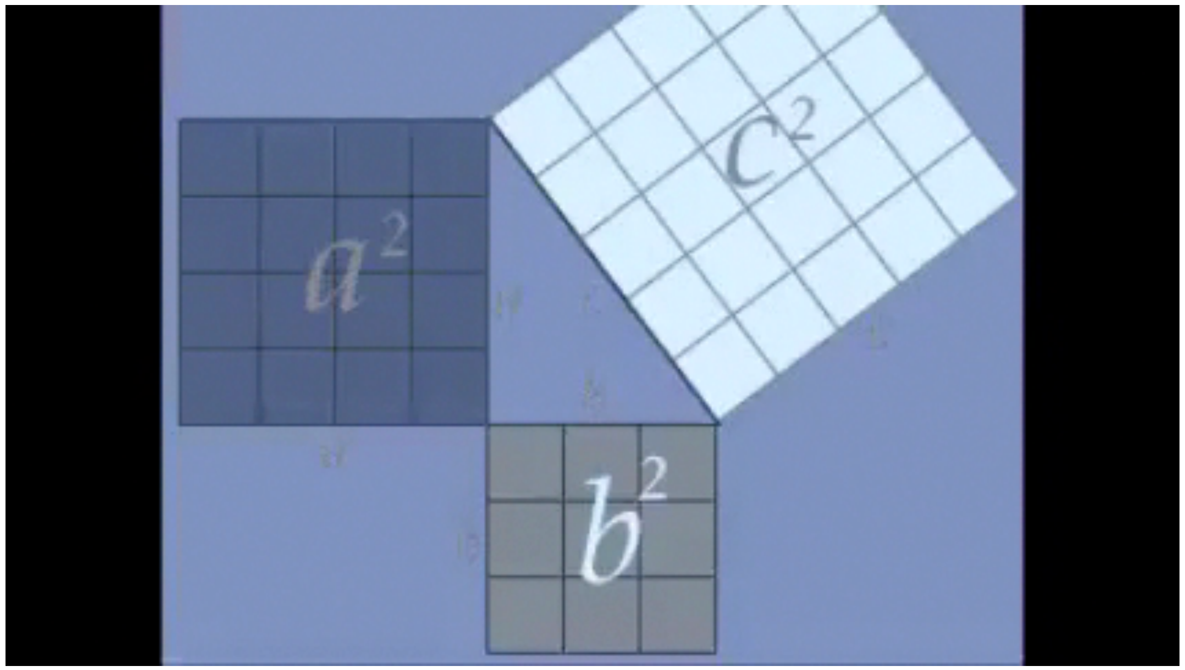
**2.27. Proposition 29** *If a,b are a pair of parallel lines then the corresponding angles at a transversal are congruent.*

**2.28. Proposition 30** *If a is parallel to b, and b is parallel to c, then a is parallel to c.*

The famous next theorem contains the important fact that the angle sum of a triangle is $180°$.

**2.29. Proposition 32** *In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.*

MOVIE 1.2 Triangles




*Why they are usefull..*

**Exercise 2.30**. *Prove that the three heights (or altitudes) of a triangle meet in a single point using that the perpendicular bisectors of the three sides of a triangle meet in a single point (Exercise 2.25).*

Propositions 33 to 36 deal with parallelograms.

Exercise 2.31. *Recall that a* ***parallelogram*** *is a quadrilateral with opposite sides parallel.*

1. (**Proposition 33**) *Let $\square ABCD$ be a quadrilateral with sides AB, BC, CD, DA such that AB is opposite CD, and BC is opposite DA. Suppose that $AD \equiv BC$ and $\overleftrightarrow{AD}$ is parallel to $\overleftrightarrow{BC}$. Prove that $AB \equiv CD$ and $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{DC}$.*

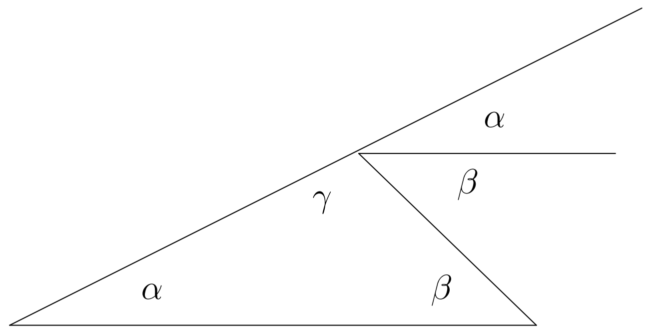



Figure 8

2.(**Proposition 34**) Let *$\square ABCD$ be the parallelogram with sides AB, BC, CD, DA such that AB is opposite CD, and BC is opposite DA. Prove that $AB \equiv CD$, $BC \equiv AD$ and that the diagonals BD and AC bisect one another.*

The following proposition deals with area for the first time. When two plane figures are called “equal” in Euclid, it means in modern terms that they have equal areas. The concept of area is treated as a known, unquestioned concept, which is not satisfactory nowadays. It is interesting, however, to observe which properties of area are used in the proofs.

**2.32. Proposition 35** *Parallelograms which are on the same base and in the same parallels are equal to one another.*

**Proof**. Let *ABCD, EBCF* be parallelograms on the same base *BC* and in the same parallels *AF, BC;*

I say that *ABCD* is equal to the parallelogram *EBCF.*

For, since *ABCD* is a parallelogram, *AD* is equal to *BC*. For the same reason *EF* is equal to *BC*, so that *AD* is also equal to *EF* [C.N. 1]; and *DE* is common; therefore the whole *AE* is equal to the whole *DF* [C.N. 2]. But *AB* is also equal to *DC* [I. 34]; therefore the two sides *EA, AB* are equal to the two sides *FD, DC* respectively, and therefore the angle *FDC* is equal to the angle *EAB*, the exterior to the interior [I. 29]; therefore the base *EB* is equal to the base *FC,* and the triangle *EAB* will be equal to the triangle *FDC* [I, 4]. Let *DGE* be subtracted from each; therefore the trapezium *ABGD* which remains is equal to the trapezium *EGCF* which remains [C.N. 3]. Let the triangle *GBC* be added to each; therefore the whole parallelogram *ABCD* is equal to the whole parallelogram *EBCF* [C.N. 2]. □

**2.33. Proposition 38** *Triangles which are on equal bases and in the same parallels are equal to one another.*

**Remark 2.34.** Propositions 37 and 38 serve as a replacement of our area formula

**area of a parallelogram = base times height**

**area of a triangle = 1/2 times base times height**

**2.35. Proposition 47 (Theorem of Pythagoras)** *In right–angled triangles the square on the side sub-tending the right angle is equal to the squares on the sides containing the right angle.*

The following is the **converse of the Pythagorean Theorem**. This converse can be used to check whether an angle is truly a right angle.

**2.36. Proposition 48** *If in a triangle the square on one of the sides be equal to the squares on the remaining two sides of the triangle , the angle contained by the remaining two sides of the triangle is right.*

**Example 2.37.** *A right triangle has sides of lengths 21 and 29. How long is the third side?*

**Solution.** Let $x$ be the length of the third side. We are not told which of the sides is the hypotenuse, so we must consider two cases.
*Case 29 is the hypotenuse*. Then $x^2 + 21^2 = 29^2$ i.e. $x^2 = 841-441 = 400$, so $x = 20$.
*Case x is the hypotenuse.* Then $21^2 + 29^2 = x^2$, i.e., $x^2 = 1282$, so $x = \sqrt{1282}$

*Example 2.38. A right triangle has a side of length x, another side is 3 units longer and the third side is 5 units longer than x. How long are the first side x?*

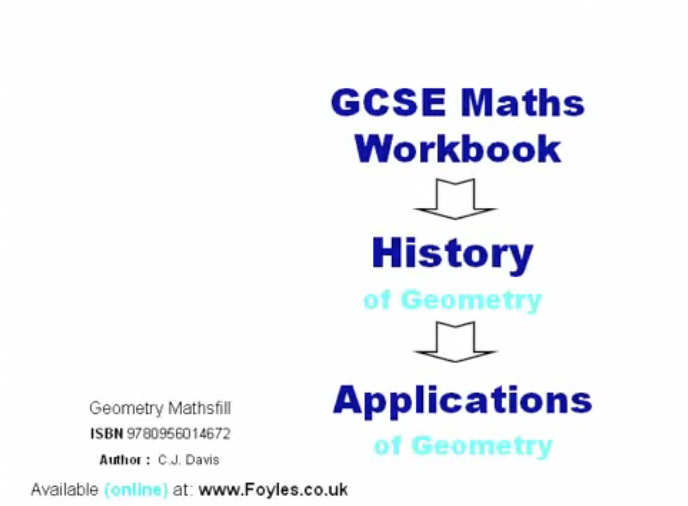
Solution. The sides are $x$, $x + 3$, and $x + 5$. Here $x + 5$ must be the hypotenuse and we must have $x^2 + (x+3)^2 = (x+5)^2$, i.e., $x^2 + x^2 + 6x + 9 = x^2 + 10x + 25$, which simplifies to $x^2 - 4x - 16 = 0$. By the quadratic formula

$$x = \frac{1}{2}(4 \pm \sqrt{16+64}) = \frac{1}{2}(4 \pm \sqrt{5 \cdot 16} = \frac{1}{2}(4 \pm 4\sqrt{5}) = 2 \pm 2\sqrt{5}.$$

The positive solution is $x = 2(1 + \sqrt{5})$

**Remark 2.39. The results in Book I that you must know are the basic concepts, Propositions 5 and 6 on isosceles triangles, the congruence theorems, the Triangle Inequality, Propositions 27 and 28 on corresponding and alternating angles, and the converse Proposition 29, Proposition 32 on angle sums in triangles, then the Theorem of Pythagoras and in converse (Propositions 47 and 48).**

MOVIE 1.3 About the link between Geometry and reality.

