Calculus Concepts Using Derive For Windows

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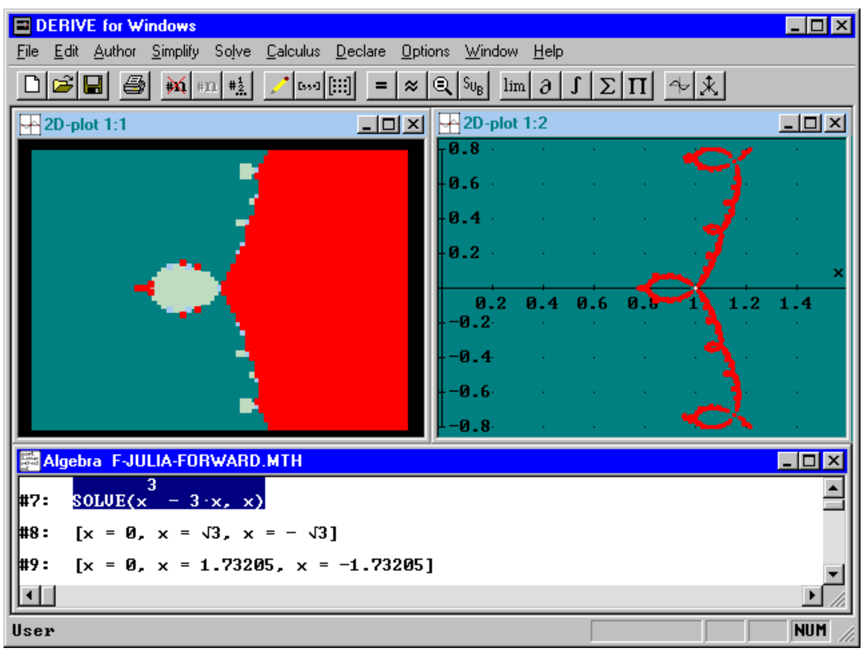

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Contents

Preface **ix**
Calculus Reform and Computers ix

0 Introduction and Derive Basics **1**
0.1 Overview 1
0.2 Starting Derive 2
0.3 Entering an Expression 3
0.4 Special Constants and Functions 5
0.5 Editing 5
0.6 Simplifying and Approximating 8
0.7 Solving Equations 8
0.8 Substituting 10
0.9 Calculus 10
0.10 Plotting 13
0.11 Defining Functions and Constants 17
0.12 Defining The Derivative Function 20
0.13 Functions Described By Tables 20
0.14 Vectors 22
0.15 Printing and Saving to a Disk 24
0.16 Help 25
0.17 Common Mistakes 26

1 Basic Algebra and Graphics **29**
1.1 Introduction 29
1.2 Finding Extreme Points 29
1.3 Zooming and Asymptotes 31
1.4 Laboratory Exercises 33

2 The Derivative **35**
2.1 The Derivative as a Limit of Secant Lines . . . 35
2.2 Local Linearity and Approximation . . . 39
2.3 Laboratory Exercises . . . 40

3 Curve Sketching **45**
3.1 Introduction . . . 45
3.2 Working with Graphs . . . 46
3.3 Exponential vs Polynomial Growth . . . 47
3.4 Laboratory Exercises . . . 49

4 Curve Fitting **53**
4.1 Introduction . . . 53
4.2 Fitting Polynomials to Data Points . . . 53
4.3 Exponential Functions and Population Growth . . . 58
4.4 Approximation Using Spline Functions* . . . 59
4.5 Laboratory Exercises . . . 61

5 Finding Roots Using Computers **67**
5.1 Introduction . . . 67
5.2 Newton's Method . . . 67
5.3 When Do These Methods Work . . . 71
5.4 Fractals and Chaos* . . . 73
5.5 Bisection Method* . . . 79
5.6 Laboratory Exercises . . . 82

6 Numerical Integration Techniques **87**
6.1 Introduction . . . 87
6.2 An Example . . . 88
6.3 Theorem on Error Estimates* . . . 92
6.4 More on Error Estimates* . . . 94
6.5 Deriving Simpson's Rule* . . . 96
6.6 Laboratory Exercises . . . 97

7 Differential Equations **101**
7.1 Introduction . . . 101
7.2 Examples . . . 101

7.3 Approximation of Solutions 105
7.4 Euler's Approximation Method* 109
7.5 Linear First Order Differential Equations 110
7.6 Laboratory Exercises 112

8 Polar and Parametric Graphs **115**
8.1 Introduction 115
8.2 Polar Coordinates 115
8.3 Rotating Polar Curves 118
8.4 Parametric Curves 120
8.5 Laboratory Exercises 124

9 Series **129**
9.1 Introduction 129
9.2 Geometric Series 130
9.3 Applications 131
9.4 Approximating Infinite Series 135
9.5 Laboratory Exercises 145

10 Taylor Polynomials **149**
10.1 Polynomial Approximations 149
10.2 Examples 151
10.3 Taylor's Theorem with Remainder 154
10.4 Computing the Sine Function 156
10.5 Computing the Exponential Function. 157
10.6 Taylor Expansions About $x = c$ 159
10.7 Interval of Convergence 162
10.8 Laboratory Exercises 164

11 Approximating Integrals **169**
11.1 Introduction 169
11.2 The Basic Error Estimate 170
11.3 The Logarithm Series 170
11.4 An Integral Approximation 172
11.5 Laboratory Exercises 174

12 Harmonic Motion **177**
12.1 Introduction 177
12.2 Applications 179
12.3 Systems of Differential Equations 183
12.4 Laboratory Exercises 189

A Utility Files **193**
A.1 The Functions 194
A.2 Listings of the Utility Files 199

B Instructors' Manual **203**

List of Figures

0.1 Author entry form with special symbols 5
0.2 Using the Calculus menu 11
0.3 Examples of Limits, Products and Sums 12
0.4 Using Plot for graphics 15
0.5 Examples of Declare, Simplify and approximating 19
0.6 Functions defined by tables of expressions 21
0.7 Using the Calculus/Vector command 23

1.1 Finding critical points 30
1.2 Zooming to find the horizontal asymptote 32

2.1 Secant lines approximating the tangent line 37
2.2 Approximating derivatives using the difference quotient 40

3.1 Using `vector` to plot several graphs 46
3.2 The functions x^4 and e^x 48
3.3 The functions x^4 and e^x, rescaled 50

4.1 Fitting a polynomial to data points 55
4.2 The algebra behind fitting polynomials to data points 57
4.3 Approximation using spline functions 61

5.1 Newton's method for finding roots 68
5.2 Each iteration gives twice as many digits 71
5.3 Newton's method with complex starting point 75
5.4 Chaos 76
5.5 Basins of attraction of $x^3 - 1$ in the complex plane 77
5.6 Bad Newton starting points for $x^3 - 3x = 0$ in the complex plane 78
5.7 Basins of attraction for $x^3 - 3x = 0$ 79

5.8 Bisection method for finding roots 81

6.1 Approximating $\ln 2$ with left endpoint method 90
6.2 Rectangular vs trapezoidal approximation 92
6.3 Trapezoid and midpoint rule for concave functions 96

7.1 Solving Newton's cooling equation 104
7.2 Slope field for the Newton cooling problem 107
7.3 A graph of a Verhulst population curve 108
7.4 Euler's method for approximating solutions 111

8.1 Polar Coordinates 116
8.2 Plotting points in polar coordinates 117
8.3 Showing that $y = 1/x$ is a hyperbola 119
8.4 Parametric plot of a semi-ellipse 121
8.5 More parametric plots 124
8.6 The cycloid curve and the rolling wheel 127

9.1 Ratio test example 138
9.2 The geometric estimate used in the integral test 141
9.3 Summing the series $\sum 1/i^2$ 143

10.1 Basic examples of Taylor polynomials 151
10.2 Taylor polynomials for $\sin x$ 153
10.3 Approximating $\sin x$ with its Taylor polynomials 154
10.4 Approximating $\sin 100$ within 6 decimals 158
10.5 Approximating e^5 within 6 decimals 160
10.6 Taylor expansion of the logarithm function 161
10.7 Graphically finding the radius of convergence 163

11.1 Using Taylor series to approximate integrals 174

12.1 Spring motion starting at different positions 180
12.2 Under Damped Oscillations 182
12.3 The pendulum 183
12.4 Euler and the 2nd Runge-Kutta methods 185
12.5 Rabbits and Foxes 187
12.6 Pendulums 188

Preface

Calculus Reform and Computers

This lab manual is our latest effort after more than a decade of experimenting with the use of computers as an enhancement to learning calculus. In the beginning we were working with Albert Rich and David Stoutemyer, founders of Soft Warehouse Inc. here in Honolulu, and their muMATH computer program. This program was the precursor to Derive. It was a PC-version of the big mainframe computer program macsyma. They could symbolically integrate, differentiate and do other calculus type problems. There were no menus or graphics at that time, so we developed a small enhancement program which included these features and distributed it to several university mathematics departments in the United States and elsewhere.

This effort took place in the mid to late 1980's. Since then, there has been a national movement to include computers in the teaching of calculus and in fact to *reform* the teaching of calculus by discussing new ideas using not only the traditional algebraic approach but also by exploring the ideas graphically and numerically. In response to this movement, new computer programs were written such as Derive, Mathematica and Maple. Computer calculus labs were created at most universities and colleges to take advantage of this new technology and to start experimenting with new ideas for teaching calculus.

Calculus text books are now starting to include substantial supplements on computer experiments and some have been completely rewritten to involve computers as an integral component of the course. It is hard to say right now what calculus instruction will look like in ten years but there is no doubt that computers are completely changing the teaching approach to certain topics with intensive graphics or computation components.

In this book we try to highlight those areas of calculus which are best studied by using the computer to explore, to visualize and to suggest further directions to study. We also try to convey that studying calculus can be fun to do and is very important to understanding other topics in mathematics and other fields.

This manual uses DERIVE because of its ease of use due to its menus and on-line help. It turns out that there are also functions which are equivalent to the various menu commands; for example, in place of Calculus/Integrate to integrate $f(x) = x^2$ one can alternately Author `int(x^2,x)`. However, there is no need to know these functional equivalents if you use DERIVE as a *calculus calculator*. On the other hand, knowing these functions enables one to write programs which extend the power of DERIVE. We will give numerous examples demonstrating both the simple calculator mode and the powerful programming language.

Chapter 0

Introduction and Derive Basics

0.1 Overview

In this course you will learn to use the computer mathematics program Derive. This program, along with others such as Maple and Mathematica, are very powerful tools for doing calculus. They are capable of doing exact computations with arbitrary precision. This means that you can work with numbers of any size or number of decimal places (most spreadsheets only use 10-20 significant digits). These programs can simplify mathematical expressions by canceling common factors and doing other algebraic operations. They can do symbolic calculus such as differentiation and integration, solve equations and factor polynomials. When possible these programs solve these problems exactly and when exact solutions do not exist, such as factoring high degree polynomials or integration of some non-polynomial expressions, then numerical methods are applied to obtain approximate results.

Probably the most important numerical technique is to graph and compare functions. This will be a key feature of the labs. Typically we will explore a topic by first graphing the functions involved and then trying to do symbolic calculus on them using the insight gained from the picture. If the problem is too difficult algebraically we then try numerical techniques to gain further insight into the problem. It is this combination of graphics, algebra and numerical approximation that we want to emphasize in these labs.

Calculus is a hard subject to learn because it involves many ideas such as slopes of curves, areas under graphs, finding maximums and minimums, analyzing dynamic behavior and so on. On the other hand, many computations involve algebraic manipulations, simplifying powers, dealing with basic trig expressions, solving equations and other techniques. Our goal is to help you understand calculus better by concentrating on the ideas and applications in the labs and let the computer do the algebra, simplifying and graphing.

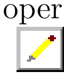
Another important goal of the lab is to teach you a tool which can used from now on to help you understand advanced work, both in mathematics and in subjects which use mathematics. There are many features such as matrices and vector calculus which we will not discuss but can be learned later as you continue with your studies in mathematics, physics, engineering, economics or whatever. Any time you have a problem to analyze you can use the computer to more thoroughly explore the fundamental concepts of the problem, by looking at graphs and freeing you from tedious calculations.

This chapter contains a brief introduction on how to use DERIVE. We suggest you sit down at the computer and experiment as you look over the material. DERIVE is very easy to learn thanks to its system of menus. The few special things you need to remember are discussed below and can also be found using the help feature in DERIVE.

0.2 Starting Derive

The computers in the Bilger labs are IBM-PCs running the Window 95 operating system. In some of the other labs such as K214 and CLIC the older Window system and the DOS operating system are still being used although they are going to be upgraded by the Fall 1997. We will mostly describe how to use the new DfW (DERIVE for Windows) software in the Window 95 environment. The DOS version which is also available in the Bilger lab is currently the only version available in some of the other labs and so we will say a few words about its use also. On a DOS based computer start DERIVE FOR DOS by just typing `derive` on the command line. If you are in Windows, either look for the DERIVE icon or else open a DOS window and type `derive`. Finally, in Win95 we start both versions by double clicking on it's startup icon which is located on the desktop. To start DfW look for

and double click it. The DERIVE FOR DOS startup is nearby.

In DfW we use the drop down menus on the top strip or else click an appropriate button. If you move the cursor onto a button and leave it there a brief explanation of what the button does will appear. All possible options can be found on the drop down menus but the buttons provide a quick way of doing most common operations. For example, to enter a mathematical expression you click the  button which represents a pencil. Alternately, you click the Author menu and then click Expression. In this manual we will indicate that two step combination by simply Author/Expression.

DERIVE FOR DOS uses a menu at the bottom of the screen from which we make selections by pressing the capitalized letter. For example, we type `a` (or `A` but uppercase doesn't matter to DERIVE on input so don't bother) to select **A**uthor. Each menu item has one capital letter (usually, but not always, the first). You can choose that menu item by pressing that letter. We denote this by showing the capital letter in bold; for example, **S**implify or so**L**ve. You can also choose this by hitting the Tab key until it is highlighted and then pressing the Enter key. Note that the mouse is not used in DERIVE FOR DOS so all selections are done by typing.

In this manual we use a typewriter like font, eg., `a(b + c)` to indicate something you might type in. We use a sans serif font for special keys on the keyboard like Enter (the return key) and Tab. The special keys are mostly used DERIVE FOR DOS whereas in DfW the analogous procedures are accomplished by clicking the OK or Simplify boxes on various data entry forms. Most of DERIVE has easy to use menus described below.

0.3 Entering an Expression

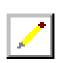
After clicking the  button (or selecting **A**uthor in DERIVE FOR DOS), you enter a mathematical expression, i.e., you type it in and then press the Enter key or else click OK. You enter an expression using the customary syntax: addition `+`-key, subtraction `-`-key, division `/`-key, powers `^`-key and multiplication `*`-key (however; multiplication does *not* require a `*`, i.e., `2x` is the same as `2*x`). DERIVE then displays it on the screen in two-dimensional form with raised superscripts, displayed fractions, and so forth. You should always check to make sure the two-dimensional form agrees with what you thought you entered (see **Editing** below to see how to correct typing errors). Table 1 gives some examples.

Table 1:

	You enter:	You get:
(1)	`25`	25
(2)	`x^2`	x^2
(3)	`a^2x`	a^2x
(4)	`a^(2x)`	a^{2x}
(5)	`sin x`	$\sin(x)$
(6)	`sin a x`	$\sin(a)x$
(7)	`sin(a x)`	$\sin(ax)$
(8)	`(5x^2 - x)/(4x^3 - 7)`	$\frac{5x^2-x}{4x^3-7}$
(9)	`(a + b)^(1/2)`	$(a+b)^{1/2}$
(10)	`√ (a + b)`	$\surd(a+b)$

If you get a syntax error when you press enter (or click OK) the problem is usually mismatched parentheses. Carefully check that each left parenthesis is matched with a corresponding right one. Also be careful to use the round parentheses and not the square brackets since they are used for vector notation; see Section 0.14 on page 22.

Note from (3) and (4) and from (6) and (7) of Table 1 that it is sometimes necessary to use parentheses. Also note in (8), that to get the fraction you want, it is necessary to put parentheses around the numerator and denominator. See what happens if you enter (8) without the parentheses. Also try entering some expressions of your own. There are two ways to enter square roots. One way is using the 0.5 or 1/2 power as in (9) and the other is to enter the special square root character as in (10). In DfW you enter special characters by clicking on them in the author form, see Figure 0.1 on the next page. In DERIVE FOR DOS you enter Alt-q which means hold down the Alt-key and press q.

0.4 Special Constants and Functions

In DfW all the special characters are on the author form and you just click on them to enter them in an expression. There are also key equivalent such as Ctrl-p for π and Ctrl-e for Euler's constant[1] e which is displayed by DERIVE as $\hat{e}$.

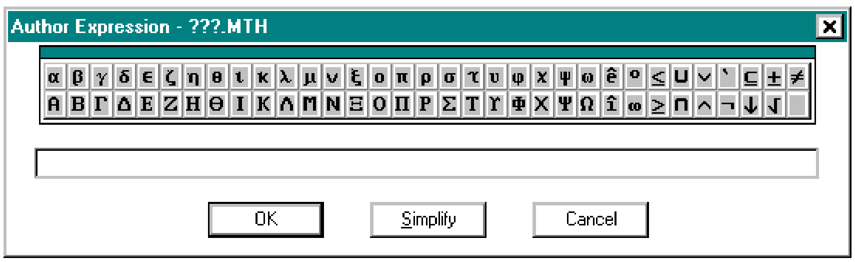
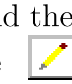


Figure 0.1: Author entry form with special symbols

In DERIVE FOR DOS a slightly different key combination is used for special characters. To get ∞, type `inf`. To get π, type `pi` or Alt-p. Euler's constant e is obtained by typing either `Alt-e` or `#e`. The help feature in DERIVE FOR DOS can be used at any time to remind you how to type these constants. Just select **H**elp or `F1` when authoring and then select either constants or functions. The list of functions is very large and you might want to avoid that in the beginning.

It either version it is important to distinguish $\hat{e}$ from just e. DERIVE takes e to just be some constant like a. To get the functions $\tan^{-1} x = \arctan(x)$, $\sin^{-1} x = \arcsin(x)$, etc., you type `atan x` and `asin x`.

0.5 Editing

Suppose that you author an expression, click OK and then observe that you typed something wrong. In DfW you would click the  button again and then *click the right mouse button*. A menu opens up with several options, one of which is Insert Expression. Clicking this option puts the highlighted expression of the current algebra window into the author box. You edit

[1]Leonhard Euler (*oi'lar*) was a 18$^{\text{th}}$ century Swiss mathematician.

this expression as you would in any windows program. That is you position the cursor either by clicking or using the arrow keys. By highlighting or selecting a subexpression and typing you replace the selected text with the new text. One can use the Edit/Copy Expressions menu or Cntl-C to place a highlighed expression from any algebra window onto the clipboard and then in the authoring form right click the mouse and click Paste to copy the clipboard contents. The simpler method of just right clicking the mouse and then Insert is the best way as long as your expressions are in a single window. There is also an option for inserting the expression enclosed in parenthesis. A key equivalent to these techniques, which is the same for DERIVE FOR DOS, is F3 and F4.

You select or highlight expressions in the algebra window by clicking on them. For more complicated expressions you can click several times until the desired subexpression is selected. This requires a little practice but you can, for example, select the $x+2$ part of the expression $\sin \frac{x^2}{x(x+2)}$ by clicking on it 4 times (each click takes you deeper into the expression).

When you are '**A**uthoring' an expression in DERIVE FOR DOS, you can use the left and right arrow keys, the End-key and the Home-key to move forward and back. The Delete key will delete characters. The Insert-key toggles between insert and overstrike mode. If you press the F3-key, the expression highlighted on the screen will be inserted; F4 will insert it with parentheses around it. You can use the up and down arrows to change which expression is highlighted on the screen. The help feature explains these techniques, just select **H**elp and then choose **E** for edit.

The displayed expressions are numbered. You can refer to them as `#n`. So, for example, with the expressions in Table 1, you could get $\sin(x)/x^2$ by **A**uthoring `#5/#2`.

When you start DERIVE it is in a character mode. This means it treats each single character as a variable, so if you type `ax` DERIVE takes this to be a times x. This mode is what is best for calculus. The exception to this are the functions DERIVE knows about. If you type `xsinx`, DERIVE knows you want $x\sin(x)$. Actually on the screen you will see $x\,\mathrm{SIN}(x)$: DERIVE displays all variables in lower case and all functions in upper case.

Table 2: Special Keys and Function Names

Expression/Action	Type:	Menu:
e	Alt-e (DOS) or #e	Author entry form
π	Alt-p (DOS) or pi	Author entry form
∞, $-\infty$	inf, -inf	Author entry form
The square sign: $\surd$	alt-q (DOS)	Author entry form
$\ln x$, $\log_b x$	ln x , log(b, x)	
Inverse trigonometric functions	asin x, atan x, *etc.*	
$\frac{d}{dx}f(x)$	dif(f(x), x)	Calc/Differentiate
$\frac{d^n}{dx^n}f(x)$	dif(f(x), x, n)	Calc/Differentiate
$\int f(x)\,dx$	int(f(x), x)	Calc/Integrate
$\int_a^b f(x)\,dx$	int(f(x), x, a, b)	Calc/Integrate
Simplify an expression	S-*key* (DOS)	Simplify
Approximate	X-*key* (DOS)	Simplify/Approx
Cancel a menu choice	Esc-*key*	
Move around in a menu	Tab-*key* (DOS)	
Change highlighted expression	▲, ▼-*key*	Click expression
Insert highlighted expression	F3 , F4 with ( )'s	Right mouse button

0.6 Simplifying and Approximating

After you enter an expression, Derive displays it in two-dimensional form, but does not simplify it. Thus, integrals are displayed with the integral sign and derivatives are displayed using the usual notation. To simplify (that is evaluate) the expression, click the $\boxed{=}$ button. The alternate method is the Simplify/Basic drop down menu. In Derive for DOS you choose **S**implify from the menu by pressing the s-key.

Derive uses exact calculations. If you Author the square root of eight, $\sqrt{8}$ will be displayed in the algebra window. If you simplify this, you get $2\sqrt{2}$. If you want to see a decimal approximation, you click the $\boxed{\approx}$ button. In Derive for DOS choose the appro**X**imate menu item by pressing the x–key. See Figure 0.5 on page 19 for several examples. The number of decimal places displayed can be changed to any number. In DfW you choose Declare/Algebra State/Output and then reset the number of decimals places. In Derive for DOS it's done by choosing **O**ptions/**P**recision and changing **D**igits by pressing the Tab-key and entering a number. This results in a change in the State variables for Derive and in DfW you will be prompted on whether you want to save these changes when you exit the program. Since you can't change files on the system directory you should click No.

An alternate way to do this is choose Simplify/Approximate from the drop down menus and enter a new number of decimals. The only trouble with this method is that if you save your file the extra decimals will be ignored unless you set the Output decimal places appropriately. When you open the file later you will also need to reset the Output decimal place accuracy.

0.7 Solving Equations

An important problem is to find all solutions to the equation $f(x) = 0$. If $f(x)$ is a quadratic polynomial such as $x^2 - x - 2$, then this can be done using the quadratic formula or by factoring. To factor in DfW you choose Simplify/Factor from the menu bar and click Simplify on the entry form. In Derive for DOS we choose **F**actor, press the Enter-key and ignore the other options for now. The result is that $(x+1)(x-2)$. This means that the roots of $f(x)$ are $x = -1$, 2, i.e., these are the only solutions to $f(x) = 0$.

We can also do this by using the `SOLVE` function. To do this in DfW we highlight the equation, say $x^2 - x - 2$ (it's assume to be equal to zero), and click the  . If you forget the function of a button just hold the cursor on it and a brief explanation will appear. An alternate method is to choose Solve/Algebraically from the drop down menu. Similarly, in Derive for DOS we choose so**L**ve with the quadratic expression highlighted. The quadratic formula is used to solve for the roots so it is possible the answer will involve square roots (and even complex solution, e.g., $x^2 + 1$ has no *real* roots but it does have two complex ones, namely, $x = \pm i$).

If $f(x)$ is not a quadratic polynomial then Derive may not be able to factor it; nevertheless, it may be able to solve the equation $f(x) = 0$. As an example, $\sin x = 0$ has infinitely many solution $x = m\pi$ where m is any integer. If we use Derive to solve this equation it gives the 3 solutions corresponding to $m = -1, 0, 1$ (these are the principle solutions and all others are obtained by adding or subtracting multiples of 2π).

Finally, the simple equation $\sin x - x^2 = 0$ cannot be solved exactly in Derive although it is obvious that $x = 0$ is one solution and by viewing the graph we see another one with $x \approx 1$. In order to approximate this solution we need to choose Solve/Numerically. We will then be asked for a range of x's (initially it is the interval $[-10, 10]$). Since we have (at least) 2 solutions in $[-10, 10]$, we should restrict the interval to say $[.5, 1]$ which seems reasonable based on the graphical evidence. The result is that Derive gives the solution $x = .876626$. We will discuss how this computation is done later in Chapter 5.

Note that Solve/Numerically will only give one solution (or none if there are none) even if the interval you choose cantains several solutions. To find additional solutions you need to use Solve/Numerically again but with an interval avoiding the first solution.

In Derive for DOS we have to proceed a little differently and **A**uthor the expression

```
solve(sin x - x^2, x, .5, 10)
```

directly and then simplify. Of course, we could do this in DfW too but the menu method is easier. Yet another method is to change the state variables so that all simplification is done numerically instead of exactly but this is less desirable because you have to remember to switch modes back to exact mode for other calculations.

0.8 Substituting

If you have an expression like $\sqrt{x^2+1}/x$ and you want to evaluate this with say $x = 3$ or if you solved an equation $f(x) = 0$ and you want to substitute that value of x back into $f(x)$, you start by highlighting the desired expression. Next you click the SUB button and the substitution form opens up. You need to fill in the substitution value so you would just type 3 in the first example. On the other hand, if the substitution value is large, say lots of digits or some other complicated expression in the algebra window, the easiest way is to move the form out of the way (just hold down the left mouse button in the top strip and drag to another location) and select the desired expression by clicking on it. Then, paste it into the form by right clicking and choosing Insert. If there happen to be other variables in the expression you may have to change the variable in the variable list box.

Again the method in Derive for DOS is a little different. You highlight the expression and then choose **M**anage/**S**ubstitute. This will ask you for the expression. It will guess the highlighted expression, which is usually what you want so you can just hit return in this case. It then gives the name of a variable occurring in the expression. In the first example x is the only variable. You then type over x with the value you want to substitute, in this case 3. You can then **S**implify or appro**X**imate. You do not have to substitute a number for x; you can substitute another expression.

0.9 Calculus

This menu item is very important for us. After choosing the Calculus drop down menu, you get a submenu with Differentiate, Integrate, Limit, Product, Sum, Taylor and Vector. After you have authored an expression, you can differentiate it by either clicking the ∂ button or choosing Calculus/Differentiate from the drop down menu. The form will have entries for what variable to use and how many times to differentiate, but it usually guesses right so you can just click OK. Then simplify.

To integrate an expression, first author it or highlight it if it is already in the algebra window, then either click the $\int$ button or else choose Calculus/Integrate. The form will have entries for what expression to integrate; it will guess you want to integrate the highlighted expression. It will have an entry for what variable you what to integrate with; again it will probably guess right. It will also have entries for the limits of integration. If you want an indefinite integral, just click the appropriate button and click OK. For a definite integral click the appropriate button, type in the upper/lower limit, then click OK. (Note: the procedure is similar in DERIVE FOR DOS except that we need to use the Tab-key to get to the other menus options such as the upper/lower limits). See Figure 0.2 for several examples using Differentiate and Integrate on the Calculus menu.

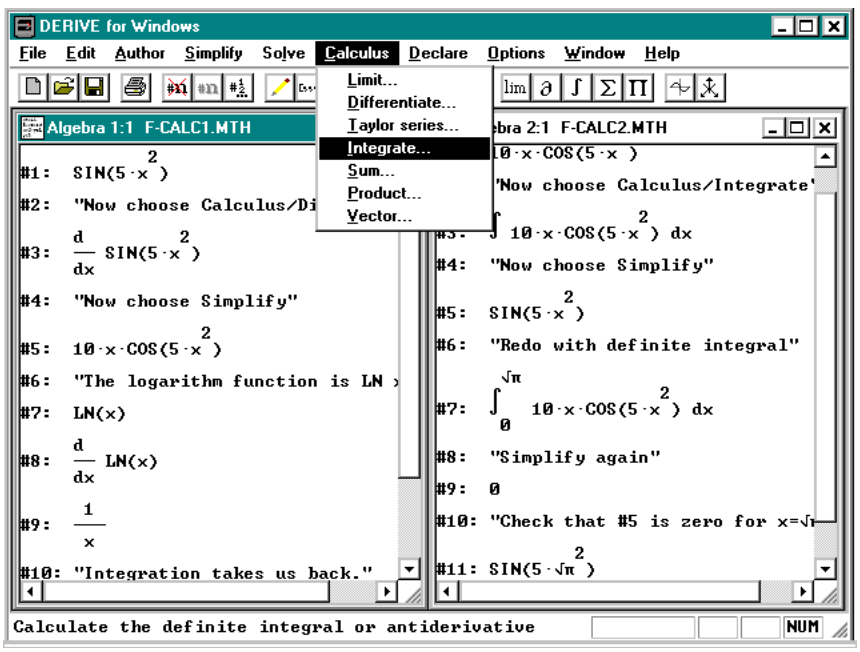


Figure 0.2: Using the Calculus menu

The options Calculus/Limits is similar to the above. To find

$$\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$$

you enter the expression, then either click lim or choose Limits from the Calculus menu. You fill in the variable (which is x) and the limit point which is -1 since $x \to -1$. Then click = or choose Simplify to get the answer.

In a similar manner DERIVE does summation and product problems. Special notations are used; namely,

$$\sum_{i=1}^{n} a_i = a_1 + \cdots + a_n \quad \text{and} \quad \prod_{i=1}^{n} a_i = a_1 \cdots a_n.$$

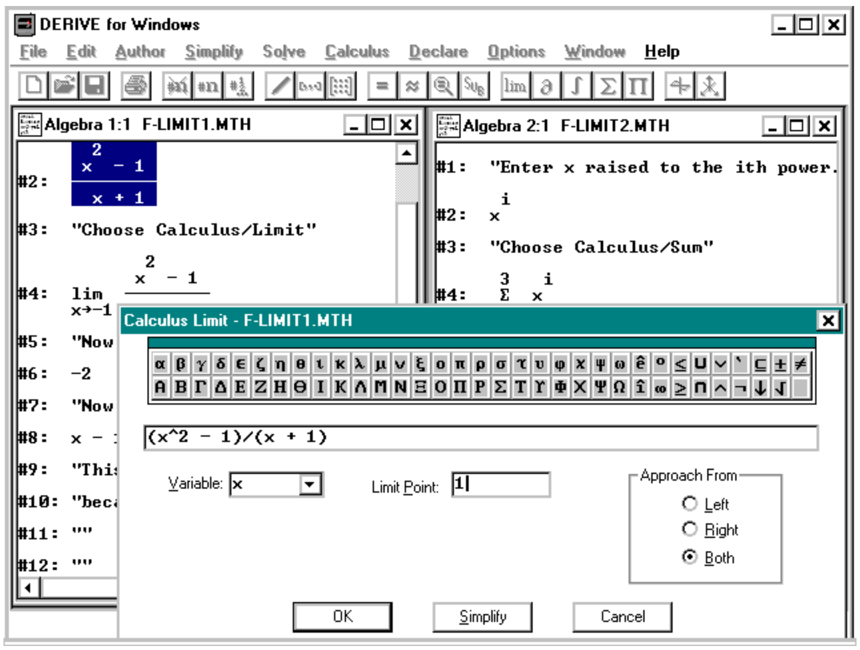


Figure 0.3: Examples of Limits, Products and Sums

Let us discuss the summation notation which may be new to you. If $a_1, \ldots, a_n$ are numbers then

$$\sum_{i=1}^{n} a_i = a_1 + \cdots + a_n.$$

The symbol on the left, $\sum_{i=1}^{n} a_i$, is read as "the sum of a_i as i runs from 1 to n." Often a_i is a formula involving i. So

$$\sum_{i=0}^{5} i^{\,2} = 0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

You can do this computation in DERIVE by clicking the $\boxed{\Sigma}$ or using the Sum option on the Calculus menu. Just author `i^2` then click $\boxed{\Sigma}$. Fill in the required variable i along with the starting value 0 and end value 5. Simplify to get 55. As an interesting aside, edit the above sum and have DERIVE Simplify $\sum_{i=1}^{n} i$ to get the formula:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

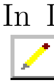
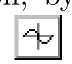
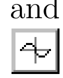
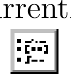
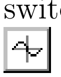
This formula is used in many calculus texts to evaluate certain Riemann sums.

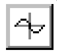
See Figure 0.3 on the facing page for some examples. Note that in Figure 0.3

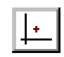
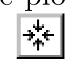
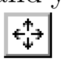
$$\prod_{i=1}^{3} x^i = x \cdot x^2 \cdot x^3 = x^6.$$

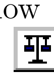
The option Calculus/Taylor will be explained in Chapter 10.

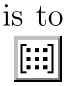
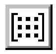
0.10 Plotting

Supposed you wanted to graph the function $x \sin x$. In DfW you simply author the expression, by clicking the pencil button , to be plotted and then click the  button. A plot window will then opens up and the icon-bar will change to a new set of buttons. You then click the  button again (it's position is different in the plot window) and the graph will be drawn. There are several different ways to view the algebra and plot window together. The one we used to produce the pictures in the manual is to first select the algebra window (if you are currently in the plot window you can go to the algebra window by clicking the  button) and then choose Window/Tile Vertically from the menus. This will split the screen into two windows: an algebra window on the left and a plotting window on the right. These windows each have a number in their upper left hand corner. You can have several plot windows associated with a single algebra window but you cannot plot together expressions from different algebra windows. You can switch windows by either clicking the top strip of the window or clicking the  or  buttons. Actually you can click anywhere in the window to select it but the top strip avoids changing the highlighted expression in the algebra window or moving the cross in the plot window.

You can plot several functions in the same plot window. Move to the algebra window, highlight the expression you want to plot, switch to the plot window and then click the  in the plot window. Now both expressions will be graphed. You can plot as many as you want this way. The plot window also has a menu option, Edit/Delete Plot, for removing some or all of the expressions to be plotted. Pressing the Delete-key also removes the current plot.

When you plot, there is a small crosshair in the plot window, initially at the $(1, 1)$ position. You can move the cross around using the arrow keys or by clicking at a new location. The coordinates of the cross are give at the bottom of the screen. This is useful for such things as finding the coordinates of a maximum or a minimum, or where two graphs meet. In order to center the graph so that the cross is in the center of the window, click the  button. This is useful for zooming in and out to get a better view of the graph. There are several buttons for doing this in the plot window. Take a look and you will see a button for zooming in, namely , and for zooming out  and various ways of changing just the x-scale or just the y-scale. You should try clicking these buttons to see exactly what happens.

In general, these buttons change the scale of the plot window by either doubling or halving it. You can customize these by using the  button (that's a picture of a balance scale). Just click this button and fill out the form the appears with your own numbers. You can see the current scale at the bottom of the screen.

We mentioned above how to plot any number of graphs simultaneously by repeatedly switching between the algebra window and the graphics window. Another technique for plotting three or more functions is to plot a vector of functions. This just means authoring a collection of functions, separated by commas and surrounded by square brackets. For example, plotting the expression `[x, x^2, x^3]` will graph the three functions: x, x^2, and x^3. In order to plot a collection of individual points one enters the points as a matrix, for example authoring the expression `[[-2,-2], [0,-3], [1,-1]]` and then plotting it will graph the 3 points: $(-2, -2)$, $(0, -3)$ and $(1, -1)$. A quick way of authoring a vector is to use the  button and a quick way to enter a matrix is to use the  button. One then just fills out the form that open up. So for example with the 3 points above we would click the matrix button  and select 3 rows and 2 columns. The form will open up and we then fill in the 6 numbers above in the obvious order. You move between fields by either clicking or using the Tab-key.

When plotting points you have a choice of connecting the points with a line segment so that it appear like the graph of a function. You do this by choosing Options from the menu bar. There are lots of interesting items on this menu that will allow you to customize plotting colors, the size points are plotted, axes and so on. To connect points we choose Points and then check the Yes button. We can also modify the size of the points by clicking the appropriate button. See the Figure 0.4 where each of these techniques is demonstrated. The color of a plot is controlled by choosing Option/Plot Color and then making sections on the menu.

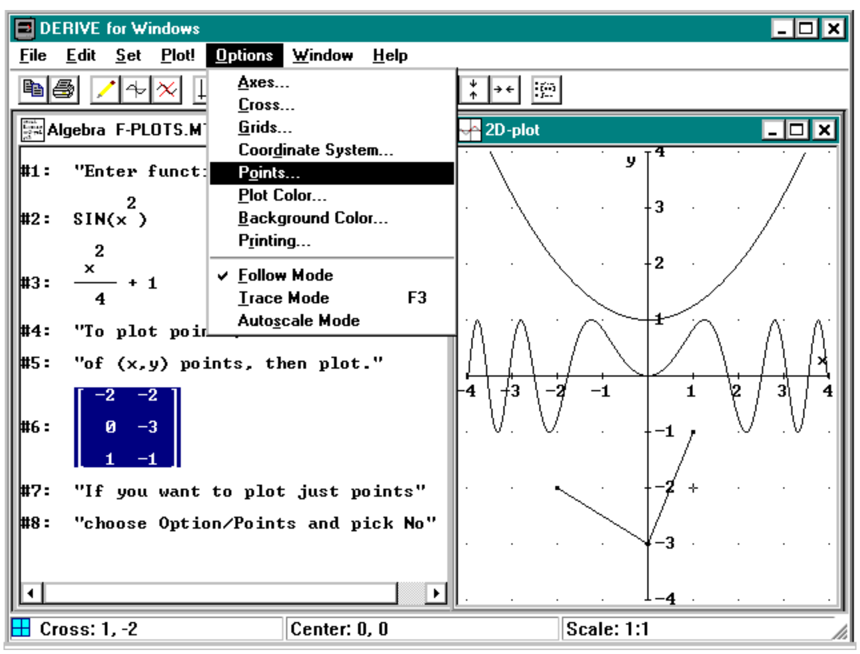


Figure 0.4: Using Plot for graphics

Graphing with Derive for DOS. In DERIVE FOR DOS the analogous procedure are as follows: You would first **A**uthor the expression to be plotted and then choose **P**lot. You then get the submenu: **B**eside, **U**nder, **O**verlay. You will usually want **B**eside. After choosing this (by pressing the b–key or pressing the Enter key), you are asked for the column. You can press Enter to get column 40. This will split the screen into two windows: an algebra window on the left and a plotting window on the right. These windows each have a number in their upper left hand corner. You can tell which window you are in by which number is highlighted. You can switch windows by pressing the F1-key or choosing **A**lgebra when you are in the plot window or choosing **P**lot from the algebra window.

After you have created the plot window, you are in that window. You need to choose **P**lot from that window to actually do the plotting. This will plot the expression highlighted in the algebra window. You can plot several functions in the same plot window. Move to the algebra window, use the up and down arrows to highlight the expression you want to plot, switch to the plot window (by pressing F1 or choosing **P**lot), and then choose **P**lot from the plot window. Now both expressions will be graphed. You can plot as many as you want this way. The plot window also has a **D**elete option for removing some or all of the expressions to be plotted.

When you plot, there is a small crosshair in the plot window, initially at the $(1, 1)$ position. You can move it around using the arrow keys. The coordinates of the position of the cross are give at the bottom of the screen. This is useful for such things as finding the coordinates of a maximum or a minimum, or where two graphs meet. The **C**enter option will redraw the graph so that the cross is in the center of the window. You can use the **Z**oom option to move in or out.

We mentioned above how to plot any number of graphs simultaneously by repeatedly switching between the algebra window and the graphics window. Another technique for plotting three or more functions is to plot a vector of functions. This just means **A**uthoring a collection of functions, separated by commas and surrounded by brackets. For example, **P**lotting the expression `[x, x^2, x^3]` will graph the three functions: x, x^2, and x^3. In order to plot a collection of individual points one enters the points as a matrix, for example **A**uthoring the expression `[[-2,-2], [0,-3], [1,-1]]` and then **P**lotting it will graph the 3 points: $(-2, -2)$, $(0, -3)$ and $(1, -1)$. In the graphics window choose **O**ption/**S**tate then press the Tab key followed by **C**onnected. Then choosing **P**lot again will graph the 3 points above but also draw the line segment between them. See the Figure 0.4 on the preceding page where each of these techniques is demonstrated. The color of a plot is controlled by choosing **O**ption/**C**olor/**P**lot and then making sections on the menu.