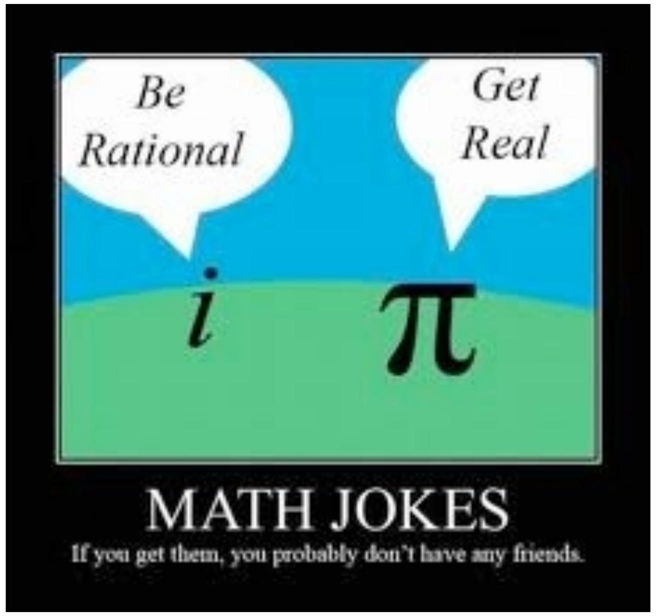



*A world of applications!*

MOVIE 1.4 About the pythagorian theorem..

*and how to use it in everyday life.*

[A link to learn about the connections between geometry and jewelry]



## SECTION 3
# Book II-XIII

## Book II

Book II contains a number of propositions on area which is the way to deal with products in Euclidean mathematics. Some propositions amount to algebraic identities which are very simple in today's algebraic language; some propositions use the Pythagorean Theorem to solve quadratic equations. An example is Proposition 14.

**3.1. Proposition 14** *To construct a square equal to a given rectilineal figure.*

**Exercise 3.2. (Theorem of Thales)** *Let AB be a diameter of a circle and C any point on the circle. Prove that ∠ACB is a right angle.*

**Exercise 3.3. (Proposition 14)** *Let a rectangle with sides a and b be given. By compass and ruler alone, construct a square which has the same area as the given rectangle. In other words, given line segments a,b construct a line segment x such that* $x^2 = a \cdot b$.

**3.4. Corollary** For any positive real number a, construct $\sqrt{a}$.

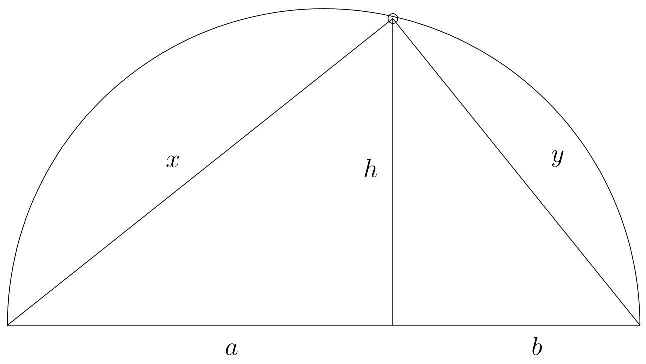



Figure 9

## Book III

This part of the Elements deals with circles and their properties. Here is a sampling of definitions from Book III.

**Definition 3.5.**

1. *A straight line is said to* ***touch a circle*** *which, meeting the circle and being produced, does not cut the circle.*

2. *A* ***segment of a circle*** *is the figure contained by a straight line and a circumference of a circle.*

3. *An* ***angle in a segment*** *is the angle which, when a point is taken on the circumference of the segment and straight lines are joined from it to the extremities of the straight line which is the base of the segment, is contained by the straight lines so joined.*

Recall that a straight line cuts a circle in at most two points.

**Exercise 3.6.** (Proposition 10) *Show that two circles intersect in at most two points.*

**Definition 3.7.** *Let* ***C*** *be a circle with center Z, and let A, B be points on the circle, i.e., A,B* $\in$ ***C****.*

1. *The line segment AB is a chord of* ***C****.*

2. *A straight line which intersects the circle in two points is called a* ***secant*** *of the circle.*

3. *A straight line which intersects the circle in exactly one point is said to* ***touch*** *the circle, and to be* ***tangen***t *to the circle.*

4. *An **arc** of a circle is the intersection of the circle with a half–plane of a secant.*

5. *The **central angle** over the chord AB is the angle $\angle AZB$.*

6. *An **inscribed angle** is an angle $\angle ACB$ where C is some point on the circle.*

7. *Two circles which intersect in exactly one point are said to **touch** one another.*

**Exercise 3.8 (Proposition 1)** *Given three (distinct) points of a circle, construct the center by compass and ruler alone.*

**3.9. Proposition 10** *A circle does not cut a circle at more than two points.*

**3.10. Proposition 16** *The straight line drawn at right angles to the diameter of a circle from its extremities will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semi-circle is greater, and the remaining angle less, than any acute rectilineal angle.*

**3.11. Proposition 18** *The tangent at a point A of a circle is perpendicular to the radius vector through A.*

**3.12. Proposition 20** *Let AB be a chord of a circle with center Z. Then the central angle over the chord AB is twice the size of any inscribed angle $\angle ACB$ when C and Z are on the same side of $\overleftrightarrow{AB}$.*

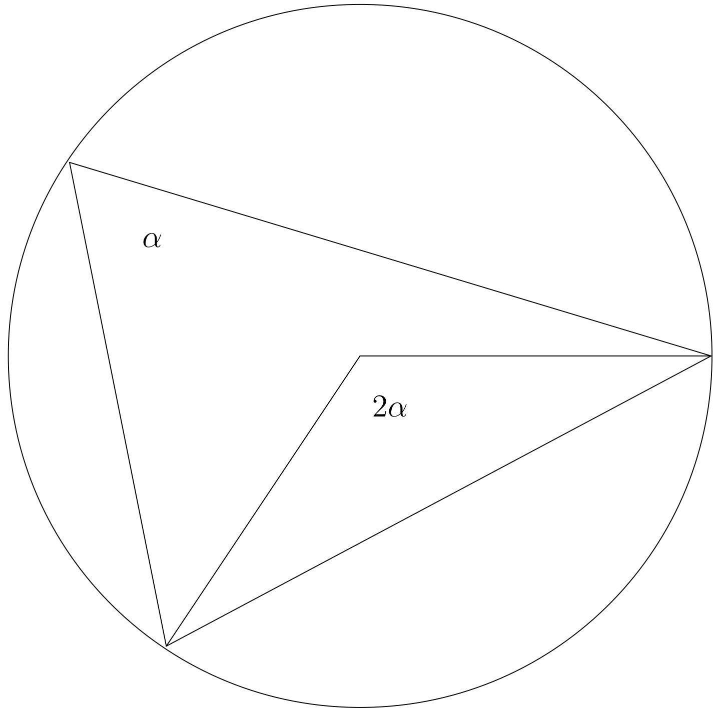



Figure 10

**Exercise 3.13.** *Find and prove the relationship between the central angle $\angle AZB$ and an inscribed angle $\angle ACB$ when C and Z are on different sides of $\overleftrightarrow{AB}$.*

**3.14. Proposition 21** *In a circle the inscribed angles over the same chord AB and on the same side of  are congruent.*

**Exercise 3.15.**

1. *Describe and verify a compass and ruler construction of the tangents to a circle passing through a given exterior point of the circle.*

2. *Describe and verify a compass and ruler construction of the common tangents of two circles.*

## Book IV

This book deals with connections between circles and triangles essentially. Here are some sample theorems.

**3.16. Proposition 4** *In a given triangle to inscribe a circle.*

This proposition essentially uses the following fact which was shown in Exercise 2.24.2.

**Proposition 3.17.** *The angle bisector is the locus of all points equidistant from the legs of the angle.*

**3.18. Proposition 5** *About a given triangle to circumscribe a circle.*

This proposition can be done easily using that the perpendicular bisector of a line segment is the locus of all points equidistant from the endpoints of the line segment (Exercise 2.25).

**3.19. Proposition 11** *In a given circle, inscribe a regular pentagon.*

MOVIE 1.5 Right angles and reality

*A mix of geometry, quantum physics and chaos theory.*

## Book V

This book contains the theory of proportions and the algebra of line segments. Already the definitions are hard to understand and the propositions are complicated, especially when compared with the elegant algebraic language which is available to us today. However, this Book throws considerable light on the Greek substitute for real number. Here are some sample definitions.

**Definition 3.20**.

1. *A magnitude is a* ***part*** *of a magnitude, the less of the greater, when it measures the greater.*

2. *The greater is a* ***multitude*** *of the less when it is measured by the less.*

3. *A* ***ratio*** *is a sort of relation in respect of size between two magnitudes of the same kind.*

4. *Magnitudes are said to* ***have a ratio*** *to one another which are capable when multiplied, of exceeding one another.*

5. *Magnitudes are said to be* ***in the same ratio,*** *the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and the fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.*

6. *Let magnitudes which have the same ratio be called* ***proportional.***

There are 11 more definitions at the start of the book. Note that 3.20 .1. defines factor, and 3.20.2. multiple. Definition 3.20.3. says that a certain relationship between the sizes of magnitudes may or may not exist; if it exists it is called “ratio”. If $a$ and $b$ are magnitudes “of the same kind”, then $a : b = a/b$ is their ratio, so some real number by our comprehension. The next Definition 3.20.4. says when such a relationship exists: For any integral multiple $ma$ there is an integral multiple $nb$ such that $nb > ma$ and conversely. This definition says that the ratio $a : b$ can be approximated to any degree of precision by rational numbers. Definition 3.20.5., due to Eudoxos of Knidos (408?–355?), then says when two ratios $a : a'$ and $b : b'$ are equal in terms of rationals: $a/a' = b/b'$ if and only if for every rational $m/n$, we have

$$a/a' \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} m/n \quad \Leftrightarrow \quad b/b' \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} m/n.$$

This is a valid criterion for the equality of the real numbers $a/a'$ and $b/b'$.