Tips for graphing with Derive for DOS. The main tools for manipulating the view of your graph are:

- Use **Z**oom to zoom in or out on either the x or y–axis or both. You can also use F9 to zoom in and F10 to zoom out. You can choose **S**cale to control the scale exactly.
- Use **C**enter to reposition the view so that the crosshair is in the center.
- **R**ange can be used to control the range. By changing the four numbers, **L**eft, **R**ight, **B**ottom, and **T**op, you can control where the view window is. You can either type in the values you want or use the arrow keys to visually change a range box on the screen.
- Choose **S**cale/**A**uto get auto-scaling mode. In this mode DERIVE chooses a good y-scaling to fit your graph for you.
- You can get the crosshair to follow along the curve (it changes to a small square when it does this) by choosing **O**ption/**S**tate and choosing **Y**es in the **T**race field. If you have more than one graph, the up and down arrows will change which graph the cursor is on.
- For some graphs you might need more accuracy in the plotting to see what is going on, for example $x\sin(1/x)$. **O**ptions/**A**ccuracy controls this.
- One easy way go back to the default coordinates in a plot window is to choose **W**indow/**D**esignate/**2**d-plot.

0.11 Defining Functions and Constants

If you Author `f(x)`, DERIVE will put $f\,x$ on the screen because it thinks both x and f are variables. If you wish to *define* say $f(x) = x^2 + 2x + 1$ for example, you could Author `f(x) := x^2 + 2x + 1`. Note that we use := for assignments and = for equations. Alternately, you could choose

Table 3: Special Keys for the DOS Graphing Window

Effect:	Type:	Menu:
Switch windows	F1	**A**lgebra or **P**lot
Zoom in, zoom out	F9, F10	**Z**oom
Move Crosshair	4 arrow keys	
Move crosshair quickly	Ctrl-◄, Ctrl-►, PgUp, PgDn	
Center on crosshair		**C**enter
change the scale		**S**cale

Declare/Function, and then fill in the form with `f(x)` for the function name and `x^2 + 2x + 1` for its value. DERIVE will then enter this as above with the :=-sign. See Figure 0.5 on the next page. The procedure in DERIVE FOR DOS is similar.

Constant are treated just like functions except there are no arguments. In order to set $a = 2\pi$ for example you type `a := 2 pi`. Then, whenever you simplify an expression containing a, each occurrence is replaced with 2π.

In many problems you find it useful to have constant names with more than one letter or symbol, which is the default in DERIVE. For example variables with names like `x1`, `y2`, etc. will be used frequently as are names like "gravity". This can be done by *declaring the variable*, for example, to use the variable $x1$ we author `x1:=`. Now any use of these letters will be treated as the single variable $x1$.

Alternately, this can be done by changing DERIVE to *word input mode* by choosing Declare/Algebra State/Input and then clicking the Word button. In DERIVE FOR DOS you would choose **O**ptions/**I**nput/**W**ord. In this mode variables can have several letters but when in word input mode you have to be more careful with spaces: to get ax^2 you should enter `a x^2`, not `ax^2` (otherwise ax will be treated as a variable). DERIVE indicates multiplication with a centered dot. So on the screen you should see $a \cdot x^2$, not ax^2. Due to these side effects it is usually best to use the previous method for multi-letter variables and not make any changes to State Variables.

An interesting function defining technique is provided by the factorials.

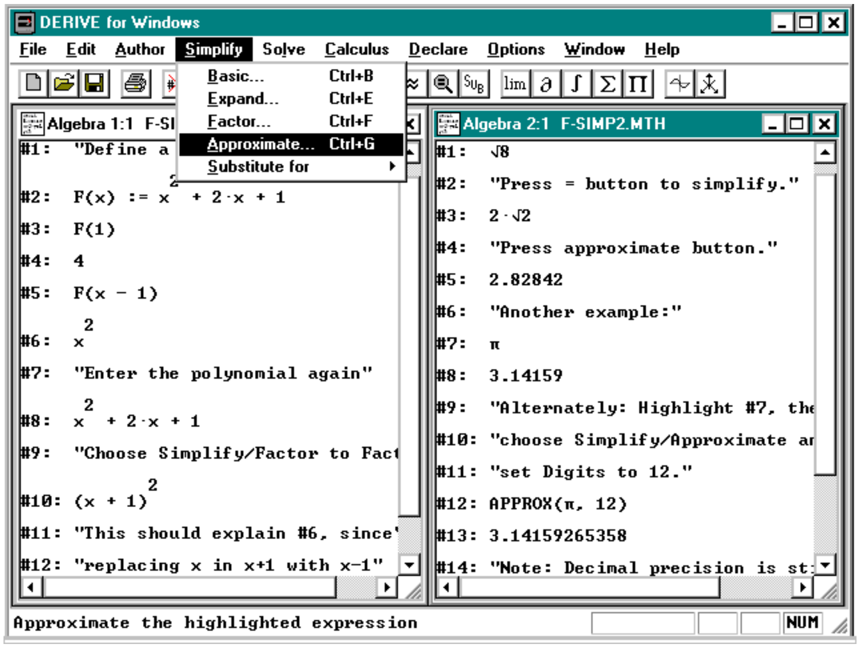


Figure 0.5: Examples of Declare, Simplify and approximating

For $n = 1, 2, \ldots$ we define n-factorial, denoted by $n!$, as

$$n! = n \cdot (n-1) \cdots 2 \cdot 1 \qquad n = 1, 2, \ldots$$

and for completeness we define $0! = 1$. These numbers are important in many formulas, e.g., the binomial theorem. One observes the important *recursive* relationship $n! = n(n-1)!$ which gives the value of $n!$ in terms of the previous one $(n-1)!$. Thus, since $5! = 120$ we see immediately that $6! = 720$ without multiplying all 6 numbers together.

In DERIVE we can recursively define a function `F(n)` satisfying $F(n) = n!$ by simply typing

```
F(n) := IF(n=0, 1, n F(n-1))
```

where the properties of the DERIVE function `IF(test, true, false)` should be clear from the context. The definition forces the function to circle back over and over again until we get to the beginning value at $n = 0$, i.e.,

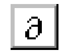
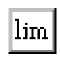
$$F(n) = n \cdot F(n-1) = n \cdot (n-1) \cdot F(n-2) = \cdots = n \cdot (n-1) \cdots 2 \cdot 1 \cdot F(0) = n!$$

We will give several other examples of this technique in the text.

0.12 Defining The Derivative Function

A common application of defining functions is to have $f(x)$ defined but the calculus problem requires a formula involving both $f(x)$ and $f'(x)$. For example, the equation of the tangent line at the point $(a, f(a))$ is given by $y = f(a) + f'(a)(x - a)$.

If you try to define the derivative of $f(x)$ by `g(x) := dif(f(x), x)` and then evaluate $g(2)$, you get `DIF(F(2), 2)`, which is not what you want. Of course we could also just compute the derivative and define `g(x) :=` to be that expression. The advantage of defining it as a function is that if we change the definition of $f(x)$, then $g(x)$ will also change to the derivative of the new $f(x)$. Thus, we get to use the formula for more than one application.

Here's the correct way to define the derivative as a function: Start by Authoring `f(x):=` and we can enter the specific definition of $f(x)$ now or wait until later. Next, click the derivative  button and enter `f(u)` in the form (note the variable is u not x). Select the Variable `u` and press OK. Now click the limit  button (with the previous expression highlighted) and enter the Variable `u` and the Point `x`. Finally, Author `g(x):=` and insert the previous expression by right-clicking and selecting Insert.

The result is the expression `G(x):= LIM( DIF( F(u), u), u, x)`. Actually, you could have just Authored this expression directly but the syntax and the number of parentheses is a little confusing in the beginning so the above method is easier and probably faster. See all this worked out on page 38 in Chapter 2 where a more technical discussion of this issue is given.

0.13 Functions Described By Tables

In calculus functions are typically described by giving a formula like $f(x) = 2x^3 + 5$ but another technique is to describe the values restricted to certain intervals or with different formulas on different ranges of x-values. As an example, consider the function

$$f(x) = \begin{cases} 2x+1 & \text{for } x < 1 \\ x^2 & \text{for } 1 \le x \le 2 \\ 4 & \text{for } 4 < x \end{cases}$$

which defines a unique value $f(x)$ for each value of x. The problem is how do we define such a function using DERIVE?

One basic technique is to use the logical `IF` statement. The syntax is `IF(test, true, false)`. For example, if we enter and simplify `IF(1 < 2, 0, 1)` we get 0 whereas `IF(1 = 2, 0, 1)` simplifies to 1. Now our function above is entered as:

```
f(x):= IF(x < 1, 2x + 1, IF(x <= 2, x^2, 4))
```

Notice how we use nested `IF` statements to deal with the three conditions and that with four conditions even more nesting would be required. Now once $f(x)$ has been defined we can make computations such as **S**implifying `F(1)` (should get 1), computing limits such as the right-hand limit $\lim_{x\to 1+} f(x)$ (should get x^2 evaluated at $x = 1$) or definite integrals using appro**X** to simplify. We can also plot $f(x)$ in the usual manner described in the previous section.

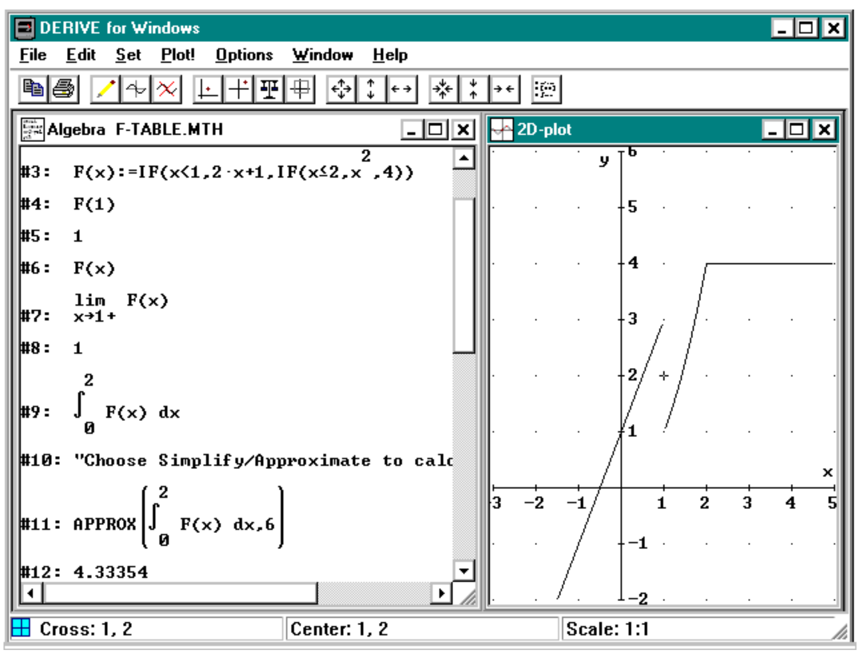


Figure 0.6: Functions defined by tables of expressions

Notice from Figure 0.6 that the function $y = f(x)$ is continuous at all

$x \neq 1$. At $x = 1$, both left and right limits exist but they are not equal so the graph has a jump discontinuity.

As the number of table entries increases we are forced into using nested `IF` statements and the formulas become quite difficult to read and understand. An alternate approach is to use the DERIVE function `CHI(a,x,b)` which is simply

$$\mathrm{CHI}(a, x, b) = \begin{cases} 0 & \text{for } x \le a \\ 1 & \text{for } a < x < b \\ 0 & \text{for } b \le x \end{cases}$$

Then except for $x = 1$ our function $f(x)$ above satisfies:

```
F(x):=(2x+1) CHI(-inf,x,1) + x^2 CHI(1,x,2) + 4 CHI(2,x,inf)
```

This technique works for graphing and limit problems and moreover gives the exact result at each point where the function is continuous.

0.14 Vectors

Vectors are quite useful in DERIVE, even for calculus. They are also useful in plotting. To enter the 3 element vector with entries a, b, and c, we can Author `[a, b, c]` directly. It is important to note the square brackets which are used in DERIVE for vectors; commas are used to separate the elements. An easier approach is just click the  button and fill in the three values on the vector input form.

DERIVE also provides a useful function for constructing vectors whose elements follow a specific pattern. The `vector` function is a good way to make lists and tables in DERIVE. For example, if you Author `vector(n^2, n, 1, 3)`, it will simplify to `[1, 4, 9]`. The form of the `vector` function is `vector(u,i,k,m)` where `u` is an expression containing `i`. This will produce the vector $[u(k), u(k+1), \ldots, u(m)]$. You can also use the Calculus/Vector menu option to create a vector. So, for example, to obtain the same vector as before, you start by authoring `n^2`. Now choose Calculus/Vector and fill in the form setting the Variable to n (not x), the start value to 1, the end value to 3 and the step size to 1 (that's the default value).

A table (or matrix) can be produced by making a vector with vector entries. If we modify the previous example slightly by replacing the expression `n^2` with `[n, n^2]` and then repeating the above we get `[[1, 1], [2, 4], [3, 9]]` which displays as a table with the first column containing the value of the index n and the second column containing the value of the expression n^2. This is a good technique for studying patterns in data. See Figure 0.7 for some examples.

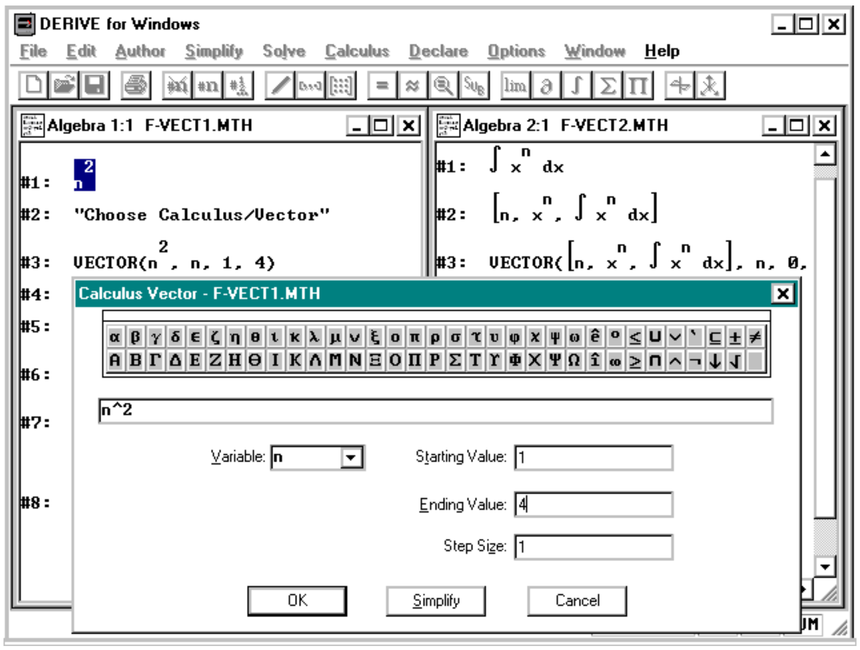


Figure 0.7: Using the Calculus/Vector command

We have already seen two important applications of vectors in Section 0.10; namely,

- Plotting a vector of 3 or more functions $[f(x), g(x), h(x), \ldots]$ plots each of these functions in order.

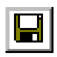
- Plotting a vector of 2-vectors $[[x_1, y_1], [x_2, y_2], \ldots]$ will plot the individual points (x_1, y_1), $(x_2, y_2), \ldots$.

We will have other application that will require us to refer to the individual expression inside of a vector. This is done with the DERIVE `SUB` function (which is short for *subscript*). Thus, for example, `[a,b,c] SUB 2` simplifies to the second element b. DERIVE will display this as $[a, b, c]_2$ which explains the name. For a matrix or vector of vectors then double subscripting is used so that, for example, if

$$y := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

then **A**uthoring `y SUB 2 SUB 1` will be displayed as $y_{2,1}$ and simplify to 3 (because it's on row 2 and column 1).

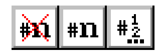
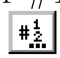
0.15 Printing and Saving to a Disk

You can save the expressions in an algebra window to either a floppy disk or the network hard drive H: and come back later to continue working on them. Unfortunately, the plot windows are not saved but the pictures can be put on the clip board and saved as graphics files using suitable graphics software. To save to a floppy, put a the diskette in say the A: drive and activate the algebra window that you want to save. Click File/Save As and fill out the menu of options with the drive A: and a file name such as Lab5 or just save to A:LAB5 and enter. DfW will add the extension .MTH to indicate that this is a file consisting of Derive expressions. If you are using Derive for DOS then instead choose **T**ransfer**S**ave, then **D**erive and enter a file name such as A:LAB5 or A:LAB5.MTH or H:LAB5 for the hard drive. In DfW after the file name has been established you can update it by simply pressing the  button.

You will most likely save your files to the network harddisk. The H: drive (H is for 'Home') is *your private area* which is accessible only using your password. To save a file, just refer to it as say H:LAB5 or switch to the H: drive and view your files.

Later, you can recall these expressions by using either the File/Open or File/Load/Math options. The second method is used primarily to add expressions to an existing window. In Derive for DOS you do this by choosing **T**ransfer/**L**oad/**D**erive, and then entering the desired file name A:LAB5 (or A:LAB5.MTH). If you forget the name of your files just type either A: or H: and press the `F1`–key to select from a listing of your files.

When you do a file operation you will notice that the default directory L:\DfW\M206L has lots of files of the form F-*.MTH. These files are the algebra window expressions from the various figures in the manual. For example, Figure 0.7 on the preceding page has two algebra windows F-VECT1.MTH and F-VECT2.MTH which are identified in the top stripe of the corresponding algebra windows. You can load these files at any time to see how expressions are entered or to experiment with the material.

During the course of your session with the computer you will make lots of typing and mathematical mistakes. Before saving your work to a file or before printing and turning your lab in for grading you should erase the unneeded entries and clean up the file. The three buttons  can be used for this purpose. For example, if you select several expressions by say dragging the mouse pointer over them with the left button held down and then press the  button these expression will be removed. Clicking  will undo the last delete. You can move a block of highlights lines by holding down the right mouse button and dragging the block to a new location. Of course, when you delete or move some lines then the line numbers will no longer be in a proper sequence of #1, #2, ... You can correct this by pressing the renumbering button . In DERIVE FOR DOS you do these operations with the **R**emove, the **U**nremove and the mo**V**e commands. You should practice these commands on some scratch work to make certain you understand them.

One way to use the move command is to write comments in the file and placing them before computation. Many of the *.MTH files that we wrote for this lab manual use this technique. To do it, just author a line of text enclosed in double quotes, for example, `"Now substitute x=0."`. Then, move this comment to the appropriate location.

You can print all the expressions in the algebra window (even the ones you can't see) by pressing the  button. You do the same thing to print a graph. Just activate the plot window and press . In DERIVE FOR DOS to print just a window with a graph in it, make it the current window, and then press Shift-F9. Typically, students turn in the labs by printing out the algebra window and penciling in remarks and simple graphs. More extensive graphs can be printed out. Some combination of hand writing and printouts should be the most efficient.

0.16 Help

You can obtain on-line help by choosing Help. This help feature provides information on all DERIVE functions and symbols. Suppose that you want to know how to enter the second derivative of a functions $f(x)$ by typing. For example, maybe this expression is to be used as part of another function. There are three techniques for learning how to do this.

The first method starts by authoring `F(x):=` to declare $f(x)$ to be a function of x. Next we use the menus with Calculus/Differentiate to calculate the second derivative by entering `F(x)` for the function and 2 for the order. Then, press Author followed by the pull-down key F3 or right click in the author box and select Insert. This will enter DERIVE's way of typing the expression, in this case it's `DIF(f(x),x,2)`. The second method is to use the online help by choosing Help on the main menu (in DERIVE FOR DOS you can press the F1-key while authoring an expression). One then searches for a topic like differentiation or vector to get further information. In DERIVE FOR DOS one has to page through the help pages since there is no searching feature. For example, one selects F (for functions) and then by pressing Enter several times one finds the appropriate page of explanations.