Here are some sample theorems which are translated into modern algebraic formulas. They should be interpreted geometrically in order to reflect the Greek original. Also note that $ma$ where $m$ is a positive integer and a a magnitude (line segment, area, volume), means “$m$ copies of a added together”, and does not mean a product. This is analogous to the definition of powers. In the following $m,n,p,\ldots$ stand for positive integers while $a,b,c,\ldots$ stand for magnitudes.

**3.21. Proposition 1** $ma + mb + mc + ... = m(a + b + c + \ldots)$.

**3.22. Proposition 2** $ma + na + pa + ... = (m + n + p + \ldots)a$.

**3.23. Proposition 3** $n(ma) = (nm)a$.

**3.24. Proposition 4** *If* $a : b = c : d$ *then* $ma : nb = mc : nd$.

**3.25. Proposition 5** $(ma) - (nb) = (m - n)b$.

There are 25 propositions of this nature altogether.

## Book VI

The results of this book which deals with similarity contains very useful and important results.

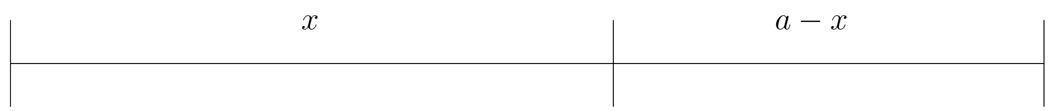
**Definition 3.26**. *Similar rectilineal figures are such as have their angles severally equal and the sides about the equal angles proportional.*

We specialize and rephrase this definition to triangles. Note the analogy to “congruent”.

**Definition 3.27.** *Two triangles are similar if they can be labelled* $\Delta ABC$ *and* $\Delta A'B'C'$ *in such a way that* $\angle A \equiv \angle A'$, $\angle B \equiv \angle B'$, $\angle C \equiv \angle C'$, $\rho = A'B' : AB = B'C' : BC = C'A' : CA$. *We will call the value* $\rho$ *the* ***similarity factor.***

Interesting is the following definition.

**Definition 3.28..** *A straight line is said to have been cut in the extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less.*



It is required that $a : x = x : (a - x)$, i.e., $a(a - x) = x^2$ or $x^2 + ax - a^2 = 0$. The solution is

$$x = \tfrac{1}{2}(-a \pm \sqrt{a^2 + 4a^2}) = \tfrac{a}{2}(-1 \pm \sqrt{5}).$$

The ratio $a : x$ is the so-called ***golden ratio*** and the division is called the ***Golden Section***.

**3.29. Proposition 2** *Let $\angle CAB$ be cut by a transversal parallel to BC in the points $B'$,$C'$ where the notation is chosen so that $B' \in \overrightarrow{AB}$ and $C' \in \overrightarrow{AC}$. Then*

$$AB : BB' = AC : CC' \quad \textit{if and only if} \quad BC \parallel B'C'.$$

MOVIE 1.6 The Golden ratio

*also known as the divine number!*

**Proof.** (Euclid) For let B′C′ be drawn parallel to BC, one of the sides of the triangle ABC; I say that, as BB′ is to B′A, so is CC′ to C′A. For let BC′, CB′ be joined. Therefore the triangleBB′C′ is equal to the triangle CB′C′, for they are on the same base B′C′ and in the same parallels B′C′, BC [I. 38]. And the triangle AB′C′ is another area. But equals have the same ratio to the same; therefore as the triangle BB ′C′ is to the triangle AB′C′, so is the triangle CB′C′ to the triangle AB′C′. etc. □

**3.30. Remark** In the situation of Proposition 2, AB : AB′ = AC : AC′ is equivalent to AB : BB′ = AC : CC′. The next four proposition are "similarity theorems" analogous to the "congruence theorems". Recall our definition of similar triangles at this point. For us only one of the similarity theorems is of importance.

**3.31. Proposition 4** (~aa) *Let $\Delta ABC$ and $\Delta A'B'C'$ be triangles such that $\angle A \equiv \angle A'$ and $\angle B \equiv \angle B'$. Then $\Delta ABC \sim \Delta A'B'C'$.*

This is the "Theorem on similar Triangles" that is by far the most important similarity theorem.

We are now in a position to prove a number of propositions that are essential.

**3.32. Proposition 8** *Let $\Delta ABC$ be a right triangle with $\angle C \equiv R$. Let the foot of the perpendicular from C to AB be H. Then $\Delta ABC \sim \Delta ACH \sim \Delta CBH$.*

**Exercise 3.33.** *Let $\Delta ABC$ be a triangle as in Proposition 8. Set $a = BC$,$b = CA$,$c = AB$,$p = AH$,$q = HB$. Use Proposition 8 to give a new proof of the formulas $h^2 = pq$, $a^2 = qc$, $b^2 = pc$ and of the Pythagorean Theorem.*

**3.34. Proposition 9** By compass and ruler alone, a given line segment can be divided into a prescribed number of congruent line segments.

Example 3.35. *In the figure below the lines* $\overleftrightarrow{BC}$ *and* $\overleftrightarrow{B'C'}$ *are parallel. Using the entered data, compute x and y. The similarity factor taking* $\Delta ABC$ *to* $\Delta AB'C'$ *is* $\rho = (3+5)/3 = 8/3$. *Hence* $y = 2(8/3) = 16/3$. *Furhter* $x/4 = 3/2$, *so* $x = 6$.

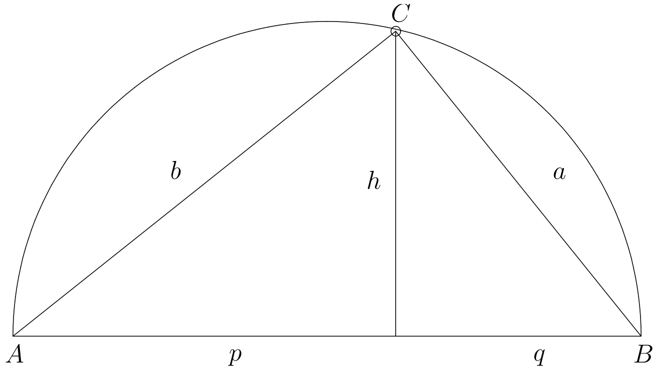



Figure 11

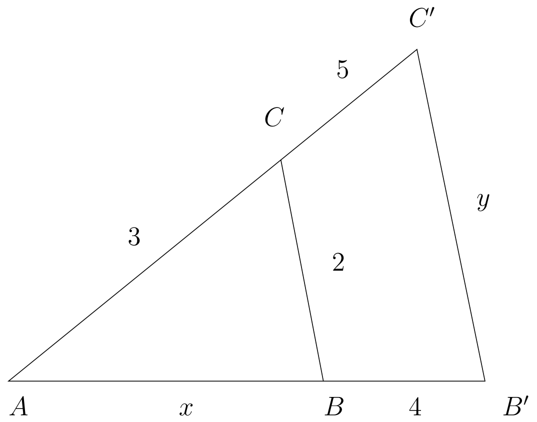



Figure 12

Example 3.36. *To measure the height of a tower (tree, pole, building) one might employ the set-up pictured below where h is the height to be found, a is the known height of the observer, b is the known height of a pole, c′ is the measured distance of the observer from the pole, c is the measured distance of the observer from the tower, and things are arranged in such a way that the eye of the observer, the top of the pole and the top of the tower form a straight line.*

*Say all being measured in feet,* $a = 6$, $b = 15$, $c' = 8$, *and* $c = 56$. *We have similar triangles and find that*

$$\frac{h-6}{56} = \frac{15-6}{8}, \quad h - 6 = 63, \quad h = 69.$$

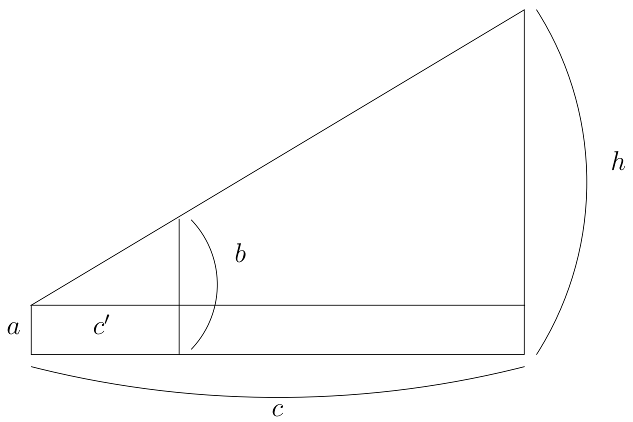



Figure 13

Example 3.37. *In the diagram below the ratios are all equal to the similarity factor $\rho$:*

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3} = \cdots = \rho.$$

*This means that the ratio depends only on the angle $\alpha$ and the common ratio is by definition the **tangent of $\alpha$**, $\tan(\alpha)$. This is to demonstrate that similarity is the source of trigonometry which is the essential tool of surveyors, and is pervasive in mathematics, physics, astronomy, and engineering.*

## Books VII, VIII, IX

These books deal with **natural numbers** which are defined as a "multitude composed of units". Ratios of numbers are what are rational numbers for us. A good deal of important and standard number theory is contained in these books.

## Book X

"Book X does not make easy reading" (B. van der Waerden, Science Awakening, p. 172.) It deals via geometry and geometric algebra with what we call today rational and irrational numbers. In fact, 13 different kinds of irrationalities are distinguished.