We have included a few quick reference tables with common keys used for entering things like π, ∞ and Euler's constant e. Table 2 on page 7 gives a summary of commands that can be issued from the algebra window and Table 3 on page 18 gives a summary of useful commands that can be used in the plot window.

0.17 Common Mistakes

Here are a few common mistakes that everyone makes, including the authors, every once in a while. It just takes practice and discipline to avoid these problems, although, it is human nature to blame the computer for *your own* mistakes. Fortunately, the computer never takes insults personally and it *never* takes revenge by creating sticky keys, erasing files, locking up, or anything else like that ... or does it?

Q1. I tried to plot the line $ax + b$ and instead I got an error message about "too many variables". What did I do wrong? You must define a, b to have numerical values, otherwise DERIVE treats your function as $f(a, b, x)$ which it cannot plot.

Q2. I tried to plot the family of parabolas $x^2 + c$ in DERIVE FOR DOS and instead I got a picture of some surface. What did I do wrong? Same problem as above, except now DERIVE is plotting a surface $z = f(x, c)$.

In DfW you would get an error message. You probably want to enter and Simplify a vector of functions such as

`VECTOR(x^2 + c, c, 0, 4).`

Now Plot this vector of 5 functions: x^2, x^2+1, x^2+2, x^2+3, and x^2+4.

Q3. I entered the expression $\sqrt{5-x}$ correctly, but when I substituted $x = 9$ and simplified I got $2\hat{\imath}$. What happened? You took the square root of a negative number which is not allowed when you are working with the real number system. DERIVE treats this as a computation with complex numbers and uses the complex number i (where $i^2 = -1$).

Q4. I solved for the 3 roots of the cubic x^3-2x^2+x-2 and I got $x = 2$ which I guessed from the graph but the other two solutions were $x = \hat{\imath}$ and $-\hat{\imath}$. Where do these last two come from? If you factor the cubic instead of using Solve you would get $(x-2)(x^2+1)$. The complex solutions come from that quadratic term. In calculus, we just ignore those complex solutions. For example, numerically solving the above cubic will give only real solutions.

Q5. I differentiated e^x and I got $e^x \ln e$, what's wrong? Nothing, DERIVE is treating the letter e as an ordinary symbol like a or b. You probably wanted Euler's constant e which can be entered with #`e`.

Q6. I tried to author the inverse tangent function `arctan x` and I got $a \cdot r \cdot c \cdot \tan x$ instead. What's wrong? DERIVE recognized the `tan x` part but treated the other symbols as individual constants. Use `atan x`.

Q7. I entered the vector $[v_1, v_2, v_3]$ by typing `[v1,v2,v3]` and I got $[v \cdot 1, v \cdot 2, v \cdot 3]$ instead. What happened? You must declare these multi-letter variables first before they can be treated as a single variable. To do this just author `v1:=`, `v2:=` and `v3:=`. A quick way to do this is to simply author the vector `[v1:=, v2:=, v3:=]`.

Q8. I tried to author `x^n` and I got a syntax error! How was this possible? The problem here is that either x or n is previously defined as a function. For example, maybe you had authored $x(t) := \sin t$. You can check on this by scrolling up to find the definition. If instead, you know the problem is that $x(t)$ is defined and you want to remove that definition, then just author `x:=`. In extreme cases you might just open a new window and copy over some of your expressions using the Copy and Paste technique. In DERIVE FOR DOS if you are sure that neither definition is needed you can select **T**ransfer/**C**lear and then choose **F**unctions. This will clear *all* function definitions.

Q9. I entered and simplified sin(2π) and I got `SIN([2π])` instead, what happened? You authored `sin[2pi]` instead of `sin(2pi)`. DERIVE treats square brackets not as parenthesis but as a device for defining *vectors*, see Section 0.14.

Q10. I tried to show that $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$, instead DERIVE returns a question mark indicating that it can't do this problem. What's wrong? Same as above, check your parenthesis. This last example is a little tricky because DERIVE FOR DOS uses square brackets to display some expressions, when in fact, those expressions must be entered with parenthesis. This situation has been corrected in DfW.

Chapter 1

Basic Algebra and Graphics

1.1 Introduction

Calculus is a beautiful and important subject. It derives its importance from its ability to describe and model basic phenomena in so many fields. Besides physics, chemistry and engineering, it is used in biology, economics, and probability. In order for calculus to be useful to you, you will need to understand calculus graphically and numerically as well as algebraically. Algebraically you learn how to differentiate functions given as complicated expressions. But you also need to understand the derivative visually as a rate of change.

With DERIVE it is easy to learn all three of these aspects and to see the relations between them.

1.2 Finding Extreme Points

As an example, consider the function $f(x) = 2x^4 - 3x^3$. In order to understand the behavior of this function we can plot it using DERIVE's plot window (see Section 0.10 for instructions on plotting.). The resulting graph can be seen in Figure 1.1 on the next page. The graph suggests $f(x)$ has one local minimum which is the absolute minimum. Using the crosshair in the plot window (see Section 0.10) we can determine that the location of the minimum point has coordinates approximately given by $x = 1.125$ and $y = -1.0625$. We can get exact results by switching to the algebra window and doing some calculus. In DERIVE's algebra window we choose Calculus/Differentiate or click the button to find the derivative. You get the answer by clicking the simplify button . We can then choose Solve/Algebraically to find where the derivative is 0, i.e., these are the *critical points*. Alternately, just click the solve button . The critical points occurs when $x = 0$ and when $x = 9/8$. Pressing the approximate button to get decimal answers we see that approximately $9/8 = 1.125$, which is exact equality in this case. Now if we substitute this value for x into $f(x)$ by highlighting the expression $2x^4 - 3x^3$ and then using the substitution button to replace x with $9/8$ we get, after approximating, that $y = -1.06787$ which is close to our first estimate.

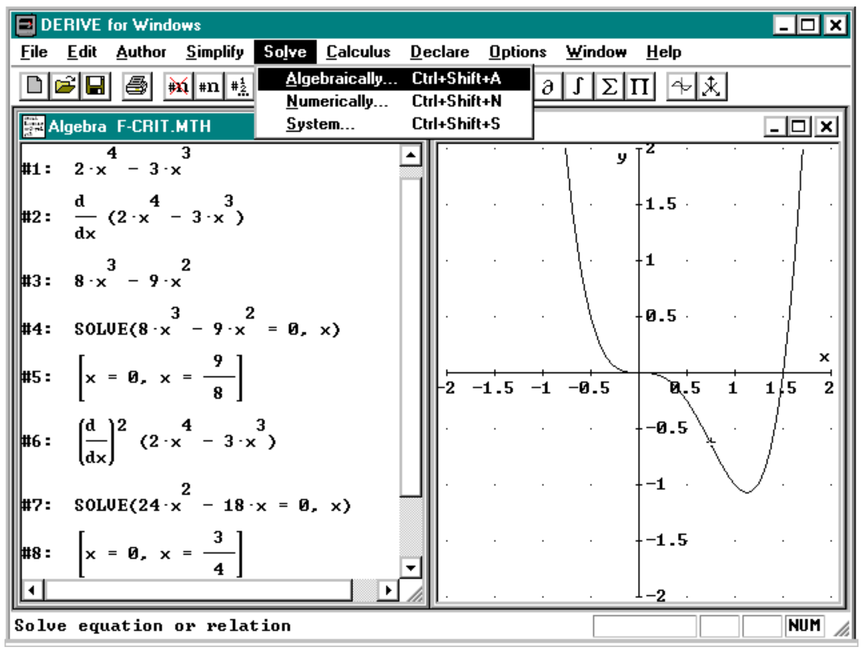
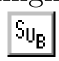


Figure 1.1: Finding critical points

Looking further at the graph we can see that $f(x)$ does not have a local minimum or maximum at $x = 0$; in fact $f'(x) < 0$ on both sides of 0. The graph also shows that $x = 9/8$ is where the minimum occurs and that $f(x)$ is decreasing on $(-\infty, 9/8]$ and increasing on $[9/8, \infty)$. If we highlight the first derivative and click (or choose Calculus/Differentiate), then we get the second derivative. We can then solve to find the points of inflection, i.e., the places where the bending in the graph changes. The second derivative must be zero at these points but this criteria alone is not sufficient. Solving $f''(x) = 0$ yields $x = 0$ and $x = 3/4$. Again looking at the graph we see that both of these are indeed points of inflection since the graph is concave up on $(-\infty, 0]$, concave down on $[0, 3/4]$, and concave up on $[3/4, \infty)$.

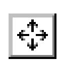
Insertion Tip In a typical problem the critical point will be complicated and retyping the expression in the substitution form is difficult and slow. A quicker method is to highlight the $f(x)$ expression, $2x^4 - 3x^3$ in our example, and then click the 

button. Now click on parts of the answer vector several times until the desired quantity, say 9/8 is highlighted. In the Substitution box click the *right* mouse button and select Insert. The highlighted quantity will be inserted in the form. You will find this a particularly useful technique when doing critical point problems algebraically instead of numerically. For example, $f(x) = ax^2 + bx + c$. In this case the critical point is a large expression involving the parameters a, b, and c.

1.3 Zooming and Asymptotes

As another example of using both plotting and calculus operations, consider the problem *does the function* $g(x) = \frac{3x^3+5x^2-x+1}{x^3-1}$ *have a horizontal asymptote?* In other words, we are interested in the behavior of the graph $y = g(x)$ for very large values of x and we want to know whether the y-values tend to a limit. To solve this you begin by entering the function by choosing Author and typing

```
(3x^3 + 5x^2 - x + 1)/(x^3 - 1).
```

Now plot this. Zoom out by clicking  and see if it appears that $g(x)$ has a horizontal asymptote. A nice technique to do this is to leave the vertical scale alone and zoom out in the horizontal direction. There are several ways to do this; one way is use the zoom buttons on the menu bar. Can you guess which button does this? Another way is change the scale, say $x = 100$ (click the button with the balance scale on it). Use the cross-hair to estimate the value of y that $g(x)$ is tending to for large x. You should get $y = 3$ (see Figure 1.2).

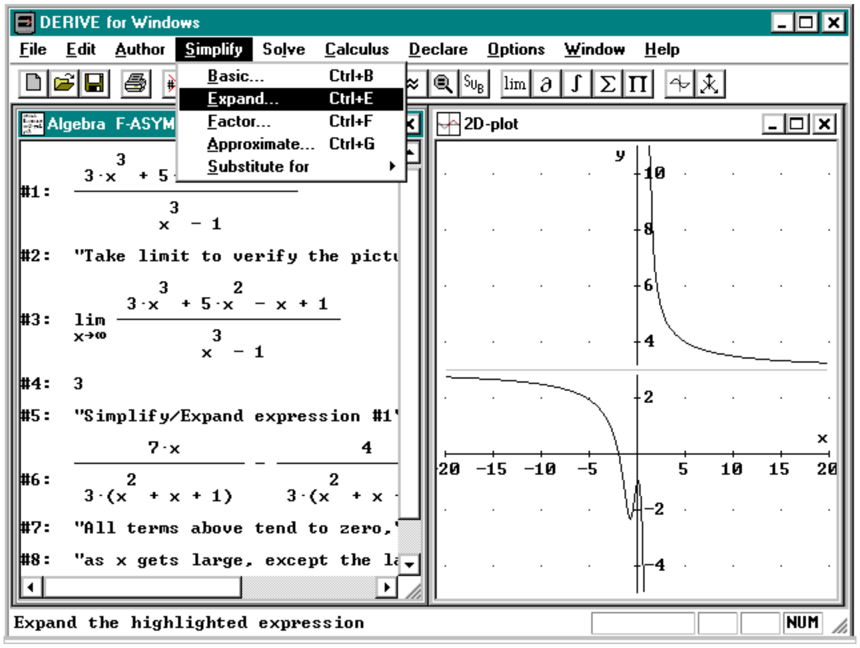


Figure 1.2: Zooming to find the horizontal asymptote

Now return to the algebra window by clicking . We want to verify that $\lim_{x\to\infty} g(x) = 3$. Now choose Calculus/Limit or click lim . When asked for the "point," type in `inf` or click ∞ symbol on the form. Simplify the answer; you should get 3. This means that as x gets large, $g(x)$ approaches 3, i.e., the line $y = 3$ is a horizontal asymptote. We can check this calculation by the method of polynomial division which is accomplished in DERIVE using the Expand option. As we see at the bottom of Figure 1.2,

$$\frac{3x^3 + 5x^2 - x + 1}{x^3 - 1} = 3 + \frac{8}{3(x-1)} + \cdots$$

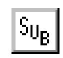
where all the terms other than the 3 are small near $x = \infty$. This is because the denominator of each term has a larger power of x than the numerator. (The answer above is too wide for the window to display so you have to see the 3. Use the horizontal scroll bar at the bottom of the window.)

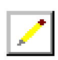
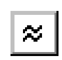
1.4 Laboratory Exercises

1. Enter the expression $x^3 - x + 1$ and plot it. Using the crosshair as described in Section 0.10, find an approximation for the both the x and y–coordinates of the local minimum of $x^3 - x + 1$.

2. Using the same method find an approximation for the unique x satisfying $x^3 - x + 1 = 0$, i.e., the place where the graph crosses the x–axis.

3. In the algebra window click  or use Solve to find the root exactly. Approximate this and see how good your answer for the last problem is.

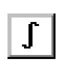
4. In the algebra window find the derivative of $x^3 - x + 1$ and solve to find the exact coordinates of the local minimum you approximated in Exercise 1. (This will give you the x–coordinate; to get the y–coordinate substitute the value of the x–coordinate into the cubic by clicking  or by using Simplify/Substitute/Variable; see Section 0.8.)

5. Use the author button  and the approximate button  (or Simplify/Approximate) to get decimal approximations for each of the following.

a. $8^{1/2}$

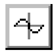
b. $\sin(\frac{\pi}{4})$

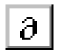
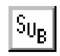
c. $\sin(\frac{\pi}{4})/5^{1/2}$

6. Integrate each expression using the  button (or Calculus/Integrate). (See Section 0.9 for instructions.)

a. $\displaystyle\int \frac{x^2}{(x^3-1)^2}\,dx$ **b.** $\displaystyle\int_0^{\pi/2} (1+\cos x)^2\,dx$

c. $\displaystyle\int_0^{\pi/4} x\sin(x^2)\,dx$ **d.** $\displaystyle\int x\sqrt{1+x}\,dx$

7. Graph the function $f(x) = (\frac{x}{1+x^2})^7$. At first there appears to be no part of the graph showing in the graphics window but this can not be since $f(0) = 0$. Try replotting the graph in another color by either just clicking  several times or using the Options/Plot Color menu. Now the graph appears to the horizontal line $y = 0$ but this can not be since clearly $f(x) = 0$ only for $x = 0$.

- **a.** In the Algebra window, find the critical points of $f(x)$ by using the  and  buttons.
- **b.** Determine the x and y coordinates of the local maximums and local minimums by using the  button to substitute the values in part (a) into the function.
- **c.** In the Graphics window, use the Zoom buttons or else the Set/Range menu in such a way that both the local maximum and local minimum points are visible. Furthermore, make the y-scale comparable to the y-coordinate of the local maximum.
- **d.** After you get a good looking graph, print out the result.

Chapter 2

The Derivative

2.1 The Derivative as a Limit of Secant Lines

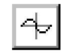
Geometrically the derivative of a function $f(x)$ at a point a is the slope of the tangent line of $f(x)$ going through the point $(a, f(a))$. We can approximate the tangent line by the 'secant' line which goes through the points $(a, f(a))$ and $(a+h, f(a+h))$. The slope of this line, the rise over the run, is $(f(a+h)-f(a))/h$, and so, by the usual point-slope formula for a line, the equation of this secant line is

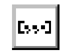
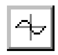
$$y - f(a) = \left[\frac{f(a+h) - f(a)}{h}\right](x - a).$$

As $a+h$ gets closer to a, i.e., as h gets smaller, this secant line approximates the tangent line at a better and better, and so its slope approaches the derivative $f'(a)$. We can visualize this with DERIVE by entering the following expressions:

```
F(x) := x^3/3
SL(a, h) := f(a) + (f(a+h) - f(a))/h (x - a)
```

The first step declares $f(x)$ to be the function $x^3/3$. The second defines a function $\mathrm{SL}(a, h)$ which gives the secant line through the points $(a, f(a))$ and $(a+h, f(a+h))$. For example, if we Simplify `SL(1,1)` we get $\frac{7x-6}{3}$ so that the equation of the secant line determined by $x = 1$ and $x = 2$ is $y = \frac{7x-6}{3}$.

Now we want to fix $a = 1$ and plot several secant lines corresponding to different h's. We can, for example, just Author and Simplify $\mathrm{SL}(1, 1)$, SL$(1, 1/2)$, SL$(1, 1/4)$, SL$(1, 1/8)$, and SL$(1, 1/16)$, and plot these lines and $f(x)$ on the same graph. This simply means you highlight each of the simplified expression and then click the plot button  in the plot window, see Section 0.10 on page 13 in Chapter 0 for the details on how to do this. This is illustrated in Figure 2.1 on the next page.

A nice way to calculate and plot these secant lines is to use vector techniques. Here's how you would do it: Click the vector button  and set the number of elements to 5. Now enter the 5 expressions above starting with `SL(1,1)` and using the `Tab` to move from entry to entry. Finally, simplify and click  on the resulting vector `[SL(1,1),...,SL(1/16)]`. All five lines will be plotted one at a time. It all happens very quickly but if you stare at the screen carefully just as you click the plot button you possibly can see an animation-like effect.

If the drawing is too quick to see the animation, try the following method instead. Erase the 5 secant lines in the plot window by pressing the `Del` key 5 times. In the algebra window select an individual line in the vector by repeatedly clicking on it. Then, activate the plot window and press . Finally, repeat this process several times to see the pattern evolving in the plot window.

Since there is a pattern to the values $1, \frac{1}{2}, \frac{1}{4}, \ldots$; namely $\frac{1}{2^n}$, we can use another approach involving the `VECTOR` function on the Calculus/Vector menu. Select this menu option and enter `SL(1,1/2^n)` in the form. Note that using uppercase letters is not necessary and that the highlighted expression will be replaced with whatever you type. For the Variable, scroll down and select `n`. Next we take the Starting value to be 0 since $2^0 = 1$ and the Ending value to be 4 since $2^4 = 16$. Click OK and simplify the resulting expression `VECTOR(SL(1, 2^-n), n, 0, 4)`[1]. The result is a vector of five secant lines as above. You will find that this is a convenient method of producing a large number of expressions without typing them individually.

We can later change the definition of $f(x)$ to a different function and use the SL(a, h) function to get secant lines to the new function. The file F-SECANT.MTH contains the definitions of SL(a, h) and the tangent line function, TL(a), discussed below.

In Figure 2.1 on the next page the secant lines tend to the tangent line by rotating in a clockwise manner, i.e., with decreasing slope. We can use

[1]See Section 0.14 on page 22 for more information about the `vector` function

DERIVE to illustrate this effect using calculus. That is, as h tends to 0, the secant line tends to the tangent line by taking the limit: Author `SL(1,h)` and choose Calculus/Limit, taking the variable to be h (not x). We get $\lim_{h \to 0} \mathrm{SL}(1, h)$, which Simplifies to

$$\frac{3x - 2}{3} = x - \frac{2}{3}.$$

which, in fact, yields the tangent line to $x^3/3$ at $x = 1$. Check this out for yourself by plotting this function on your previous graph. Since the slope of the secant line is $(f(a + h) - f(a))/h$, this explains why we define the derivative as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \tag{1}$$

and why the derivative is the slope of the tangent line.