**Definition 9.1.**

1. *Those magnitudes are said to be **commensurable** which are measured by the same measure, and those **incommensurable** which cannot have any common measure.*

2. *Straight lines are **commensurable in square** when the squares on them are measured by the same area, otherwise they are **incommensurable in square.***

3. *Line segments are **rational** if they are commensurate with a fixed line segment (or unit), otherwise **irrational**.*

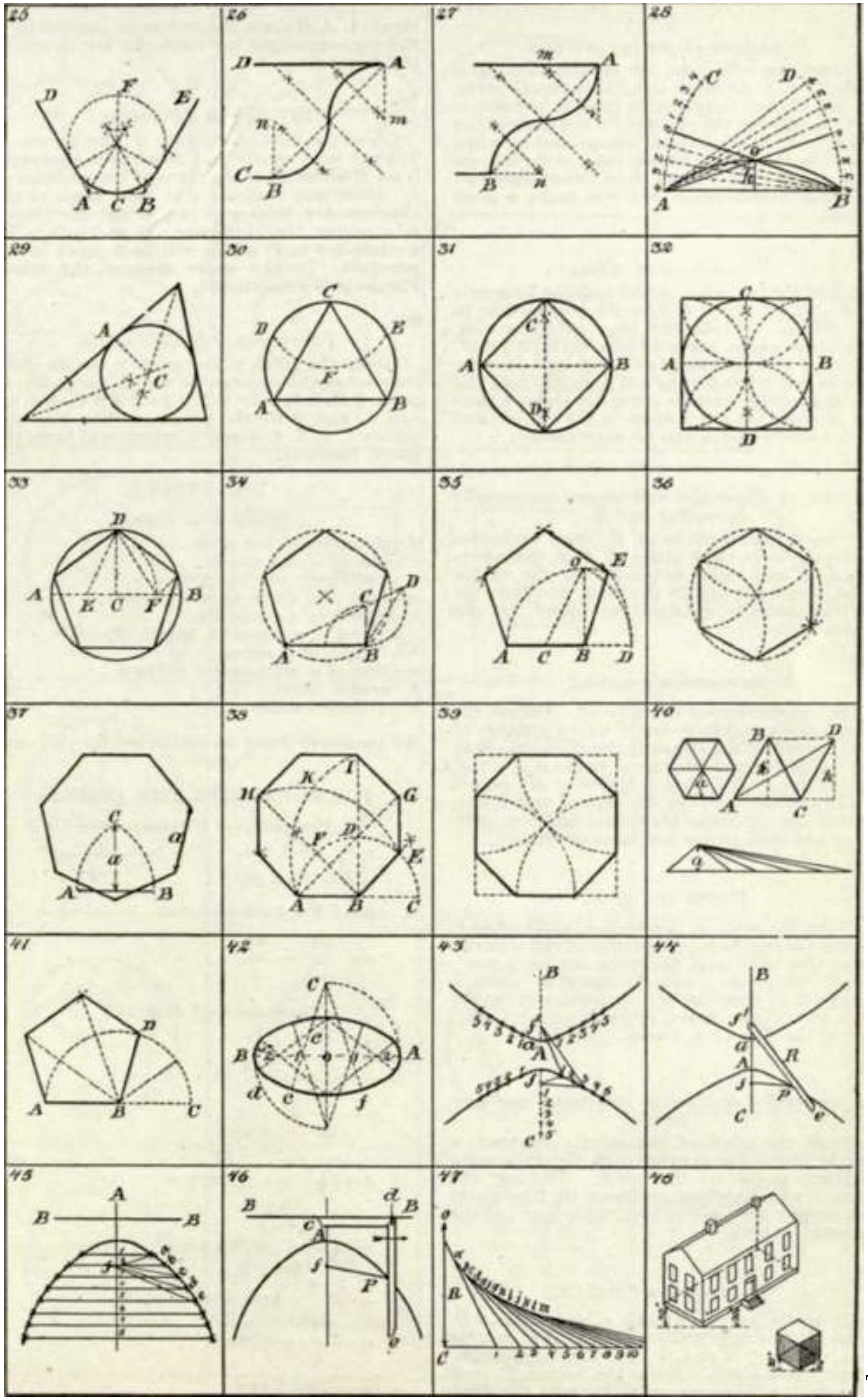
The book contains 115 propositions none of which is recognizable at first sight. There is general agreement that the difficulty and the limitations of geometric algebra contributed to the decay of Greek mathematics (Van der Waerden, Science Awakening, p.265.) Author like Archimedes and Apollonius were too difficult to read. However, Van der Waerden disputes that it was a lack of understanding of irrationality which drove the Greek mathematicians into the dead-end street of geometric algebra. Rather it was the discovery of irrationality, e.g. the diagonal of a square is incommensurable with the side of the square, and a strict, logical concept of number which was the root cause.

## Books XI, XII, and XIII

Book XI deals with solid geometry and theorems on volumes, in geometric language, of course. Book XII uses the **method of exhaustion** to discuss the area of curved figures, e.g. the circle. Finally, Book XIII contains a discussion of the five Platonic solids (regular polyhedra).

SECTION 4

# Constructions, Axiomatic Math, and the Parallel



The

## The Famous Problems of Antiquity

We have seen that many propositions in Euclid are of the form "To construct this or that". These constructions had to be achieved with no other tools than the straight edge (without measurements) and the compass. We have seen many examples of such constructions. Some such are bisecting an angle, bisecting a line segment, drawing perpendiculars, constructing incenter and circumcenter of a given triangle, constructing the center of a given circle, constructing a regular (= equilateral) triangle and a regular pentagon. There are some construction problems that nobody could do and nobody could ever do. These are:

1. Trisecting a given angle.
2. Squaring the circle.
3. Doubling the cube.

It was shown in the 20th century by tools of modern algebra, that these constructions are actually impossible to do.

**A Fragment of Geometry.** *The objects of plane geometry are **points** and **lines**. Points and lines may be related by a relation called **incidence.** If a point and a line are related in this way, we say that **the point is on the line** or **the line is on the point.** In this language, we have the following incidence axioms:*

1. *There exists at least one line.*
2. *On each line there exist at least three points.*
3. *Not all points lie on the same line.*
4. *There is a unique line passing through any two distinct points.*
5. *There is a unique point on any two lines.*

*These axioms suffice to prove the following theorems.*

1. *Each point lies on at least two distinct lines.*
2. *Not all lines pass through the same point.*
3. Two distinct lines intersect in at most one point.

MOVIE 1.7 A question of modern geometry.

*How to theoretically turn a sphere inside?*

*Historically, one of the most important questions in geometry was whether Euclid's Parallel Postulate could be derived from his other axioms.*

## Story of the parallel axiom

Inspite of all the rigor introduced into mathematics by Greek mathematicians and in particular by Euclid, they took for granted a number of things that cannot be taken for granted by today's standards.

• Certain properties were read off diagrams, such as the existence of points of intersection, and whether a point is between two points or not.

• The area concept was accepted and used without question, although in the "common notions" some basic principles satisfied by area measures are incorporated.

However, it was a good thing in a sense because completely rigorous axiomatic geometry is very tedious and much less enjoyable than what Euclid presented. The strong points of the Elements are

• It contains all of elementary geometry that is good and useful.

•The notions and results are arranged in way that has never been surpassed and can be taught today essentially as it was 2300 years ago.

•The proofs are clever and sophisticated.

•The late use of the Parallel Axiom seems like an uncanny anticipation of things to come in the twentieth century.

We exclusively deal with plain geometry here and two lines are parallel if and only if they do not intersect.

**1. Statements equivalent to Euclid's Parallel Axiom**

a. (John Playfair (1748-1819)) Given a line $\ell$ and a point P not on $\ell$ there is at most one line parallel to $\ell$ and passing through P.

b. There exists a triangle whose angle sum equals two right angles.

c. There exist two triangles that are similar but not congruent.

d. There exists a line $\ell$ such that the locus of all points on one side of $\ell$ with equal distance from $\ell$ is a straight line.

e. Every triangle has a circumscribed circle.

**2. Attempts to prove the Parallel Axiom**

a. Ptolemy (85?-165?)

b. Nasir al-din (1201-1274)

c. Girolamo Saccheri (1667-1733) "Euclides ab omni naevo vindicatus"

d. Johann Heinrich Lambert (1728-1777)

e Adrien-Marie Legendre (1752-1833)

**3. Discoverers of Non-Euclidean Geometry**

a. Carl Friedrich Gauss (1777-1855)

b. Janos Bolyai (1802-1860)

c. Nicolai Ivanovitch Lobachevsky (1793-1856), University of Kasan

**4. Consistency Proofs**

a. Eugenio Beltrami (1835-1900)

b. Arthur Cayley (1821-1895)

c. Felix Klein (1849-1925)

d. Henri Poincaré (1854-1912)

**5. Geometries galore**

a. Bernhard Riemann (1826-1866)

## Janos Bolyai (1802-1860)

János was born in Kolozsvár (now renamed Cluj in Romania) but soon went to Marosvásárhely where his father Farkas had a job at the Calvinist College teaching mathematics, physics and chemistry. Farkas Bolyai always wanted his son to be a mathematician, and he brought him up with this in mind. One might suppose that this would mean that János's education was put first in the Bolyai household, but this was not so for Farkas believed that a sound mind could only achieve great things if it was in a sound healthy body, so in his early years most attention was paid to János's physical development. It was clear from early on, however, that János was an extremely bright and observant child.

"... when he was four he could distinguish certain geometrical figures, knew about the sine function, and could identify the best known constellations. By the time he was five [he] had learnt, practically by himself, to read. He was well above the average at learning languages and music. At the age of seven he took up playing the violin and made such good progress that he was soon playing difficult concert pieces."

It is important to understand that although Farkas had a lecturing post he was not well paid and even although he earned extra money from a variety of different sources, János was still brought up in poor financial circumstances. Also János's mother was a rather difficult person and the household was not a particularly happy place for the boy to grow up.

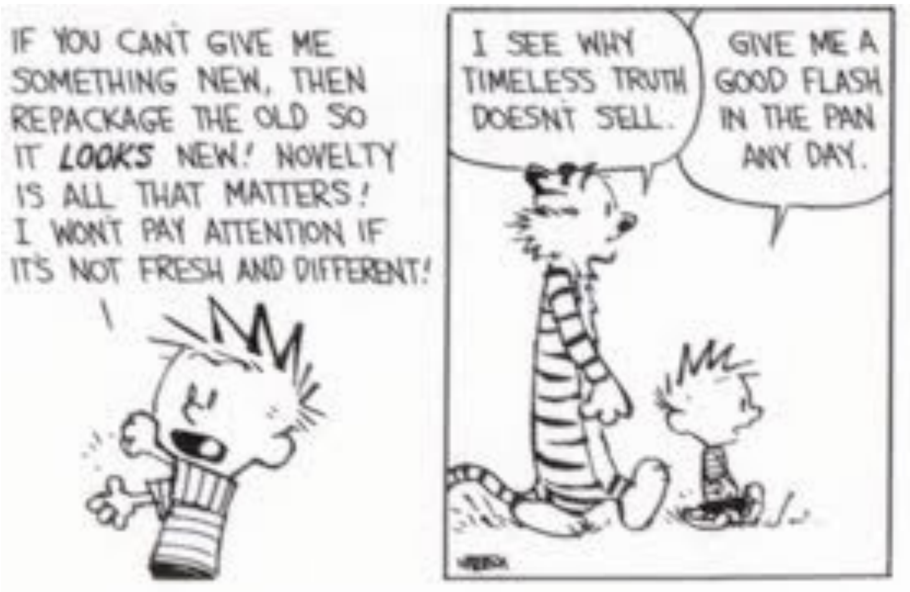
Until János was nine years old the best students from the Marosvásárhely College taught him all the usual school subjects except mathematics, which he was taught by his father. Only from the age of nine did he attend school. By the time Bolyai was 13, he had mastered the calculus and other forms of analytical mechanics, his father continuing to give him instruction. By this time, however, he was attending the Calvinist College in Marosvásárhely although he had started in the fourth year and often attended lessons intended for the senior students.

Janos studied at the Royal Engineering College in Vienna from 1818 to 1822 completing the seven year course in four years. He was an outstanding student and from his second year of study on he came top in most of the subjects he studied. When he graduated from the Academy on 6 September 1822 he had achieved such outstanding success that he spent a further year in Vienna on academic studies before entering military service. Of course he had received military training during his time in Vienna, for the summer months were devoted to this, but Bolyai's nature did not allow him to accept easily the strict military discipline.

In September 1823 he entered the army engineering corps as a sublieutenant and was sent to work on fortifications at Temesvár. He spent a total of 11 years in military service and was reputed to be the best swordsman and dancer in the Austro-Hungarian Imperial Army. He neither smoked nor drank, not even coffee, and at the age of 23 he was reported to still retain the modesty of innocence. He was also an accomplished linguist speaking nine foreign languages including Chinese and Tibetan.

Around 1820, when he was still studying in Vienna, Bolyai began to follow the same path that his father had taken in trying to replace Euclid's parallel axiom with another axiom which could be deduced from the others. In fact he gave up this approach within a year for still in 1820, as his notebooks now show, he began to develop the basic ideas of hyperbolic geometry. On 3 November 1823 he wrote to his father that he had:-

... created a new, another world out of nothing...




By 1824 there is evidence to suggest that he had developed most of what would appear in his treatise as a complete system of non-Euclidean geometry. In early 1825 Bolyai travelled to Marosvásárhely and explained his discoveries to his father. However Farkas Bolyai did not react enthusiastically which clearly disappointed János. By 1831 Farkas had come to understand the full significance of what his son had accomplished and strongly encouraged him to write up the work for publication as an Appendix to a book he was writing. .

What was contained in this mathematical masterpiece?

... denote by Sigma the system of geometry based on the hypothesis that Euclid's Fifth Postulate is true, and by S the system based on the opposite hypothesis. All theorems we state without explicitly specifying the system Sigma or S in which the theorem is valid are meant to be absolute, that is, valid independently of whether Sigma or S is true.

Today we call these three geometries Euclidean, hyperbolic, and absolute. Most of the Appendix deals with absolute geometry. By 20 June 1831 the Appendix had been published for on that day Farkas Bolyai sent a reprint to Gauss who, on reading the Appendix, wrote to a friend saying:-

I regard this young geometer Bolyai as a genius of the first order .

To Farkas Bolyai, however, Gauss wrote:-

To praise it would amount to praising myself. For the entire content of the work ... coincides almost exactly with my own meditations which have occupied my mind for the past thirty or thirty-five years.

Gauss was simply stating facts here. In a letter dated 8 November 1824 he wrote:-

The assumption that the sum of the three angles of a triangle is less than 180 degrees leads to a curious geometry, quite different from ours [i.e. Euclidean geometry] but thoroughly consistent, which I have developed to my entire satisfaction ... the three angles of a triangle become as small as one wishes, if only the sides are taken large enough, yet the area of the triangle can never exceed, or even attain a certain limit, regardless of how great the sides are.

The discovery that Gauss had anticipated much of his work, greatly upset Bolyai who took it as a severe blow. He became irritable and a difficult person to get on with. His health began to deteriorate and he was plagued with a fever which frequently disabled him so he found it increasingly difficult to carry out his military duties. He retired on 16 June 1833, asking to be pensioned off, and for a short time went to live with his father.

Bolyai continued to develop mathematical theories while he lived, but being isolated from the rest of the world of mathematics much of what he attempted was of little value. His one major undertaking, to attempt to develop all of mathematics based on axiom systems, was begun in 1834, for he wrote the preface in that year, but he never completed the work.

In 1848 Bolyai discovered that Lobachevsky had published a similar piece of work in 1829.

"János studied Lobachevsky's work carefully and analysed it line by line, not to say word by word, with just as much care as he administered in working out the Appendix. The work stirred a real storm in his soul and he gave outlet to his tribulations in the comments added to the 'Geometrical Examinations'.

The 'Comments' to the 'Geometrical Examinations' are more than a critical analysis of the work. They express the thoughts and anxieties of János provoked by the perusal of the book. They include his complaint that he was wronged, his suspicion that Lobachevsky did not exist at all, and that everything was the spiteful machinations of Gauss: it is the tragic lament of an ingenious geometrician who was aware of the significance of his discovery but failed to get support from the only person who could have appreciated his merits.

Although he never published more than the few pages of the Appendix he left more than 20000 pages of manuscript of mathematical work when he died of pneumonia at the age of 57. These are now in the Bolyai-Teleki library in Tirgu-Mures. In 1945 a university in Cluj was named after him but it was closed down by Ceaucescu's government in 1959.

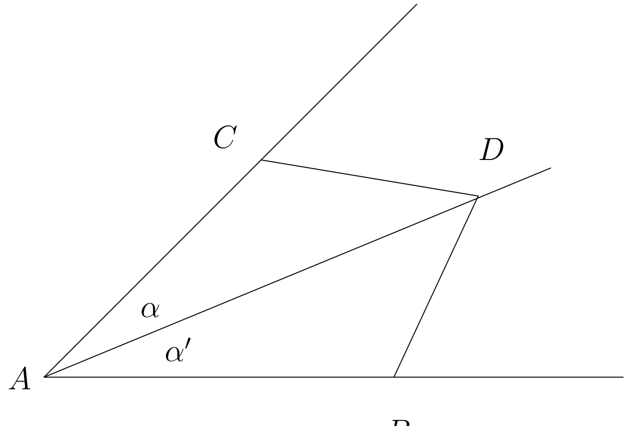
Let's read a little more about non-Euclidian geometry...

and a little bit more again!

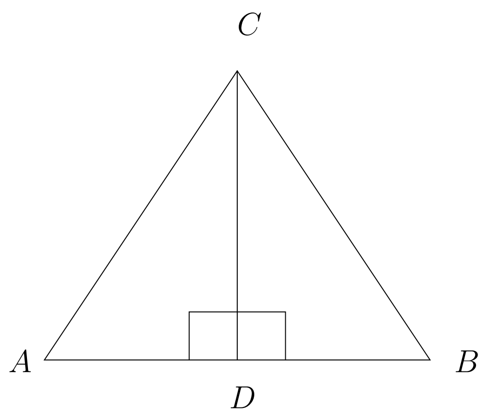
# SECTION 5
# Exercises

**Note:** The diagrams only describe the general situation by do not accurately show lengths and angles measures. You cannot read off results from the pictures. E.g. the diagram in Exercise 5.8 is way off.

**Exercise 5.1.** *It is given that $AB \equiv AC$ and $\angle BAD \equiv \angle CAD$ in the diagram below. Prove that $CD \equiv BD$ and $\angle CDA \equiv BDA$.*



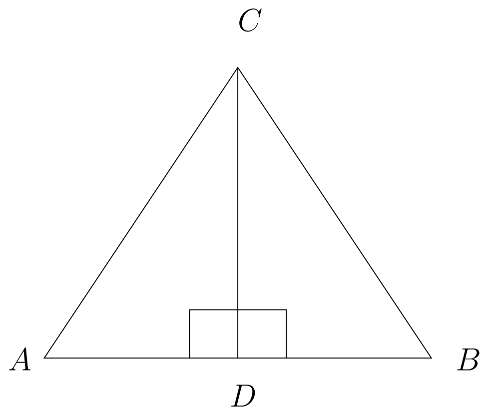

**Exercise 5.2.** *In the diagram below $\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{CD}$ and $AC \equiv BC$. Prove that $AD \equiv BD$. Give reasons.*



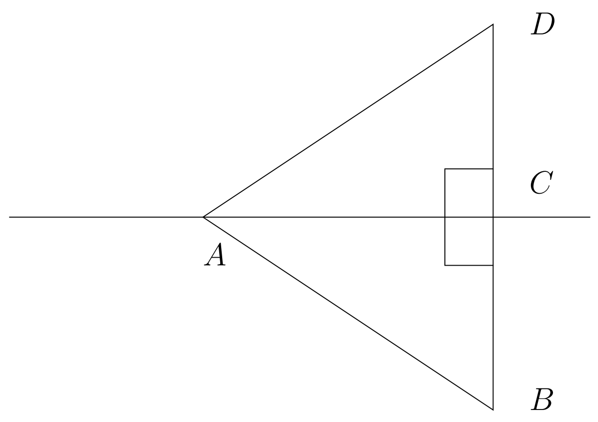

**Exercise 5.3.** *The vertices A and B of the triangle $\Delta ABC$ lie on a circle whose center is C. If $\angle CAB = 30^\circ$, what is the degree measure of $\angle CBA$? Give reasons for your conclusion.*