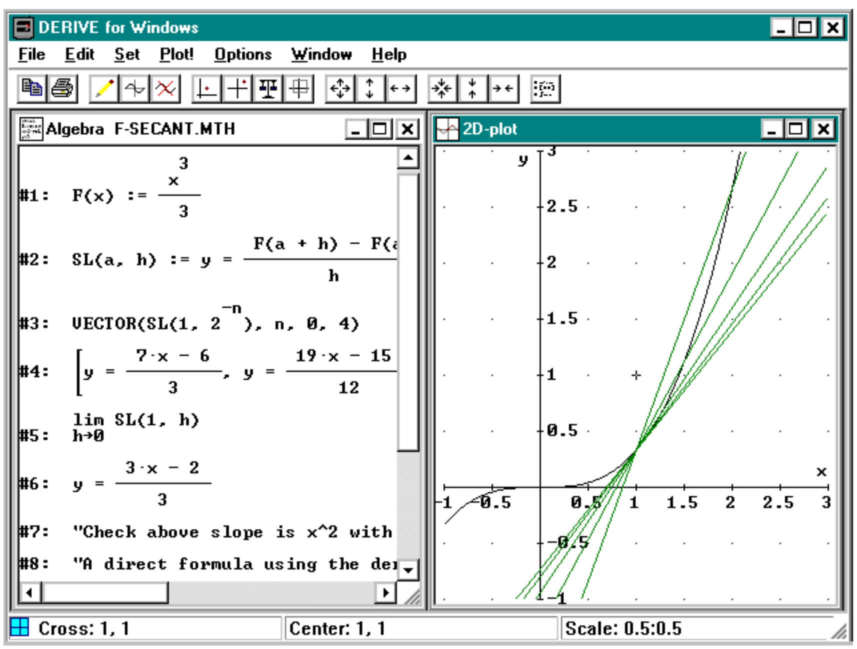


Figure 2.1: Secant lines approximating the tangent line

In order to get a function `TL(a)` for the tangent line at a analogous to the secant line function `SL(a,h)`, we need to be a little careful since the most obvious definition; namely,

```
TL(a) := F(a) + DIF(F(a),a) (x - a)
```

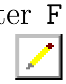
doesn't work. This is because the order of evaluation is wrong. Consider what would happen if we evaluated `TL(5)`. First 5 would be substituted for `a` and then the resulting expression,

```
F(5) + DIF(F(5),5) (x - 5),
```

would be evaluated. But `DIF(F(5),5)` doesn't make sense.

To solve this problem we use the `SUBST` function in the utility file ADD-UTIL.MTH. This file contains several new functions that we will be using for now on and so we suggest adding a few lines to start of every DfW session. This can be done by Opening or using Load/Math on the file ADD-HEAD.MTH. Once this has been done, these new functions can be used. See Appendix A for a more detailed explanation.

For example, the function `SUBST(u, x, a)` simplifies the expression u and then substitutes the value a for x in u. The three variables in the `SUBST` function are the *expression*, the *variable* and the evaluation *point*. It has the same effect as first Simplifying u and then using the  button to replace x with a.

To define the `TL` function we first make a function `DF(x)` of the derivative using the `SUBST` function. Click  and enter `F(u)` with the variable set to `u`. With this expression highlighted click  and type in `SUBST(`. Then, insert the derivative by right-clicking and selecting Insert from the menu. Finally, type in next two arguments u and x separated by commas to complete the three arguments for this function. Pressing OK, you should get the first expression below:

```
(2)            DF(x):= SUBST(DIF(F(u), u), u, x)
(3)            TL(a) := F(a) + DF(a)(x - a)
```

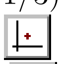
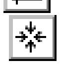
The `TL` function can then be defined as above.

The utility file contains two more functions which you can use for the exercises and that eliminate the need to reproduce the definitions we've been discussing. To find the tangent line of say $x^3/3$ at $x = 5$ you enter and simplify the expression `TANGENT(x^3/3, x, 5)`. Here again, the three variables in the `TANGENT` function are the *expression*, the *variable* and the evaluation *point*. Similarly, the secant line computed earlier can be obtained by entering `SECANT(x^3/3, x, 1, 1/16)`. Here the last variable is the h-increment.

The answer to these function is of the form $y = mx + b$ rather than just $mx + b$. You can still plot the entire equation and get the right result since DfW knows how to plot equations in addition to functions. You can test this by plotting the familiar equation $y^2 = -x^2 + 1$ to get the *unit circle*.

Notice that using these functions it is not necessary to define any functions such as `F(x):=` or `DF(x):=` in order to get an answer. This is usually a better approach because the use of variables or definitions causes problem when you forget that something is defined. As a result you get some strange answers to your problems and you don't know why. This difficulty is particularly common as you go from problem to problem in the exercises. Just remember to start off your labs by doing Load/Math to the ADD-HEAD file.

2.2 Local Linearity and Approximation

One of the properties of a function with a derivative at x is that the function can be well approximated by the tangent line. This means as you move in close the function appears to be quite flat, not differing much from the tangent line. This 'local linearity' is very useful in many applications. To see this local flatness, move the crosshair in the plot we obtained above to the point $(1, 1/3)$ where all the lines intersect, then center on the cross by clicking the  button. Now we want to zoom in several times by clicking the zoomin  button. Notice how flat the curve appears. Try clicking The zoom out button  several times and then repeating to completely visualize this process.

We can use the above approach to approximate the derivative of a function and plot the result. For example, we know that the derivative of $f(x) = x^3$ is $3x^2$ by using the standard formulas. On the other hand, the function of x, $g(x,h) = \frac{f(x+h)-f(x)}{h}$ with h fixed at some value like $h = .01$ is a good approximation to $3x^2$ as one can see from Figure 2.2 on the following page. The figure actually shows both plots although they appear to be only one curve. In DERIVE you should enter and Simplify the above expression (it sometimes helps to Expand the result to further simplify it). Then compare the graph with $3x^2$ by plotting both expressions together.

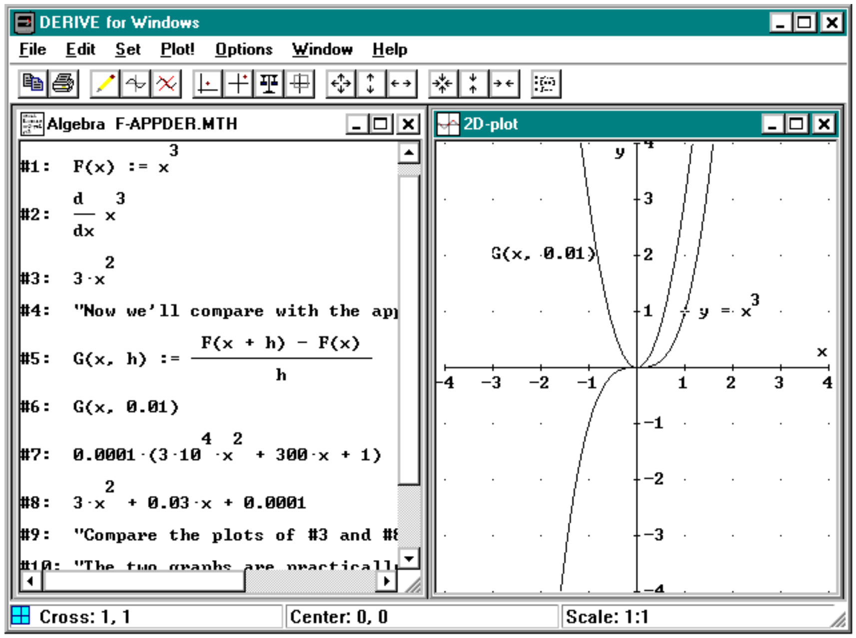


Figure 2.2: Approximating derivatives using the difference quotient

2.3 Laboratory Exercises

Start off your lab by Loading the ADD-HEAD.MTH file (use File/Load/Math). Note that the syntax of the `SECANT` and `TANGENT` functions are displayed on the second line of the ADD-HEAD file.

1. **a.** Using the `TANGENT(u,x,a)` function find the equation of the tangent line for $f(x) = \sqrt[3]{x}$ (enter cube roots as `x^(1/3)`) at the point $a = 8$ and plot it along with the graph of $f(x)$.

   b. In part **a** you found the tangent line to $\sqrt[3]{x}$ at $a = 8$. Estimate $\sqrt[3]{9}$ by finding the y–value of this line when $x = 9$. Compare your answers with DfW's own approximation to this quantity obtained by clicking the  button.

   c. Using the plot window again give a reasonably accurate interval [c,d] containing the point $x = 8$ for which the tangent line approximates the function to 2 decimal place accuracy. (**Hint:** Plot the difference between the function and the tangent line and rescale to get a good picture. It is also helpful to use the Trace Mode which can be set from the Options menu of the graphics window.)

2. Consider the class of all function of the form $f(x) = x^3 + bx^2 + cx$.

a. Author the expression form $x^3 + bx^2 + cx$. Click the substitution button SUB and enter some specific values for b, c, then plot the result[2]. Do this for several different choices for b and c and observe the critical points and inflection points of the different graphs.

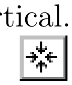
b. Using Derive's calculus facilities in the algebra window, show that the function $f(x) = x^3 + bx^2 + cx$ always has exactly one inflection point, regardless of the values of b, c.

c. Again using Derive's calculus facilities, show that $f(x)$ can have either 0, 1 or 2 critical points. Determine for what values of b and c does $f(x)$ have no critical point?

d. Choose values b, c which demonstrate that $f(x)$ may have either 0, 1 or 2 critical points and plot their graphs.

***3.** Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. In this problem you will show that $f(x)$ is continuous and differentiable for all x but $f'(x)$ is not continuous at 0. This means to find $f'(0)$ you must use the definition of the derivative; you cannot just find $f'(x)$ and take the limit as $x \to 0$.

a. Define $f(x)$ as above by Authoring `F(x) := x^2 sin(1/x)` (don't worry about $x = 0$ for now). Show $\lim_{x\to 0} f(x) = 0$. (Hint: Click lim and fill in the form.)

b. Graph $f(x)$, x^2, and $-x^2$, setting the plot scale to 0.1 horizontal and 0.01 vertical. Zoom in several times towards the origin by clicking the  button and convince yourself that $f(x)$ is continuous at $x = 0$. But notice that the curve oscillates up and down slightly.

c. Find $f'(0)$ by finding $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$.

d. Find the derivative of $f(x)$ using the ∂ button.

[2] Derive can't plot the function unless the values are provided.

e. Make a new graph of $f'(x)$ and by zooming in several times convince yourself that $f'(x)$ oscillates wildly between approximately ± 1 as x approachs zero.

4. The volume of a tin can is $V = \pi r^2 h$ where r is the radius of the top (and the bottom) and h is the height. The surface area is $A = 2\pi rh + 2\pi r^2$. (The first term is the area of the side and the second is the area of the top and bottom.)

a. A manufacturing company wants to make cans with volume 42 in^3. To minimize their costs they want to minimize the area of the can. What values of r and h do this? (Hint: Author the formula for the area, $2\pi rh + 2\pi r^2$; use the equation for the volume, $42 = \pi r^2 h$, to solve for h in terms of r and substitute this into you expression for the area. Now find the value of r that minimizes the area using calculus techniques and use this value of r to find what h is.)

b. You may have noticed that $h = 2r$ for the can of minimum area you found in part **a**. Show that this relation always holds for the can of least surface area (not just for cans with volume 42). (Hint: Do this just as in part **a** except don't replace V by 42 in the equation for the volume.)

***5.** Suppose we have the situation of the previous problem except that now the metal for the top and the bottom of the can costs 1.5 times as much as the metal for the side. What is h/r for the can of minimun cost?

6. The acceleration due to gravity, a, varies with the height above the surface of the earth. If you go down below the surface of the earth, a varies in a different way. It can be shown that, as a function of r, the distance from the center of the earth, a is given by

$$a(r) = \begin{cases} \frac{GMr}{R^3} & \text{for } r < R \\ \frac{GM}{r^2} & \text{for } r \geq R \end{cases}$$

where R is the radius of the earth, M is the mass of the earth, and G is the gravitational constant. All three of these are constants. In order to define the function $a(r)$ and examine its graph, we'll use the numerical values: $GM = 4.002 \times 10^{14}$ and $R = 6.4 \times 10^6$ meters.

a. Define $a(r)$ using the technique in Section 0.13 and plot its graph. Rescale as necessary to give a good picture.

b. Is a a continuous function of r?

c. Is a a differentiable function of r? Explain your answer.

***7.** $\Gamma(x)$ is a differentiable function for $x > 0$ which is very important in applications. DERIVE knows this function but not how to differentiate it. You can get Γ in either version of DERIVE by typing `gamma` but in DfW you can also just click on the Γ in the Author Dialog Box.

a. Graph $\Gamma(x)$ and the four secant lines to $\Gamma(x)$ through the points $(3, \Gamma(3))$ and $(3 + h, \Gamma(3 + h))$, for $h = 1/2$, $1/4$, $1/8$, and $1/16$. [It is known that $\Gamma(3) = 2$, but you don't really need this here.]

b. Use the secant line you obtained in part **a** with $h = 1/16$ to approximate $\Gamma(3.1)$.

c. Have DERIVE approximate $\Gamma(3.1)$.

d. Use the graph to verify that $\Gamma(n + 1) = n! = 1 \cdot 2 \cdots n$ whenever $n = 0, 1, \ldots, 5$. (Since factorials play an important role in many applications this explain why the Γ function is important.)

Chapter 3

Curve Sketching

3.1 Introduction

Before the widespread use of computers and graphing calculators, graphing a function $f(x)$ was done by a combination of techniques including:

- Plotting some judiciously chosen points.
- Finding solutions to $f(x) = 0$.
- Finding the local minima and maxima and where $f(x)$ is increasing and decreasing.
- Finding the inflection points and concavity.
- Finding the horizontal and vertical asymptotes.

As we have seen graphing is easier with a computer algebra system. Moreover, we can also find the local minima and maxima and the other items above if we need them. It is also possible to make a small change in $f(x)$ and graph that and see how the graph is affected. But we have also seen that in order to see the important aspects of $f(x)$ it may be necessary to zoom in or out and to move around in the graph. In this lab you will develop your skills at graphing with the computer.

3.2 Working with Graphs

In many problems involving periodic behavior, such as oscillating springs, pendulums, planetary motion and others, the solutions generally have the form

$$a \sin(b(x - x_0))$$

where a, b and x_0 are given numbers. This raises the question of how the graph of a function, such as $\sin x$, changes when subject to the above modifications. You should observe the changes by comparing with the original function but you should also think about why the changes make sense, for example, what does changing a do, what is the geometrical significance of the point $x = x_0$ on the x-axis.

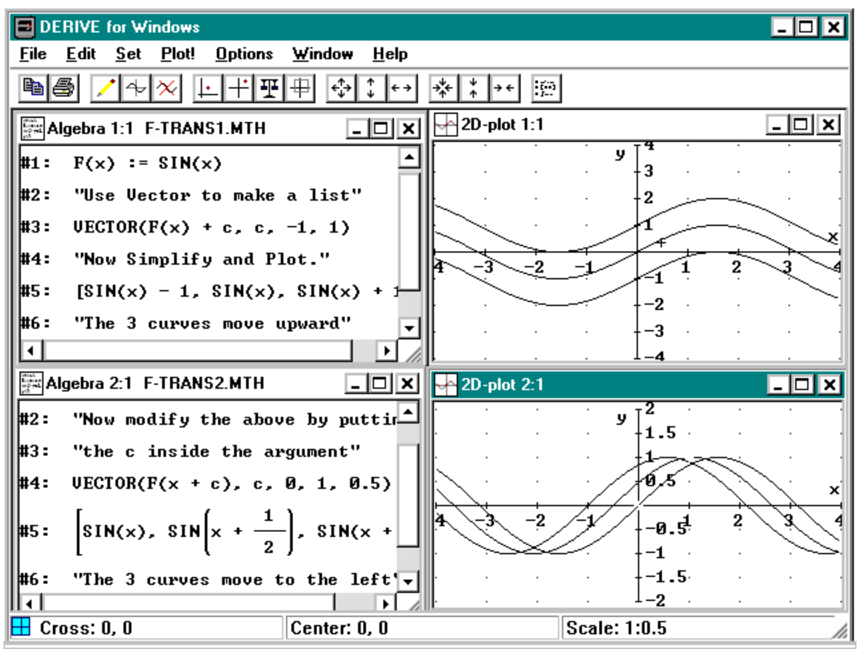
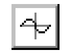


Figure 3.1: Using `vector` to plot several graphs

Now to see how the transformations $y = f(x)+c$ affect the graph $y = f(x)$ for various choices of c we start by Authoring our function, say `f(x) := sin x`. It's always a good idea to define our function using `f(x):=` because then later we can change the definition and see the effects on a different function.

Next we select the Calculus/Vector menu and enter `f(x) + c` in the form. Click the Variable list box and select the variable c. For this example, set the Starting value to -2 and the Ending value to 2. The Step Size can be left at 1 although in other examples you might want to change this. Press OK. Finally, simplify the resulting expression `VECTOR( F(x) + c, c, -2, 2)` by clicking the $\boxed{=}$ button and then plotting by clicking the plot  button twice (once to open the plot window and the second time to plot the highlighted expression).

If you look carefully at the plot window as you click  you might see the successive graphs are just that of the usual sine function but moved in the vertical direction. They start below the x-axis and then rise to a few units above the x-axis. If the drawing is too quick to see the animation, try clicking the individual functions in the vector expression in the algebra window and plotting them. By deleting the graphs and redrawing you should be able to see the pattern.

Many other options are also possible; for example, editing the `vector` formula above by replacing `f(x) + c` with `f(x + c)`gives an interesting result upon graphing. See if you can see a *traveling wave* in the plot window. Is it traveling from left to right or right to left?

Also, you can change the function by simply Authoring `f(x) :=` with a new expression. Remember that the`:=` symbol is for *assignment* whereas the `=` sign is used for equations and comparisons. It is important to note that once you define a function by this method it will *not* go away if you simply erase that line from your algebra window because it is in the computers memory. The way to completely remove a definition using the letter f is to author the expression `F:=`. This gives F an empty definition.

3.3 Exponential vs Polynomial Growth

Suppose we want to compare the behavior of the functions x^4 and e^x. [1] If we graph x^4 we see it has the same basic shape as the parabola x^2 (you probably guessed this). It is a little flatter than the parabola between -1 and 1 and seems to grow more quickly for $|x| > 1$.

If we now graph e^x on the same graph and zoom out once, we see that

[1] This problem is essentially taken from *Calculus* by Deborah Hughes-Hallett, Andrew M. Gleason, et al. It is one of the most popular ‘calculus reform’ texts.

the graph seems to get close to the x–axis as x gets larger in the negative direction (as x approaches $-\infty$); that it crosses x^4 at least twice; and it grows quickly when x is positive, but not as quickly as x^4; see Figure 3.2. One way to verify that the x–axis is a horizontal asymptote of e^x is to highlight e^x in the algebra window and choose Calculus/Limit and enter `-inf` for the 'Point,' as we did in Figure 3.2.

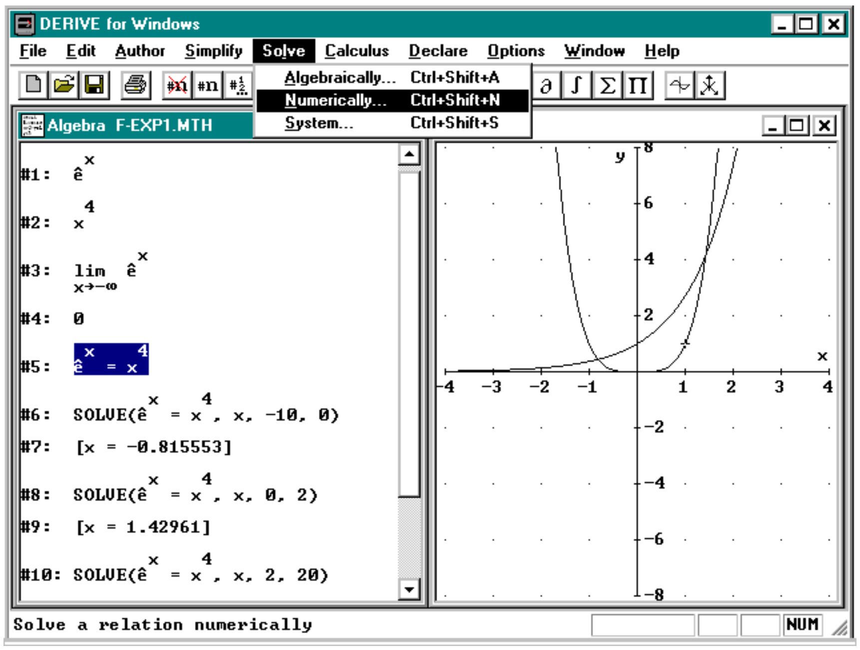


Figure 3.2: The functions x^4 and e^x

To see where the curves cross we need to solve the equation $x^4 = e^x$. We cannot solve it exactly but we can get approximate solutions. To do this with DERIVE we use the Solve/Numerically menu. Enter the equation `x^4 = e^x` and choose an appropriate interval. When you solve numerically, things work slightly differently. You need to specify an interval in which to search for a solution. If there are no solutions in the interval, DERIVE returns `[]` which means no solutions. If it finds a solution it gives that as the answer. But ***it only gives one solution, even if the interval contains several!*** To find other solutions you need to specify new intervals that do not contain the solution already found. So if you originally choose the interval from 0 to 2 and DERIVE found a solution 1.3 and you suspect there is also a smaller solution, you could solve again but this time use the interval 0 to 1.29.