**Exercise 5.4.** *The vertex A of the triangle ΔABC lies on a circle whose center is C. If ∠CAB ≡∠CBA does pass through B? Give reasons for your conclusion.*

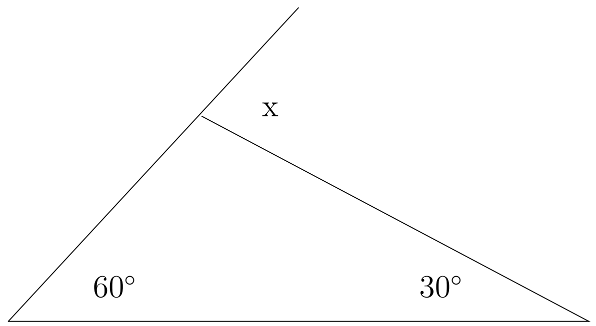
**Exercise 5.5.** *In the diagram below* $\overleftrightarrow{AB}$ *is perpendicular* $\overleftrightarrow{CD}$ *to and AD ≡ BD. Prove that AC ≡ BC. Give reasons.*



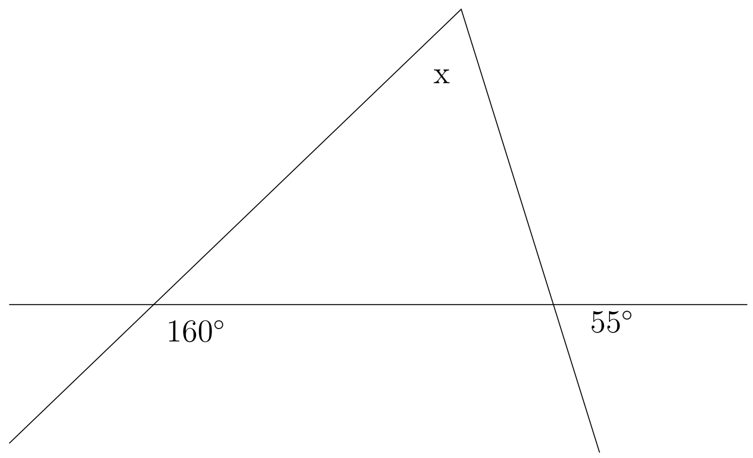

**Exercise 5.6.** *In the diagram below BC ≡ DC and the line* $\overleftrightarrow{BD}$ *is perpendicular to the line* $\overleftrightarrow{AC}$*. Prove that AD ≡ AB. Give reasons.*




**Exercise 5.7.** *Reference is made to the diagram below. Compute the angle x.*

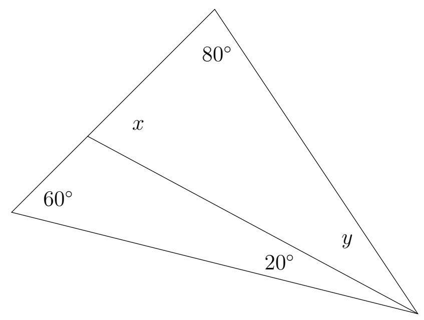



**Exercise 5.8.** *Reference is made to the diagram below. Compute the angle x.*

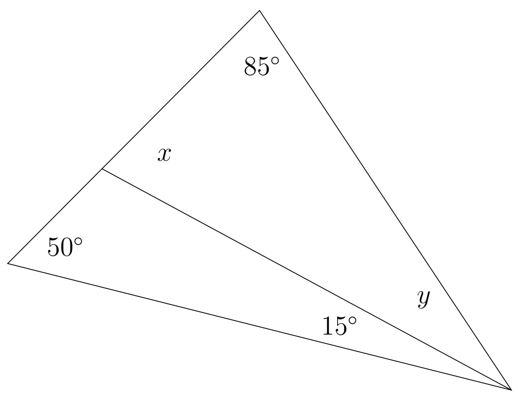



**Exercise 5.9.** *In a triangle the height is by 2 units larger than the base and its area is 6 square units. How long is the base?*

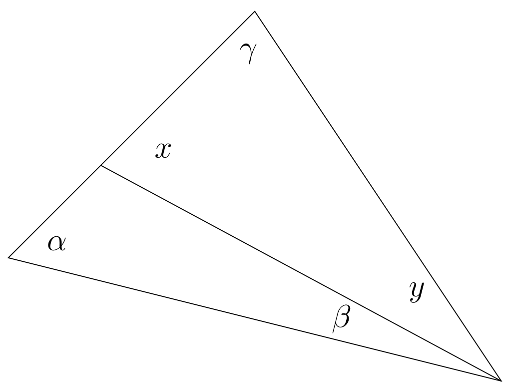
**Exercise 5.10.** *Reference is made to the diagram. Compute the angles x and y.*



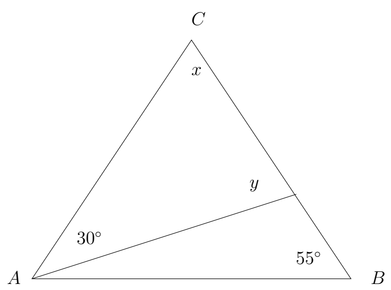
**Exercise 5.11.** *Reference is made to the diagram. Compute the angles x and y.*



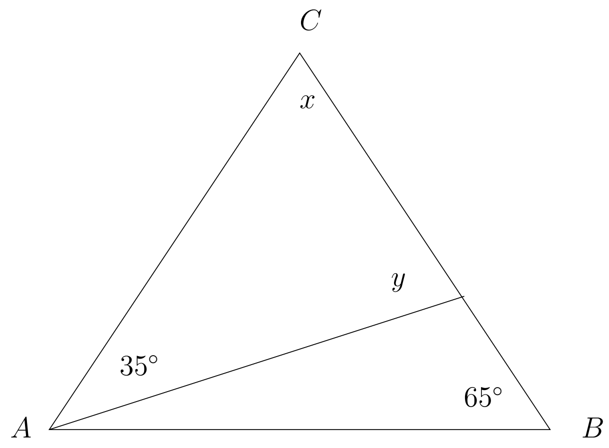

**Exercise 5.12.** *Reference is made to the diagram. The angles α, β, and γ are given, compute the angles x and y.*



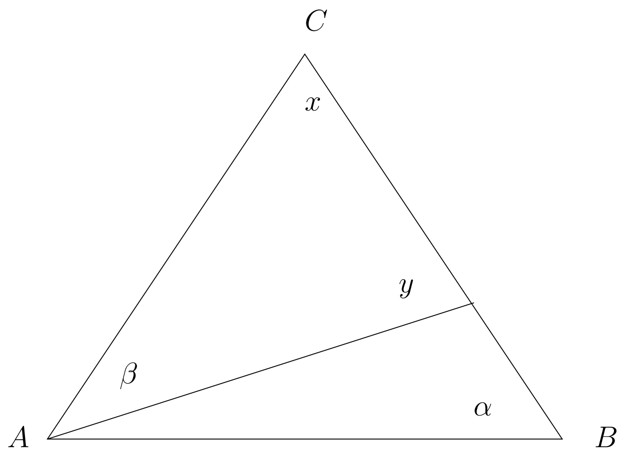

**Exercise 5.13.** *In the triangle ΔABC depicted below the sides AC and BC are congruent. Compute x and y.*



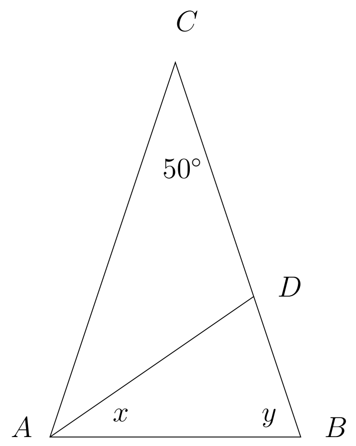

**Exercise 5.14.** *In the triangle ΔABC depicted below the sides AC and BC are congruent, and the angles α and β are given. Compute x and y.*



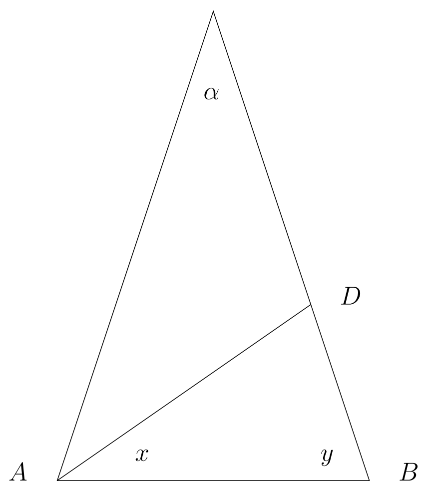

**Exercise 5.15.** I*n the triangle ΔABC depicted below the sides AC and BC are congruent, and the angles α and β are given. Compute x and y.*



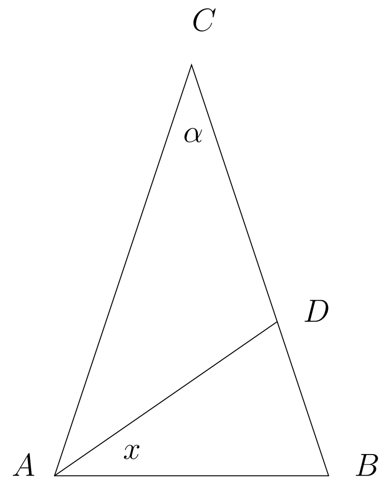

**Exercise 5.16.** *In the triangle ΔABC depicted below the sides AC and BC are congruent, and AD and CD are congruent as well. Compute x and y.*




**Exercise 5.17.** *In the triangle $\Delta ABC$ depicted below the sides $AC$ and $BC$ are congruent, and $AD$ and $CD$ are congruent as well. The angle $\alpha$ is given. Compute $x$ and $y$.*