The graph in Figure 3.2 on the facing page suggests that there is a solution between -2 and 0. If you use this interval DERIVE gives $x = -0.815553$, which seems to agree with the graph. (Another way to find an approximate solution is to move the crosshair to where the curves intersect.) The graph also shows there is a solution between 0 and 2. If we solve using that interval we get $x = 1.42961$.

Are there any other solutions? It is pretty clear that there are no other solutions for $x < 0$ but what about large x? From the graph it appears that x^4 grows much faster than e^x. But of course e^x has "exponential growth" so perhaps $e^x > x^4$ for large enough x. To test this we can try to solve $x^4 = e^x$ for larger x. If we choose the interval to be 2 to 20, we get the solution $x = 8.61316$. So the graphs cross at this point. To find the y value of this point, we use the Substitute button to substitute 8.61316 into x^4. The result approximates to 5503.64.

To see this on the graph we need to zoom out once so that the x–scale includes $x = 8.61316$. Then we need to zoom out on the y–axis without zooming out on the x–axis. We do this by choosing `Y` on the zoom menu. After zooming out several times we obtain the graph of Figure 3.3 on the next page.

There are a couple things this demonstration shows. First that in order to see the important features of a graph it may take some skill at moving around and manipulating the scale of the graph. Moreover, even though we can clearly see the two graphs intersecting at $x = 8.61316$ in Figure 3.3, we can no longer see the other two solutions. So it may not be possible to see all the important features in one plot. In this lesson you will learn how to move and scale in the plot window and to use the algebra window in order to find all the important features of one or more graphs.

3.4 Laboratory Exercises

1. Let $f(x) = 1/(1 + 2x^2)$.

a. Graph each of the following $f(x) - 1$, $f(x)$, $f(x) + 1$, and $f(x) + 2$ in a plot window. Then, use the Window/New 2D-Plot Window command to open another plot window and plot $f(x - 1)$, $f(x)$, $f(x + 1)$ and $f(x + 2)$ in that window. (Hint: The Vector menu

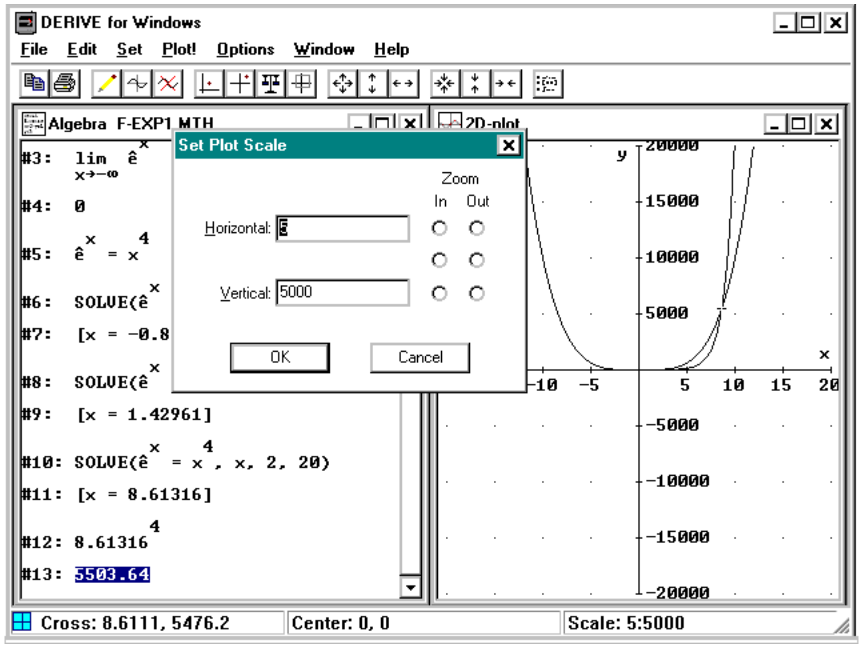


Figure 3.3: The functions x^4 and e^x, rescaled

can be used to simplify the typing.)

b. What does the transformation $f(x) \to f(x) + a$ do to the shape of the graph? to the position of the graph?

c. What does the transformation $f(x) \to f(x + a)$ do to the shape and position of the graph?

2. Graph $\cos x$, $2\cos x$, and $\cos(2x)$ and explain what the transformations $f(x) \to f(ax)$ and $f(x) \to af(x)$ do to the graph of $f(x)$.

3. What do the transformations $f(x) \to f(-x)$ and $f(x) \to -f(x)$ do? Graph $f(x) = x^5 - x^2 + 1$ and $f(-x)$ and $-f(x)$.

4. Let $g(t) = \sin t + \cos t$.

a. Graph $g(t)$.

b. What sort of transformations should be applied to $\sin t$ to make its graph look like the graph of $g(t)$?

c. Use DERIVE's crosshair to find the maximum value of $g(t)$ and to find the first root of $g(t)$ to the left of zero.

d. Use these numbers to find a and b so that $g(t) = a\sin(t+b)$, at least approximately.

***e.** Find exact values of a and b so that

$$\sin t + \cos t = a\sin(t+b).$$

Hint: Set the simplification of trigonometry functions to "Expand" using the Declare/Algebra State/Simplification menu. Then evaluate $a\sin(t+b)$.

5. Let $f(x) = e^{-ax^2}$ where a is a constant.

a. Plot $f(x)$ for $a = -2, -1, 0, 1,$ and 2. You can use the `vector` function if you like.

b. Using calculus facilities in the algebra window, find the points of inflection for e^{-ax^2}.

6. Find the points where the curves $\ln x$ and $x^{1/4}$ intersect. Make two (or more) graphs with different scales showing the places where the curves intersect.

7. Make separate graphs of each of the following functions. Using some of the graphing techniques such as zooming, centering, etc. Make sure your graphs show the main features such as the x and y–intercepts, the critical points, and the inflection points.

a. $\sin(x)\cos(20x)$ **b.** $\dfrac{3x}{\sqrt{4x^2+1}}$

c. $\dfrac{1}{1+5000(x-1)^2}$ **d.** $x\sin(1/x)$

***8.** Enter the rational function

$$\frac{x^6+3x^5+x^4+1}{2x^4-1} \tag{1}$$

a. Choose Simplify/Expand and select Rational. This gives a partial fractions decomposition of the function. (Partial fraction decompositions are used in integrating rational functions) Notice that the partial fraction decomposition consists of (a sum of) one or more proper rational functions (where the denominator has higher degree than the numerator) and a polynomial. What is the polynomial?

b. Graph the rational function given by (1) and the polynomial you found in the first part. Zoom out a few times. How are the two graphs related when $|x|$ is large? Explain why this is.

***9.** Let

$$g(x) = \frac{-2x^3 + 6x^2 - 3x + 5}{4x^2 - 6x - 7}$$

a. Graph $g(x)$ so that your graph shows the main features of this function.

b. This graph has a slant asymptote, i.e., an asymptote which is a line with nonzero slope. Zoom out a few times until you can see this slant asymptote.

c. Find the formula for the slant asymptote by using Simplify/Expand.

***10.** In reading this chapter you might have wondered if e^x and x^4 intersect some place beyond $c = 8.61316\cdots$. You could use DERIVE to verify that there is no solution say between 8.7 and 100 and this would be strong evidence that they don't intersect beyond c, but not a proof. So in this problem you are to find a proof that e^x and x^4 don't intersect beyond c (without using DERIVE). **Hint:** By taking 4^{th} roots we must show $e^{x/4} > x$ for all large x. Now show the slope of $e^{x/4} - x$ is positive for all $x \geq 8$ and use this to show $e^x > x^4$ for all $x > c$.

Chapter 4

Graphing Data and Curve Fitting

4.1 Introduction

Consider the population of a certain country, $P(t)$, as a function of time. We may not know exactly what $P(t)$ is but instead just have a table of data, for example, the population at the beginning of each year for the last few years. We are interested in finding an appropriate curve for the data. We might try comparing the data against a linear function $ax + b$, a quadratic function $ax^2 + bx + c$ or say an exponential curve of the form $P(t) = ae^{rt}$. Under a certain model of population growth $P(t)$ will have this last form. Our problem is to determine the parameters $a, b, \ldots$, from the data. Once we do this then we can use $P(t)$ to estimate the population at times between the data and predict the population in the future.

This kind of problem of fitting a function from a family of functions to numerical data arises frequently in many applied areas including statistics. In this lab we use the computer to help visualize data and fit the data to a function from a class of functions. We begin with the class of all polynomial functions.

4.2 Fitting Polynomials to Data Points

Given a finite set of data points:

$$(x_1, y_1), \ldots, (x_n, y_n)$$

let's consider the problem of finding a polynomial function $f(x)$ which goes through these points. That is, we want $f(x)$ to satisfy $f(x_i) = y_i$ for $i = 1, \ldots, n$. If $x_i \neq x_j$, for $i \neq j$, then it turns out that we can always find a unique polynomial of degree at most $n-1$ going through these points. This is quite obvious when $n = 2$ since there is a unique line passing through any 2 distinct points.

If we are given 3 points, (x_1, y_1), (x_2, y_2), and (x_3, y_3), and want to find a quadratic polynomial passing through these points, we let $f(x) = ax^2 + bx + c$ be an arbitrary quadratic. Since $f(x_i) = y_i$ for $i = 1$, 2, and 3, we obtain three (linear) equations

$$
\begin{aligned}
ax_1^2 + bx_1 + c &= y_1 \\
ax_2^2 + bx_2 + c &= y_2 \\
ax_3^2 + bx_3 + c &= y_3
\end{aligned}
\tag{1}
$$

in the unknowns a, b, and c. (Remember, we are given the points (x_i, y_i) so they are known and we want to find the unknowns a, b, and c.) We then solve this system of 3 equations for the 3 unknowns a, b, and c.

For example, suppose we want to find a quadratic polynomial $f(x) = ax^2 + bx + c$ passing through $(0, 0)$, $(1, 2)$, and $(2, 8)$. The way to do this with DfW is to first author `f(x) := ax^2 + bx + c` then choose Solve/System from the menu bar, set the number of equation to be 3, and then enter the three equations (you can either use the Tab-key after entering an equation or click the next equation box)

```
f(0) = 0    f(1) = 2    f(2) = 8
```

Click on the Equation Variables box and select the variables to solve for as a, b, and c. Click OK and then simplify the resulting expression `SOLVE( [F(0) = 0, F(1) = 2, F(2) = 8], [a, b, c])` (see Section 0.7 on page 8). DERIVE returns

```
[a = 2  b = 0  c = 0]
```

So in this case $f(x) = 2x^2$.

We can double check this result by plotting the function $2x^2$ determined above along with the 3×2 data matrix

$$
\begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 2 & 8 \end{bmatrix}
$$

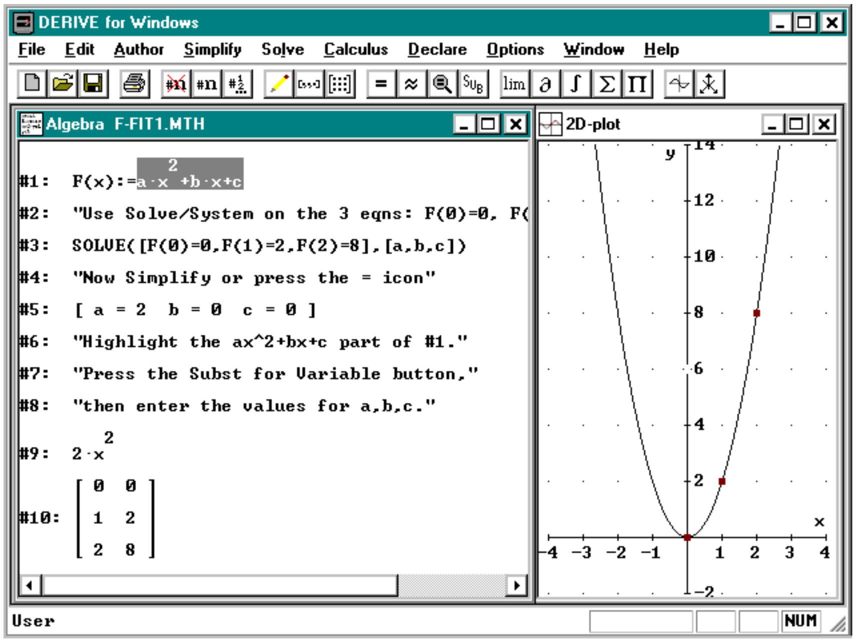
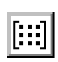
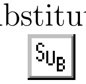
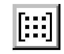


Figure 4.1: Fitting a polynomial to data points

entered by using the  button. See Figure 4.1.

For more complicated problems we would have to substitute in the values of a, b, c into the expression $ax^2 + bx + c$ using the  button. The utility file ADD-UTIL has a function `CURVEFIT(x, data)` which does this automatically. Here the x is the variable and data is the matrix of point we want to fit the curve to. The more points we use the higher the degree of the polynomial needs to be.

As we mentioned above, for 3 points with distinct x–coordinates there is a unique quadratic polynomial function passing through them. We can use DERIVE to demonstrate this by showing that the system of equations (1) can always be solved for a, b, and c, regardless of the values of (x_i, y_i). To do this we will just have DERIVE solve the system (1).

However, there is a slight problem. In its normal input mode, called character input, DERIVE treats each letter as a separate variable. So if you author `ab` DERIVE will read this as a times b and the algebra window will show it as $a \cdot b$. This is very convenient for calculus where we almost always use single characters for our variables. But to solve (1) we need the variables `x1`, `x2`, etc. When we enter `x1`, DERIVE will think of this as x times 1, which is not what we want. So we need to declare that `x1`, `x2`, etc., are variables. To declare `x1` as a variable you can author `x1 :=`. You need to do this for all three `x`'s and `y`'s. You can do this quickly by clicking the  button and selecting 3 rows and 2 columns. Then, enter `x1:=` press Tab and enter `y1:=`. Continuing, enter all the remaining x_i's and y_i's but be sure to use the assignment character `:=` and not the `=` sign alone. Click the Simplify button in the matrix form instead of the OK button and the result should be

$$\begin{bmatrix} x1 & y1 \\ x2 & y2 \\ x3 & y3 \end{bmatrix}$$

Now we Author `CURVEFIT(x,data)` where data is the above matrix of points. Simplifying yields a complicated looking answer which is a little difficult to digest. However, if you factor the answer; using Simplify/Factor where in the Factor dialog box we highlight each of the x_i variables to factor over, the result shows that the denominator cannot be 0 since we are assuming that x_1, x_2, and x_3 are distinct. See Figure 4.2 on the facing page.

As an interesting variant on the above, suppose we want to find a, b, and c for a function $f(x) = ax^2 + bx + c$ when we know

$$\begin{aligned} f(x_1) &= y_1 \\ f(x_2) &= y_2 \\ f'(x_3) &= y_3 \quad \text{(That's the derivative!)} \end{aligned} \tag{2}$$

In other words we specify that $f(x)$ must pass through (x_1, y_1) and (x_2, y_2) and that its slope at $x = x_3$ is y_3. We define `f(x)` as before and define `g(x)` as the derivative[1]

(3) `g(x) := 2ax+b`

Now if we solve the system of equations

`f(x1) = y1    f(x2) = y2    g(x3) = y3`

[1]Note that you can't simply define `g(x):=DIF(f(x),x)`. We ran into this problem earlier on page 38 (see Section 0.12 on page 20 for a complete explanation). One solution to this problem is to use the utility file and define `g(x):= SUBST( DIF(f(u),u), u, x)`.

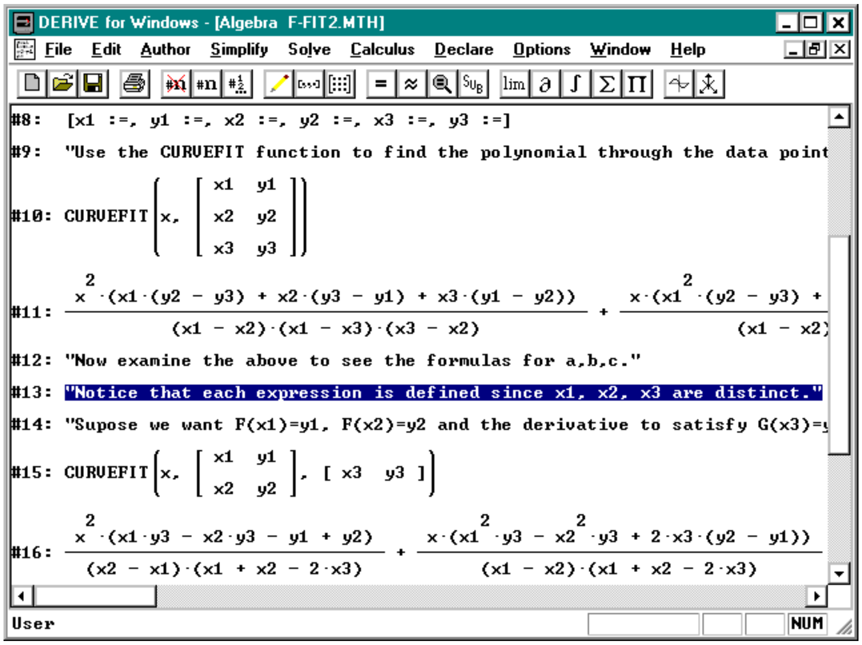


Figure 4.2: The algebra behind fitting polynomials to data points

as before and then factor the answer, we see that the denominator of each of the 3 fractions is

$$(x_1 - x_2)(x_1 + x_2 - 2x_3)$$

(You can do this just as in Figure 4.2 except you need to define `g(x)` and use `g(x3) = y3` in place of `f(x3) = y3`.) Of course we are assuming that $x_1 \neq x_2$ so the factor $x_1 - x_2$ will not be 0. The other factor is 0 when

$$x_3 = \frac{x_1 + x_2}{2}$$

This means we can always find $f(x)$ except possibly if $x_3 = (x_1 + x_2)/2$. This is somewhat surprising since one expects to be able to solve for 3 unknowns satisfying 3 equations just as one can solve for 2 unknowns satisfying 2 equations. However, in both cases there are exceptional cases that need to be considered. In this case, the difficulty is related to the mean value theorem and is explored in Exercise 3. Related results for cubic functions are examined in Exercises 4 and 7.

The solution to curve fitting problems involving the derivative can also be found using the `CURVEFIT(x,data,ddata)` function. As before x is the variable and `data` is a matrix of points satisfied by the function. The matrix `ddata` represents the points satisfied by the *derivative*. In the above example, we would author

```
CURVEFIT(x,[[x1,y1],[x2,y2]],[[x3,y3]])
```

and simplify to get the answer.

4.3 Exponential Functions and Population Growth

A good first model for population growth is

$$P(t) = ae^{r(t-t_0)} \tag{4}$$

Population models are studied more thoroughly in Chapter 7 using the theory of *differential equations* but for now we will just consider the exponential model. Here $P(t)$ is the population at time t and a is the population at the starting time, t_0. Problem 7 uses this model.

There are two parameters in (4), a and r. These parameters can be determined if we know the population at two different times, t_1 and t_2, i.e., if we know $P(t_1) = y_1$ and $P(t_2) = y_2$. This gives the equations

$$\begin{aligned} ae^{r(t_1-t_0)} &= y_1 \\ ae^{r(t_2-t_0)} &= y_2 \end{aligned}$$

but solving for a and r is a little more difficult since this is not a linear system of equations. The way to do this is to use the first equation to solve for a and then substitute that value into the second equation and then solve the resulting equation for r.

Another approach is to observe that the equations *are linear* in the quantities $\ln a$ and r because, if we let $c = \ln a$, they are equivalent to:

$$\begin{aligned} c + r(t_1 - t_0) &= \ln y_1 \\ c + r(t_2 - t_0) &= \ln y_2 \end{aligned}$$

Of course, once we find c then $a = e^c$, so you're done. Problem 6 will require solving these equations.

4.4 Approximation Using Spline Functions*

Suppose that, as before, we are given data points in the form of an $n \times 2$ matrix declared as `data`. To take a simple example let's assume that

$$\text{data} := \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}.$$

We want to find a smooth function $f(x)$ whose graph passes through the data points. One solution to this problem is to use `CURVEFIT(x, data)` which gives us a degree $n-1$ polynomial passing though the given data points. Unfortunately, for problems with a large number of data points this can take a long time to solve because it requires solving a large system of equations ($n-1$ equations and $n-1$ unknowns).

One simple technique that doesn't involve solving large systems of equations is to use a piecewise quadratic polygonal approximation to the graph. The idea is find a quadratic polynomial connecting each pair of consecutive data points but the catch is that in order for the graph to be smooth you need to make the derivatives match at each data point.

Here's how we do it: We start with an arbitrary slope, say $m = 2$, at the first data point, which is $(0,0)$ in our example, and use the second form of `CURVEFIT` to find a quadratic polynomial $f_1(x)$ which satisfies the equations

$$f_1(0) = 0, \quad f_1(1) = 1 \quad \text{and} \quad f_1'(0) = 2\,.$$

This solution is

```
CURVEFIT(x,[data SUB 1, data SUB 2],[[0,2]])
```

where we note that each data point can be referred to as `data SUB i` or alternately, using the symbol bar as `data↓i`.

Now to find our second quadratic $f_2(x)$ connecting the second pair of data points and making sure that the graph of the two functions is smooth at $x = 1$ we simply solve for $f_2(x)$ using the equation: $f_1'(1) = f_2'(1)$. Continuing in this way we get quadratics $f_1(x)$, $f_2(x)$, $\ldots$, $f_{n-1}(x)$ corresponding to each of the $n-1$ intervals: $[\text{data}_{1,1}, \text{data}_{2,1}]$, $[\text{data}_{2,1}, \text{data}_{3,1}]$, $\ldots$, $[\text{data}_{n-1,1}, \text{data}_{n,1}]$. Note that we have used the double subscript notation to get the x-values in the first column of the `data` matrix.

We combine these functions into a single function using the `CHI` function. Here `CHI(a,x,b)` is 0 unless $a \le x \le b$ in which case it is 1. Thus, the combined function is

$$f(x) = \sum_{k=1}^{3} f_k(x) \cdot \mathrm{CHI}(\mathrm{data}_{k,1}, x, \mathrm{data}_{k+1,1}) \tag{5}$$

In our example we can solve for the three quadratics and get

$$f_1(x) = 2x - x^2, \quad f_2(x) = 2x - x^2 \quad \text{and} \quad f_3(x) = 3x^2 - 14x + 16$$

Application The resulting function $f(x)$ above is called a *quadratic spline function* and is important in approximation theory and computer graphics. One important example is in generating fonts for computer screens. Computers used to view a highly stylized letter like the capital S in some fancy font as a bitmap picture which required lots of memory to store and lots of time to draw on the screen. The modern approach is to view the letter as say 10-20 carefully chosen data points and then fill in the rest of the letter using spline function techniques.

You can experiment with these techniques by using the utility function `SPLINE(x,data,m1)` which gives the quadratic spline passing the data points `data` and having derivative m1 at $x = \mathrm{data}_{1,1}$. Using our example, we enter the above with m1 = 2 and Simplify. It's instructive to plot the points `data` as a (non-connected) set of points and then plot the spline function to make sure that it passes through the points and that it indeed has a smooth graph.

The definition behind the `SPLINE` function (see the file ADD-UTIL.MTH) is fairly straightforward. The function $f_k(x)$ depends on the previous function $f_{k-1}(x)$ and more specifically on the quantity $f'_{k-1}(x_k)$, where the k^{th} interval is $[x_k, x_{k+1}]$. It turns out that it is more efficient to make a vector out of the $n-1$ slopes $m = [m_1, m_2, \ldots]$ using the formula

$$m_k = 2\frac{y_{k+1} - y_k}{x_{k+1} - x_k} - m_{k-1} \quad k = 2, \ldots, n-1\,. \tag{6}$$

which can be derived using DfW(see the file F-SPLINE.MTH). Using this formula one produces the vector of slopes using the `ITERATES` function. The formula looks a little complicated at first but should look straigtforward after some careful examination (see either the file ADD-UTIL.MTH or the `SLOPE` function in the file F-SPLINE.MTH).

Using these quantities one then computes $f_k(x)$ using

```
CURVEFIT(x, [data↓k, data↓(k+1)], [[data↓k↓1, m↓k]])
```

.

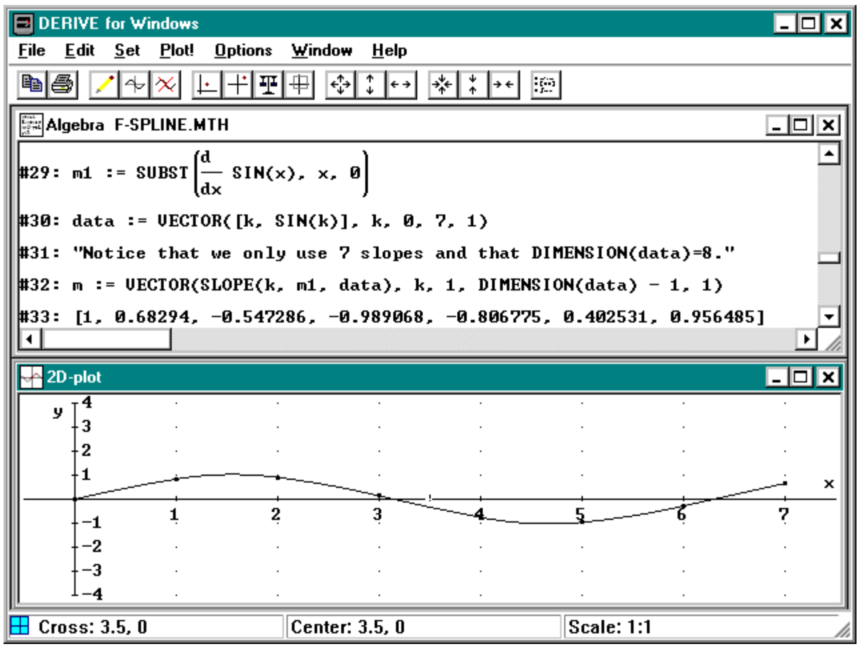


Figure 4.3: Approximation using spline functions

See Figure 4.3 where we use this method of approximation to approximate the function $y = \sin x$ using $n = 7$, which is the smallest integer greater than 2π. Thus, based on the numbers $\sin 1, \ldots, \sin 7$ plus the derivative at $x = 0$, i.e., $m_0 = 1$, we get a good approximation to the sine function.

4.5 Laboratory Exercises

Start off your lab by Loading the ADD-HEAD.MTH file (use File/Load/Math). Note that the syntax of the `CURVEFIT` function is displayed on the second line of the ADD-HEAD file. There are two possibilities: `CURVEFIT(x, data)` where data is a matrix of data points satisfied by the function or `CURVEFIT,` `data, ddata)` where now the derivative satisfies the matrix of data points ddata.

1. **a.** Use the `CURVEFIT` function to find the cubic polynomial passing through the points: $(0, 0)$, $(1, 1)$, $(2, 0)$ and $(3, 1)$.

b. What degree polynomial is required to pass through 7 points? (**Hint:** Make up a 7 point data set and examine the solution.)

2. Use the `CURVEFIT` function to find a, b, and c if $ax^2 + bx + c$ passes through

a. $(1, 1)$, $(3, 4)$, and $(4, 4)$.

b. $(1, 1)$, $(3, 4)$, and $(4, 1)$.

c. Show that the functions determined in part **a** and **b** both have the same slope at $x = 2$.

d. Do you think it is possible that

$$f(1) = 1$$
$$f(3) = 4$$
$$f'(2) = 2$$

(see equation 2). Use the second form of the `CURVEFIT` function to find the solution. Note that the ddata is a 1×2 matrix in this case. What does DERIVE tell you? What if you change the derivative to $f'(2) = 3/2$? Can you explain what the answer means?

For the following problems you will need to enter the variables `x1`, `x2`, `x3`, `x4`, `y0`, `y1`, `y2`, `y3`, and `y4`. You can declare these as variables easily by authoring

```
[x1 :=, x2 :=, x3 :=, x4 :=, y0 :=, y1 :=, y2 :=, y3 :=, y4 :=]
```

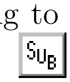
See the discussion on page 55.

3. Let (x_1, y_1) and (x_2, y_2) be two points in the plane with $x_1 \neq x_2$. Let $m = \frac{y_2 - y_1}{x_2 - x_1}$ be the slope of the line through these points. The Mean Value Theorem says that if $f(x)$ is a differentiable function which passes through these points then $f'(x_3) = m$ for some x_3 between x_1 and x_2. Show that if $f(x)$ has the form $ax^2 + bx + c$ then we can take $x_3 = (x_1 +$

$x_2)/2$, i.e., show that $f'((x_1+x_2)/2) = m$ if $f(x)$ has this form. **Hint:** Solve the system $f(x_1) = y_1$, $f(x_2) = y_2$ for b and c and substitute those values back into $ax^2 + bx + c$. Then show that the derivative of the resulting expression is m when $x = (x_1 + x_2)/2$. Of course this means that all quadratic functions through (x_1, y_1) and (x_2, y_2) have the same slope at $(x_1 + x_2)/2$.

4. Use the `CURVEFIT` function to find the quadratic function $f(x) = ax^2 + bx + c$ that satisfies

$$\begin{aligned} f(x_0) &= y_0 \\ f(x_1) &= y_1 \\ f(x_2) &= y_2 \end{aligned}$$

Integrate the resulting function over the interval $[x0, x2]$. Observe that your answer is a pretty big expression that requires scrolling to view. Now substitute in this expression $x1 = (x0+x2)/2$ using the  button and simplify. Note that $x1$ is the *midpoint* of the interval $[x0, x2]$. The answer should be a very simple formula in terms of x_0, x_2, y_0, y_1 and y_2. In the next chapter this calculation will be the basis for the *Simpson Method* of numerical integration.

***5.** Suppose we want to find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ such that

$$\begin{aligned} f(x_1) &= y_1 \\ f(x_2) &= y_2 \\ f(x_3) &= y_3 \\ f''(x_4) &= y_4 \end{aligned}$$

Show that this is always possible if x_1, x_2, and x_3 are all distinct and $x_4 \neq (x_1 + x_2 + x_3)/3$. The algebra in this problem gets fairly messy.

6. Let (x_1, y_1) and (x_2, y_2) be two points in the plane with $y_1 > 0$, $y_2 > 0$, and $x_1 \neq x_2$. Let $f(x) = ae^{rx}$ be an exponential function. Show that it always possible to find a and r so that $f(x)$ passes through these points. **Hint:** you need to solve the equations

$$\begin{aligned} ae^{rx_1} &= y_1 \\ ae^{rx_2} &= y_2 \end{aligned}$$

To do this first solve for a in one of these and substitute the answer into the other and then solve for r.

7. Table 4.1 on the next page shows the population of the US for every decade from 1800–1900. Consider two models for the data: an exponential model $P(t) = ae^{r(t-t_0)}$ (take $t_0 = 1800$) and a linear model $L(t) = bt + c$.

a. Use the data for 1800 and 1810 to determine a, r, b and c.

b. What does each model predict for 1830?

c. How do the models compare during the first 50 years? 100 years? Do this by graphing both functions and the population data. Adjust the scale to get a good picture.

d. Make up your *own* value of r in the exponential model and see if you can get a better representation of the data. Do this by plotting the model and comparing with the data.

e. The population for 1990 is 248.7 million. What value of r would yield a population model which would result in this population size after 190 years?

***8.** Suppose that `data` is an $n \times 2$ matrix of data points

$$\text{data} := \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \\ x_n & y_n \end{bmatrix}.$$

where x_1, y_1, ...have numerical values and $x_1 < x_2 < \ldots$. We know that plotting this vector in connected mode gives a piece-wise linear graph. You can test this using a sample value for `data`. Write a function `f(x)` in DfW which will have the same graph, i.e., between any two consecutive x-values, $x_k \le x \le x_{k+1}$, $f(x)$ linearly interpolates the data points. (**Hint:** Look at equation 5 on page 60 for doing spline function interpolation and use the `CHI` function as is done there. You will need to use subscript notation to refer to the x, y values. For example, `data↓1↓1` is x_1 and `data↓3↓2` is y_3.)

Table 4.1: Population of the US, 1800–1990

Year	Population (millions)
1800	5.3
1810	7.2
1820	9.6
1830	12.8
1840	17.0
1850	23.0
1860	31.4
1870	38.5
1880	50.0
1890	62.9
1900	76.2
1910	92.2
1920	106.0
1930	123.2
1940	132.2
1950	161.3
1960	179.3
1970	203.3
1980	226.5
1990	248.7

***9.** Let (x_1, y_1), (x_2, y_2), and (x_3, y_3) be three points in the plane with $x_1 \neq x_2$, $x_1 \neq x_3$, and $x_2 \neq x_3$. Show that all cubic functions, $f(x) = ax^3 + bx^2 + cx + d$ which go through all three of these points have the same second derivative at $\frac{x_1+x_2+x_3}{3}$. (Hint: Just solve the 3 equations in the 3 unknowns b, c, and d in terms of the 4^{th} unknown a. Differentiate twice and substitute in the above value of x. Check that the answer does not depend on a.)

Chapter 5

Finding Roots Using Computers

5.1 Introduction

This lab explains two techniques for numerically solving equations, Newton's famous method and the bisection method. If we have any equation we want to solve for x, we can subtract one side from the other to get an equation of the form $f(x) = 0$. Of course, in case $f(x)$ is a polynomial then solving this equation means finding the roots of $f(x)$.Thus, for quadratic polynomials we would ordinarily use the quadratic formula. However, we will be considering very general functions which typically involve trigonometric functions, logarithms and exponentials and hence algebraic methods are usually hopeless.

Newton's method is called a *dynamic process* and is related to interesting topics such as chaos and fractals. We will explore these concepts later in this chapter.

5.2 Newton's Method

Newton's method for finding a solution r to the equation $f(x) = 0$ is to start with a guess x_0 (presumably not too far from r) and form the tangent line to $f(x)$ through $(x_0, f(x_0))$. Then find the place, call it x_1, that this tangent line crosses the x–axis. Now we repeat this process with x_1 in place of x_0. (See Figure 5.1 on the next page.) In this way we obtain a sequence of numbers $x_0, x_1, x_2, \ldots$ which, under reasonable conditions, will converge to r.

Since $y - y_0 = m(x - x_0)$ is the equation of the line through (x_0, y_0) with slope m, the equation for the tangent line of $f(x)$ through $(x_0, f(x_0))$ is

$$y - f(x_0) = f'(x_0)(x - x_0).$$

Solving for x when $y = 0$ gives $x = x_0 - f(x_0)/f'(x_0)$. Thus we get the $(n+1)^{\text{st}}$ approximation from the n^{th} by the formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

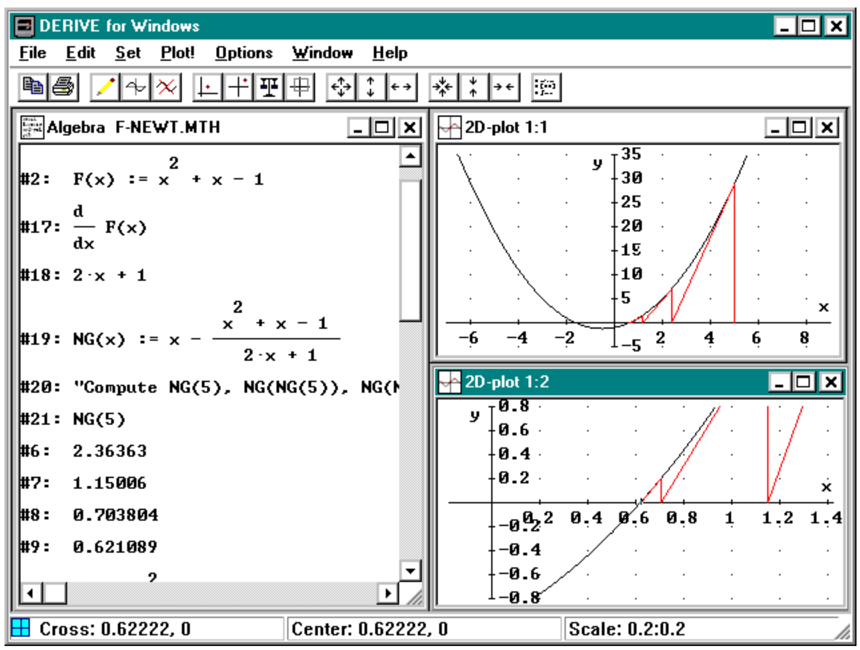


Figure 5.1: Newton's method for finding roots

In the graphics window of Figure 5.1 the first several approximations in Newton's method are shown for the equation $x^2 + x - 1 = 0$ which has the unique positive solution $x = \sqrt{5}/2 - 1/2 \approx 0.618$. The initial guess is $x_0 = 5$. From the point $(5, 0)$ we go up to the curve at the point $(5, f(5))$ and then follow the tangent line until it intersects the x–axis at the point

$(x_1, 0) \approx (2.36363, 0)$. The process is now repeated, starting with the guess x_1.

It is convenient to view the computations as an iteration process:

$$\mathrm{NG}(x) = x - \frac{f(x)}{f'(x)} \tag{2}$$

which changes a guess x into a (hopefully) better guess $\mathrm{NG}(x)$. (Note that $x_{n+1} = \mathrm{NG}(x_n)$.) You can think of NG as standing for 'Newton guess' or for 'next guess'. To make a DERIVE function to do this for our function $f(x) = x^2 + x - 1$ we define

(3)
```
NG(x) := x-(x^2+x-1)/(2x+1)
```

Now starting with x_0 and successively applying this function to the previous result produces a sequence of approximations:

$$\begin{aligned}
&x_0 \\
&x_1 = \mathrm{NG}(x_0) \\
&x_2 = \mathrm{NG}(x_1) = \mathrm{NG}(\mathrm{NG}(x_0)) \\
&x_3 = \mathrm{NG}(x_2) = \mathrm{NG}(\mathrm{NG}(x_1)) = \mathrm{NG}(\mathrm{NG}(\mathrm{NG}(x_0))) \\
&\quad\vdots
\end{aligned}$$

which we hope get closer and closer to the exact answer. In the limit we want this sequence of approximations to converge to the root.

We can compute several approximates by first Authoring `NG(5)`, and then approximating. Now we can author `NG`, press the right mouse button and then click (Insert expression) or press `F4`. This will bring down the highlighted expression in parentheses giving `NG(2.36363)` which we approximate (just press Simplify instead of OK) again and then repeat this process.

A somewhat fancier method is to use the DERIVE's `ITERATES` function. `ITERATES(u,x,a,n)` simplifies to an $(n+1)$-vector whose first entry is a and each subsequent entry is obtained by substituting the previous entry for x in u. Thus, `ITERATES(x^2,x,2, 4)` returns the vector $[2, 4, 16, 256, 65536]$. (The function `ITERATE` is similar, but just gives the last value, so `ITERATE(x^2,x,2,4)` gives 65536.) We can get the first 4 approximates by authoring `ITERATES(NG(x), x, 5, 4)` and approximating the result.

The utility file ADD-UTIL.MTH contains two functions that make computing the Newton iterations easier. The function `NEWT(u,x,a)` computes the Newton guess of the expression u, in the variable x, starting with an initial guess at $x_0 = a$. In our previous example of $f(x) = x^2 + x - 1$ with starting point $x_0 = 5$ we would enter `NEWT(x^2+x-1,x,5)`. To get a vector containing the starting point and the first 4 Newton iterates you author and simplify `NEWT(x^2+x-1,x,5,4)`. The general syntax is `NEWT(u,x,a,k)`.

Looking at the algebra window in Figure 5.1 we see the above function along with the first 4 iterates starting at $x_0 = 5$. The graphic demonstration shows the Newton method in action by plotting a part of the tangent line until it crosses the x-axis. The picture clearly shows how well the Newton method works since one has to zoom-in several times near the actual root in order to see the last two iterations. The utility function `DRAW_NEWT(u,x,a,k)` simplifies to a matrix which plots the figure shown in Figure 5.1.