**Exercise 5.18.** *In the triangle $\Delta ABC$ depicted below the sides $AC$ and $BC$ are congruent, and $AD$ and $CD$ are congruent as well. The angles $\alpha$ and $x$ are congruent. What is the angle measure of $\alpha$?*

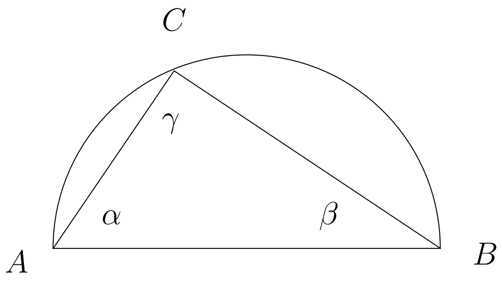



**Exercise 5.19.** *A rectangle has area 32 square units, and the base of the rectangle is by twice as long its height. How long is the base?*

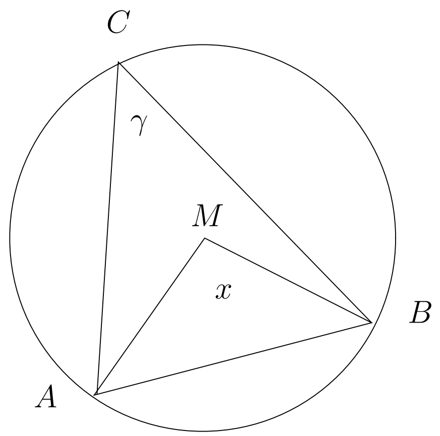
**Exercise 5.20.** *A rectangle has area $27/2$ square units, and the base of the rectangle is by one half longer than its height. How long is the base?*

**Exercise 5.21.** *In a right triangle the hypotenuse has length 15 units and the other two sides differ by 3 units of length. What are the dimensions of the triangle?*

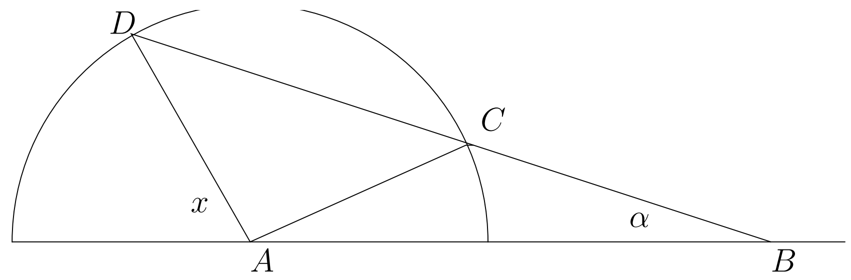
**Exercise 5.22.** *Reference is made to the diagram below. What you see is a semicircle. The points A, B, C are on the semicircle. Its center is the midpoint M of the line segment AB. The angles $\alpha$ and $\beta$ are not known. Show that the angle $\angle ACB = \gamma$ must measure 90°*. **[**Hint: Mark the center M of the semicircle and join it to C. Now use theorems on isosceles triangles and the angle sum in the triangle ΔABC.**]**



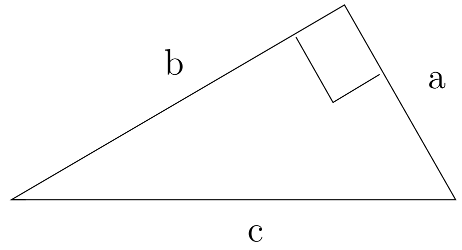

**Exercise 5.23.** *Reference is made to the diagram below. What you see is a circle. The points A, B, C are on the circle. Its center is the point M. The angle $\gamma$ is not known. Show that the angle $x = \angle AMB = 2 \cdot \gamma$.* **[**Hint: Join M to A, B, and C. Now use theorems on isosceles triangles and the angle sum in the triangles ΔAMC and ΔBMC. Finally use that the sum of the angles at M is 360°.**]**




**Exercise 5.24.** *Reference is made to the diagram below. The points C and D lie on a semicircle with center A. It is given that $AC \equiv BC$. Compute x in terms of $\alpha$.* **[**Hint: Consider in turn the isosceles triangle ΔABC, one of its exterior angles, the isosceles triangle ΔACD, and finally the triangle ΔABD of which x is an exterior angle.**]**




**Exercise 5.25.** *Two sides of a right triangle are given in the table below. Compute the remaining side marked.*




| $a$ | $b$ | $c$ |
|---|---|---|
| 4 | 3 | ? |
| ? | 5 | 13 |
| 24 | ? | 25 |
| ? | 15 | 17 |
| 4 | 4 | ? |
| 3 | 5 | ? |
| 1 | 1 | ? |

**Exercise 5.26.** *Two sides of a right triangle have lengths 5 and 6 units. What is the length of the third side?* **[**Note that this problem is ambiguous, there are two ways that this can happen.**]**

**Exercise 5.27.** *A right triangle is such that the second side is by two units larger than the first, and the third side is by 3 units larger than the first. How long are the sides?*

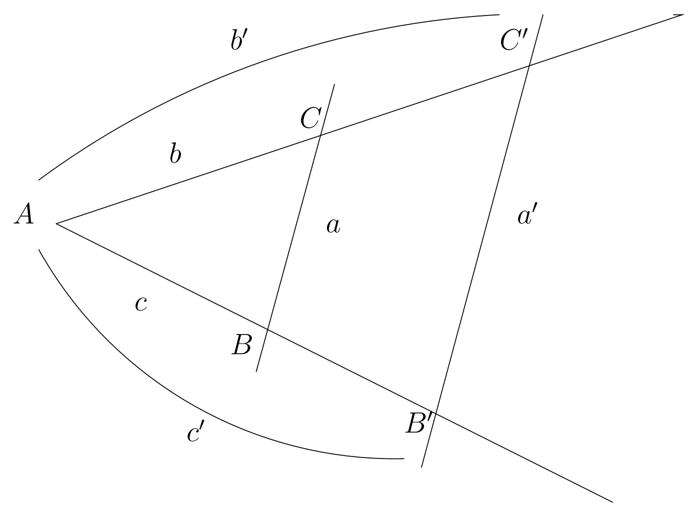
**Exercise 5.28.** *A right triangle is such that the second side is by two units shorter than the first, and the third side is by 1 units larger than the first. How long are the sides?*

**Exercise 5.29.** *A right triangle is such that the second side is twice as long as the first, and the third side is by 4 units longer than the first. How long are the sides? [Again there are two possibilities to consider.]*

**Exercise 5.30.** *A right triangle is such that the second side is by 3 units longer than the first, and the third side is by 5 units longer than the first. How long are the sides?*

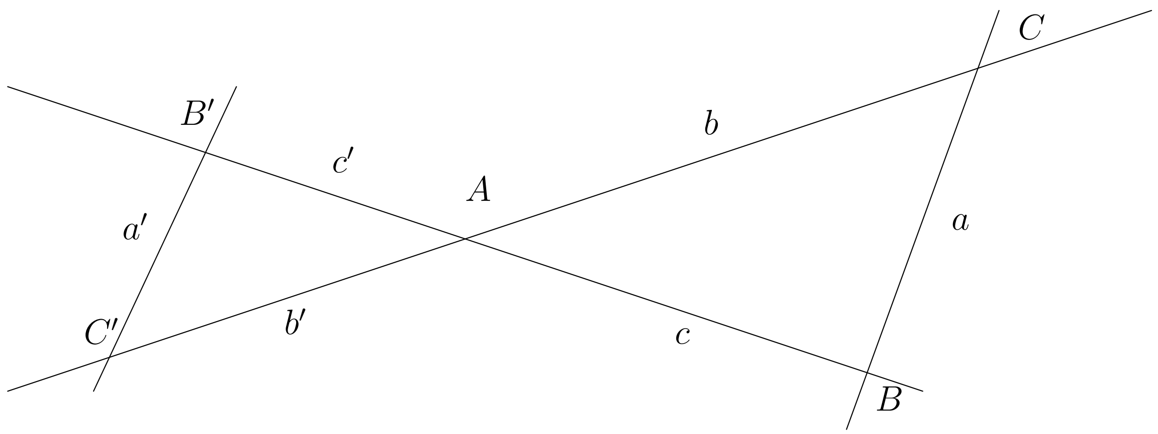
**Exercise 5.31.** *Reference is made to the diagram below in which $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ are parallel. Furthermore $a = BC$, $b = AC$, $c = AB$, $a' = B'C'$, $b' = AC'$, and $c' = AB'$. In the table below some of the values $a, a', b, b', c, c'$ are given, and you are asked to compute the remaining values indicated by ?*

| $a$ | $b$ | $c$ | $a'$ | $b'$ | $c'$ |
|---|---|---|---|---|---|
| 4 | ? | ? | 16 | 20 | 24 |
| 2 | 1 | ? | 6 | ? | 6 |
| 4 | ? | 6 | 8 | 6 | ? |
| 5 | 6 | ? | ? | 2 | 3 |
| ? | 6 | 8 | 3 | ? | 6 |
| 3 | ? | 7 | ? | 15/2 | 21/2 |
| 2/3 | 3/4 | 5/3 | 2/5 | ? | ? |



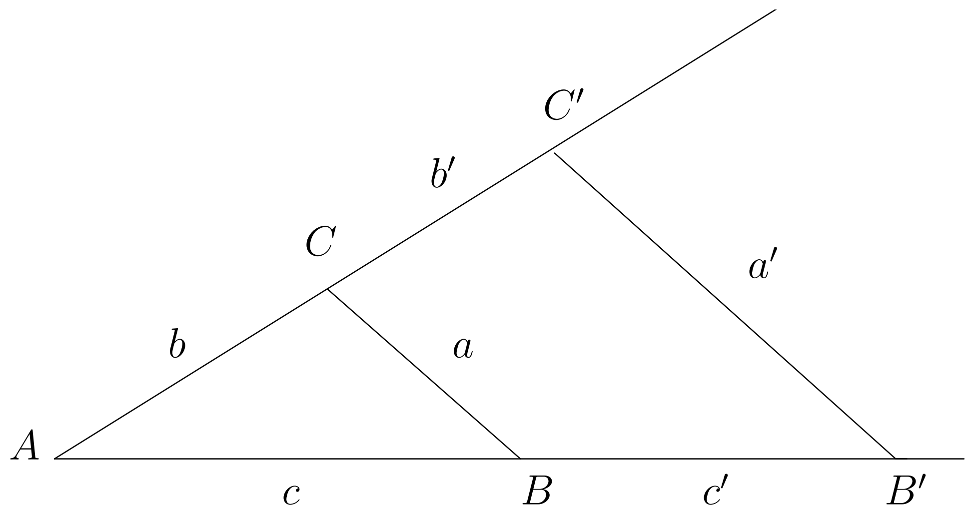
**Exercise 5.32.** *Reference is made to the diagram below in which* $\overleftrightarrow{BC}$ *and* $\overleftrightarrow{B'C'}$ *are parallel. Furthermore* $a = BC$, $b = AC$, $c = AB$, $a' = B'C'$, $b' = AC'$, *and* $c' = AB'$. *In the table below some of the values* $a, a', b, b', c, c'$ *are given, and you are asked to compute the remaining values indicated by ?*

| $a$ | $b$ | $c$ | $a'$ | $b'$ | $c'$ |
|---|---|---|---|---|---|
| 4 | ? | 6 | ? | 20 | 24 |
| 8 | 5 | ? | 36 | ? | 18 |
| ? | 5 | 4 | 3/2 | ? | 3 |
| ? | 10 | ? | 24 | 30 | 36 |
| 4 | ? | 6 | 10 | 15/2 | ? |
| 5 | 6 | ? | 15/2 | ? | 12 |




**Exercise 5.33.** *Reference is made to the diagram below in which* $\overleftrightarrow{BC}$*and* $\overleftrightarrow{B'C'}$ *are parallel. Furthermore* $a = BC$, $b = AC$, $C = AB$, $a' = B'C'$, $b' = CC'$, *and* $c' = BB'$. *In the table below some of the values* $a, a', b, b', c, c'$ *are given, and you are asked to compute the remaining values.*

| $a$ | $b$ | $c$ | $a'$ | $b'$ | $c'$ |
|---|---|---|---|---|---|
| 4 | 5 | 6 | 16 | ? | ? |
| 4 | 4 | 6 | ? | 2 | 3 |
| 7 | ? | ? | 35 | 24 | 20 |
| 1/2 | 1/3 | ? | ? | 29/3 | 29/5 |
| 3 | ? | 5 | ? | 4/3 | 5/3 |
| 5 | ? | 4 | 10 | 7 | ? |



**Exercise 5.34.** *Which of the following triples can be the sides of a right triangle?*

| | | |
|---|---|---|
| $7$ | $25$ | $5$ |
| $6$ | $4$ | $2\sqrt{5}$ |
| $\frac{3}{20}$ | $\frac{1}{5}$ | $\frac{1}{4}$ |
| $24$ | $8$ | $25$ |
| $4$ | $5$ | $6$ |
| $3$ | $5$ | $\sqrt{34}$ |
| $\frac{1}{2}(1+\sqrt{3})$ | $\frac{1}{2}(3+\sqrt{3})$ | $1+\sqrt{3}$ |

SECTION 6

# Solutions of Geometry Exercises

13.1 $\Delta ABD \equiv \Delta ACD$ by sas.

13.2 $\angle CAD \equiv \angle BDC$ because $\Delta ABC$ is isosceles. By asa $\Delta ADC \equiv \Delta BDC$, hence $AD \equiv BD$ as corresponding pieces in congruent triangles.

13.3 $\Delta ABC$ is isosceles so the base angles are congruent, so $\angle CBA = 30^\circ$.

13.4 The triangle is isosceles because the base angles are congruent. The circle passes through B.

**5.1** $\Delta ABD \equiv \Delta ACD$ by sas.

**13.2** $\angle CAD \equiv \angle BDC$ because $\Delta ABC$ is isosceles. By asa $\Delta ADC \equiv \Delta BDC$, hence $AD \equiv BD$ as corresponding pieces in congruent triangles.

**13.3** $\Delta ABC$ is isosceles so the base angles are congruent, so $\angle CBA = 30^\circ$.

**13.4** The triangle is isosceles because the base angles are congruent. The circle passes through B.

**13.5** $\Delta ADC \cong \Delta BDC$ by sas.

**13.6** $\Delta ACD \equiv \Delta ACB$ by sas.

**13.7** $x = 90^\circ$.

**13.8** $x = 105^\circ$.

**13.9** $x = -1 + \sqrt{13}$.

**13.10** $x = 80^\circ$, $y = 20^\circ$.

**13.11** $x = 65^\circ$, $y=30^\circ$.

**13.12** $x = \alpha + \beta$, y = 180 − γ − α − β.

**13.13** $x = 70°$, $y = 80°$∘.

**13.14** $x = 50°$, $y = 95°$.

**13.15** $x = 180 - 2\alpha$, $y = 2\alpha - \beta$.

**13.16** x = 15°, y = 65°.

**13.17** $x = (1/2)(180 - 3\alpha)$, $y = (180 - \alpha)$.

**13.18** $\alpha$ =36°.

**13.19** base = 8 units of length, height = 4 units of length

**13.20** base = 9/2 units of length, height = 3 units of length

**13.21** The sides measure 9, 12, 15.

**13.22** Using that in isosceles triangles the angles opposite the congruent sides are congruent, you find that $\gamma = \alpha + \beta$. Summing the angles in ΔABC you find that $180° = \alpha + \beta + \gamma = 2\gamma$.

**13.23** Call ∠MCA = $\gamma_1$ and ∠MCB = $\gamma_2$. Then ∠CAM = $\gamma_1$ and ∠CBM = $\gamma_2$. Now your have three angles at *M*, $x$, $180 - 2\gamma_1$, and $180 - 2\gamma 2$ that have to add up to 360°. Write down and solve for $x$.

**13.24** $x = 3\alpha$.

**13.25**

| $a$ | $b$ | $c$ |
|---|---|---|
| 4 | 3 | 5 |
| 12 | 5 | 13 |
| 24 | 7 | 25 |
| 8 | 15 | 17 |
| 4 | 4 | $4\sqrt{2}$ |
| 3 | 5 | $\sqrt{34}$ |
| 1 | 1 | $\sqrt{2}$ |

**13.26** Suppose the third side is the hypotenuse $x$. Then $x^2 = 5^2 + 6^2$, and $x = \sqrt{61}$. Suppose that the third side is not the hypotenuse. Then $x^2 + 5^2 = 6^2$, and $x = \sqrt{11}$.

**13.27** Let $x$ be the length of the first side. Then the others are $x + 2$ and $x + 3$. Solve $x^2 + (x+2)^2 = (x+3)^2$ to get that the sides are $1 + \sqrt{6}$, $3 + \sqrt{6}$, $4 + \sqrt{6}$

**13.28** The sides $x$, $x - 2$, $x + 1$ must satisfy $x^2 + (x-2)^2 = (x+1)^2$. The sides are $3 + \sqrt{6}$, $1 + \sqrt{6}$, $4 + \sqrt{6}$.

**13.29** The sides $x$, $2x$, $x + 4$ and either $x^2 + (2x)^2 = (x+4)^2$ or $x^2 + (x+4)^2 = (2x0^2$ depending on whether $x + 4$ is larger than $2x$ or $2x$ is larger than $x + 4$. In the first case the sides are $1 + \sqrt{5}$, $2 + 2\sqrt{5}$, $5 + \sqrt{5}$ , in the second case the sides are $2 + 2\sqrt{3}$, $6 + 2\sqrt{3}$ $4 + 4\sqrt{3}$

**13.30** The sides are $x$, $x + 3$, $x + 5$ satisfying $x^2 + (x+3)^2 = (x+5)^2$ 2. The sides are $2 + 2\sqrt{5}$ $5 + \sqrt{5}$, $7 + 2\sqrt{5}$

**13.31**

| $a$ | $b$ | $c$ | $a'$ | $b'$ | $c'$ |
|---|---|---|---|---|---|
| 4 | 5 | 6 | 16 | 20 | 24 |
| 2 | 1 | 2 | 6 | 3 | 6 |
| 4 | 3 | 6 | 8 | 6 | 12 |
| 5 | 6 | 9 | $5/3$ | 2 | 3 |
| 4 | 6 | 8 | 3 | $9/2$ | 6 |
| 3 | 5 | 7 | $9/2$ | $15/2$ | $21/2$ |
| $2/3$ | $3/4$ | $5/3$ | $2/5$ | $9/20$ | 1 |

13.32

| $a$ | $b$ | $c$ | $a'$ | $b'$ | $c'$ |
|---|---|---|---|---|---|
| 4 | 5 | 6 | 16 | 20 | 24 |
| 8 | 5 | 9 | 36 | $45/2$ | 18 |
| 2 | 5 | 4 | $3/2$ | $15/4$ | 3 |
| 8 | 10 | 12 | 24 | 30 | 36 |
| 4 | 3 | 6 | 10 | $15/2$ | 15 |
| 5 | 6 | 8 | $15/2$ | 9 | 12 |

**13.33**

| $a$ | $b$ | $c$ | $a'$ | $b'$ | $c'$ |
|---|---|---|---|---|---|
| 4 | 5 | 6 | 16 | 15 | 18 |
| 4 | 4 | 6 | 6 | 2 | 3 |
| 7 | 6 | 5 | 35 | 24 | 20 |
| $1/2$ | $1/3$ | $1/5$ | 15 | $29/3$ | $29/5$ |
| 3 | 4 | 5 | 4 | $4/3$ | $5/3$ |
| 5 | 7 | 4 | 10 | 7 | 4 |

**13.34** Which of the following triples can be the sides of a right triangle?
Answers: can, can, can, cannot, cannot, can, can.