Alternately, that file contains the necessary definition for doing the graphics directly. The basic idea is to make a vector out of several *triples* of points which have the form $(x, 0)$, the initial guess on the x–axis, $(x, f(x))$, the corresponding point on the curve, and $(\mathrm{NG}(x), 0)$, the place where the tangent to the curve at $(x, f(x))$ intersects the x–axis. When we graph these points we want the lines connecting them to be drawn. If this is not the case then adjust the Options/Points menu.

You might note that a little trick is used in the above of `DRAW_N` in the file F-NEWT.MTH. The special form of the `VECTOR(u,x,v)` function sets x equal to each value in the vector $v = [v_1, \ldots, v_n]$ and makes the new vector $[u(v_1), \ldots, u(v_n)]$.

Example. Suppose we want a numerical approximation of $\sqrt{2}$. We think of it as a solution to the equation $x^2 - 2 = 0$. Then formula (2) gives the very simple expression:

$$\mathrm{NG}(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 2}{2x} = \frac{x^2 + 2}{2x} = \frac{x}{2} + \frac{1}{x}$$

We get several approximates by clicking $\boxed{\approx}$ after authoring either

(4) `NEWT(x^2-2,x,2,5)`

or equivalently as in the file F-NEWT.MTH

(5) `ITERATES(x/2+1/x,x,2,5)`

with precision digits set to 10 decimal places. We get

$$[2, 1.5, 1.41666666, 1.414215686, 1.414213562, 1.414213562] \tag{6}$$

which is accurate to 10 decimal places. In fact, Figure 5.2 shows a remarkable property about Newton approximation: the number of decimal place accuracy approximately *doubles* with each iteration!

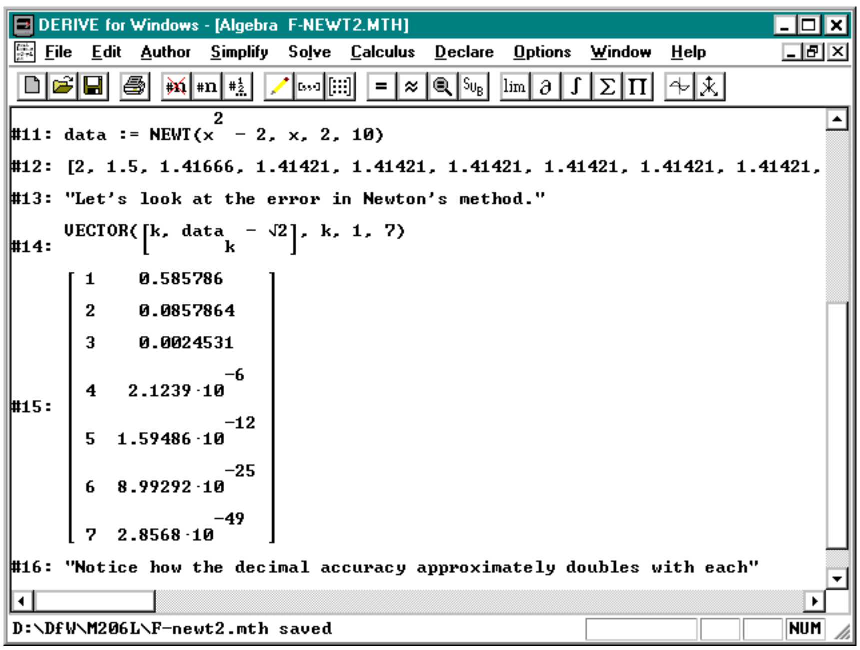


Figure 5.2: Each iteration gives twice as many digits

5.3 When Do These Methods Work

For Newton's method to work we need at least that $f(x)$ is differentiable, since the derivative appears in the formula (2). If we assume that $f''(x)$ exists we get the following theorem:

Theorem 1. *Suppose $f(r) = 0$ and that $f''(x)$ exists in some open interval containing r. If $f'(r) \neq 0$ then the iterates of*

$$\mathrm{NG}(x) = x - \frac{f(x)}{f'(x)}$$

converge to r provided the starting point x_0 is sufficiently close to r. More precisely, given $0 < \varepsilon < 1$, there exists a $\delta > 0$ such that:

$$|\operatorname{NG}(x) - r| < \varepsilon |x - r| \tag{7}$$

whenever $|x - r| < \delta$.

Proof. Since f' is continuous (because f'' exists) and is not zero at $x = r$, we know that $f'(x) \neq 0$ near $x = r$ and hence $\operatorname{NG}(x)$ is defined for these x. Clearly, $\operatorname{NG}(r) = r$ since $f(r) = 0$ and also

$$\operatorname{NG}'(r) = 1 - \left[\frac{f'(r)^2 - f(r)f''(r)}{f'(r)^2}\right] = \frac{f(r)f''(r)}{f'(r)^2} = 0.$$

Hence by the definition of the derivative, given $\varepsilon > 0$, there is a $\delta > 0$ so that

$$\left|\frac{\operatorname{NG}(x) - r}{x - r}\right| = \left|\frac{\operatorname{NG}(x) - \operatorname{NG}(r)}{x - r}\right| < \varepsilon$$

whenever $|x - r| < \delta$. This shows that (7) holds which is what we needed to prove. □

Notice how this proof works. First we showed that (under the hypotheses of the theorem) $\operatorname{NG}'(r) = 0$. This is the crux of the proof. It means for any $\varepsilon > 0$ there is an interval around r where $\operatorname{NG}'(x) < \varepsilon$ for all x in this interval. By the Mean Value Theorem we get an $\tilde{x}$ between r and x such that

$$\left|\frac{\operatorname{NG}(x) - \operatorname{NG}(r)}{x - r}\right| = |\operatorname{NG}'(\tilde{x})| < \varepsilon\,.$$

This implies that (7) holds for all x in this interval and this says that the error of the next guess, $\operatorname{NG}(x)$, is ε times smaller than the last one. If we take $\varepsilon = 1/10$ say, then each guess will be 10 times as accurate as the previous one. This implies each new guess has at least one more decimal place accuracy.

But if we look at the example above, or the examples below, we see that the convergence is much faster. This is because if we are in an interval where $\operatorname{NG}'(x) < 1/10$, then $\operatorname{NG}(x)$ is at least 10 times closer to r and, since $\operatorname{NG}'(r) = 0$, we are now likely to be in an interval where $\operatorname{NG}'(x)$ is much smaller. So not only are we getting closer but the amount by which we are getting closer is also increasing. This is why the convergence when $\operatorname{NG}'(r) = 0$ is so fast. It means the next guess after $\operatorname{NG}(x)$, i.e., $\operatorname{NG}(\operatorname{NG}(x))$, will tend to be much more than 10 times as accurate as $\mathrm{NG}(x)$. In dynamic systems like this, a point r with $\mathrm{NG}'(r) = 0$ is called a *super attractor*. Once x gets close to a super attractor r, repeated applications of N will move it towards r very quickly.

The next theorem investigates a situation where $0 < \mathrm{NG}'(r) < 1$. In this case $N^k(x)$ still tends to r but not nearly as fast. In this case r is no longer a super attractor but is simply an *attractor*.

Theorem 2. *Suppose that $f(r) = 0$ and that $f(x) = (x - r)^m g(x)$ where $g(x)$ is differentiable, m is a positive integer and $g(r) \neq 0$. Then,* $\mathrm{NG}(x)$ *is defined for all $x \neq r$ which are sufficiently close to r and the iterates converge to r.*

Proof. Since $f'(r) = 0$ for $m > 1$ (check!) it is not clear that we can even define $\mathrm{NG}(x)$ for x near r. But

$$\begin{aligned} f'(x) &= m(x-r)^{m-1}g(x) + (x-r)^m g'(x) \\ &= (x-r)^{m-1}[mg(x) + (x-r)g'(x)] \approx (x-r)^{m-1}mg(r) \end{aligned} \tag{8}$$

and since $g(r) \neq 0$ it is easy to see that the bracketed expression above can not be zero for all x near to r and hence the same is true of $f'(x)$ provided $x \neq r$.

Now using (8) to simplify $\mathrm{NG}(x)$ (do this using DERIVE) we get

$$\frac{\mathrm{NG}(x) - r}{x - r} = \frac{(m-1)g(x) + (x-r)g'(x)}{mg(x) + (x-r)g'(x)} \approx \frac{m-1}{m} < 1$$

and hence the iterates converge as before. □

5.4 Fractals and Chaos*

Which root does Newton find? Of course $f(x) = x^2 - 2$ has two roots, $\sqrt{2}$ and $-\sqrt{2}$. If our initial guess is any positive number, Newton's method will converge to $\sqrt{2}$ and, if it is any negative number, to $-\sqrt{2}$. If the initial guess is 0 the method fails since $\mathrm{NG}(0)$ is not defined.

The situation for this $f(x)$ is pretty simple but that is not always the case. To get a clearer picture of what can happen we need to discuss the *complex numbers*. Recall that a complex number has the form $a+b\,i$, where a and b are real numbers and i is a square root of -1, i.e., $i^2 = -1$. Complex numbers can be represented as points in the plane: (a, b) for $a + b\,i$. We call this the *complex plane*. In DERIVE we input i by using the symbol bar or typing `#i`. This is displayed with $\hat{\imath}$.

You probably have already encountered complex numbers in DERIVE when, for example, you try to Solve an equation such as $x^2 + 1 = 0$ or something more complicated, while trying to find extreme points. The result is that DERIVE computes the two solutions $x = \pm i$. Of course, in calculus we usually ignore complex solutions since they are not relevant to max-min theory or graphics. Nevertheless, they do play an important role in algebra since they provide a complete theory for the solution to polynomial equations.

Using the same function $f(x) = x^2 - 2$, let's see what happens if we start with a complex number for x_0 like $3 + 2\,i$. This time we Author

```
data := NEWT(x^2-2,x,3+2 #i,5)
```

If we approximate this with precision set to 6 decimal places, we obtain

$$[3 + 2\,i, 1.73076 + 0.846153\,i, 1.33170 + 0.195097\,i,$$
$$1.40099 - 0.0101504\,i, 1.41423 + 9.59747 \times 10^{-5}\,i, 1.41421]$$

so that it still converges to $\sqrt{2}$. We can get a picture of this convergence by plotting the complex number $a + ib$ as the point (a, b). To do this we need one of the utility functions. Thus, authoring

```
DRAW_COMPLEX(data)
```

and then simplifying the result will give the matrix of 6 points. We plot this matrix to observe how the iterates converge to the point $(\sqrt{2}, 0)$ on the x-axis, see Figure 5.3 on the next page.

It's a fact that the Newton method will converge to $\sqrt{2}$ whenever we start with $x_0 = a + b\,i$ where $a > 0$. We call $\sqrt{2}$ an *attractive fixed point* and the right half plane is called the *basin of attraction* for $\sqrt{2}$. If we start with $x_0 = a + b\,i$ where $a < 0$ it will converge to $-\sqrt{2}$, so $-\sqrt{2}$ is also an attractor with the left half plane as its basin of attraction.

What happens if we start with a point on the imaginary axis (the y–axis $x_0 = b\,i$? Simplify and plot the expression

```
DRAW_COMPLEX(NEWT(x^2-2,x,#i,25))
```

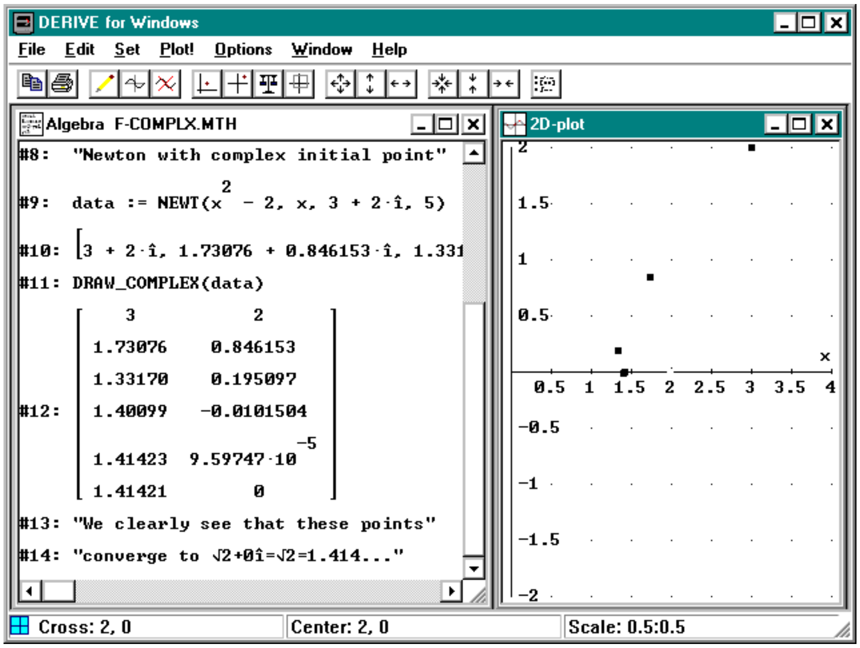


Figure 5.3: Newton's method with complex starting point

Notice that all the values are purely imaginary (they only have an i component) and that they seem to bounce around randomly. Moreover, if you author

```
DRAW_COMPLEX(NEWT(x^2-2,x,1.01#i,25))
```

you'll notice that the corresponding entries of the answers are approximately the same for the first few terms but very quickly seem to have no relation to each other. Here's a nice way to do this: Define the first set of points to be `data1` and the second set to be `data2`. Author the vector `[data1,data2]` and simplify. Then scroll though the matrix to compare entries.

In other words, even though the two starting points above; namely i and $1.01i$ are quite close together their long-term behavior seem completely different. The above phenomenon is what is known as *chaos*.

We can illustrate this last property graphically by looking at the equation $x^2+2=0$ rather than $x^2-2=0$. The former equation has roots $\sqrt{2}\,i$ and $-\sqrt{2}\,i$. Just as before if we start with any point $a+ib$ in the upper half of the complex plane ($b>0$), the Newton iterates of the function x^2+2 converge to $\sqrt{2}\,i$ and any point in the lower half plane ($b<0$) converges to $-\sqrt{2}\,i$.

Test this by plotting

```
DRAW_COMPLEX(NEWT(x^2+2,x,1+#i,5))
```

But we get chaos on the real axis. To see this chaos plot the function x^2+2 and the output to `DRAW_NEWT(x^2+2,x,2,4)` in connected mode, see Figure 5.4.

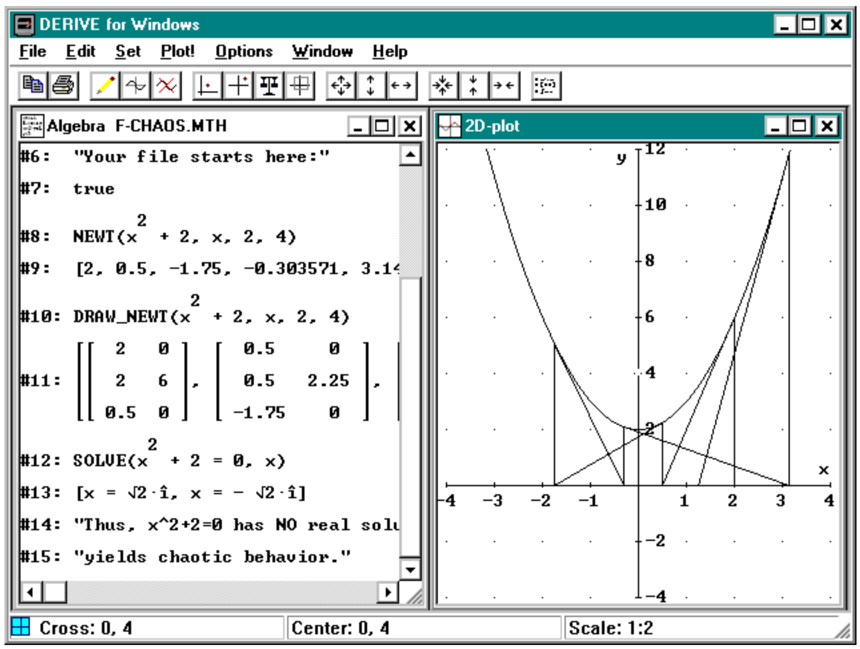


Figure 5.4: Chaos

Now consider $f(x) = x^3 - 1$. This has three roots: $x = 1$, $x = -1/2 + i\sqrt{3}/2$, and $x = -1/2 - i\sqrt{3}/2$. This is easy to do in DERIVE just Solve the equation $x^3 - 1 = 0$. Each of these is an attractor with a basin of attraction. However the shapes of these basins of attraction are really quite interesting and bizarre. Figure 5.5 on the next page shows the basin of attraction for the root $x = 1$ in white. The basins of attraction of both of the other roots are black. In Figure 5.5 the center is the origin in the complex plane and the right hand edge has $x = 2$. So the point $(1, 0)$ (i.e., $1 + 0\,i$) is between the center and the right hand edge.

A color version of this figure which indicates the number of iterates needed to converge can be viewed on our World Wide Web home page at

http://www.math.hawaii.edu/206L/

An interactive Java applet which can be used to show the iterations of Newton's method is also available at this site.

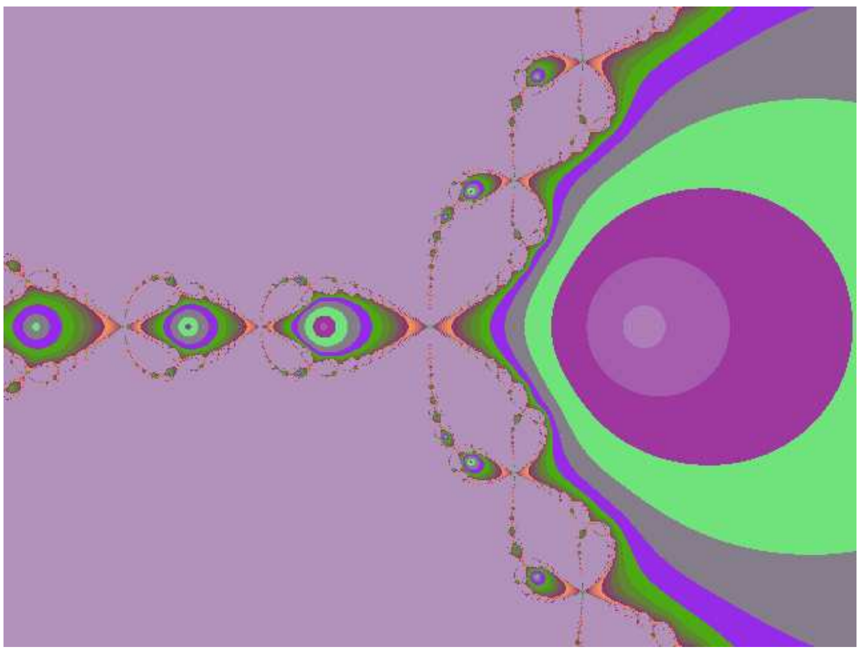


Figure 5.5: Basins of attraction of $x^3 - 1$ in the complex plane

Constructing the Julia set The set of points where Newton's method fails, that is, the set of points x_0 where the sequence

$$x_0, \mathrm{NG}(x_0), \mathrm{NG}(\mathrm{NG}(x_0)), \ldots \tag{9}$$

fails to converge, is called the *Julia set* for $\mathrm{NG}(x)$. In the example $f(x) = x^3 - 1$ these are the points on the edge or boundary of the basin of attraction. As the picture in Figure 5.5 shows this set can be very complicated, it looks a little like a necklace with infinitely many smaller and smaller loops coming out in many different directions.

There are two basic methods for constructing this set. Since $\mathrm{NG}(x_0)$ is not even defined when $f'(x_0) = 0$ this is a good place to start. If x is a solution of

$$\mathrm{NG}(x) = x_0$$

then the third term of the sequence (9) is not defined and so x will be in the Julia set. In the case when $f(x) = x^3 - 1$ the equation above has three solutions. For each of these there are three more obtained by solving a similar equation or in other words finding the points where $\mathrm{NG}(\mathrm{NG}(x)) = x_0$. Continuing in this way we get a close approximation to the Julia set. The actual set is obtained by taking limits of these points. This method is called the *backward method* and is done in the file F-JULIA-BACKWARD.MTH for the polynomial $x^3 - 3x$ which has critical points at ± 1. This function has three real roots at $x = 0$ and $x = \pm\sqrt{3}$ and the Julia set somehow has to separate the three basins of attraction corresponding to these roots. See Figure 5.6 for a picture of it's Julia "necklace".

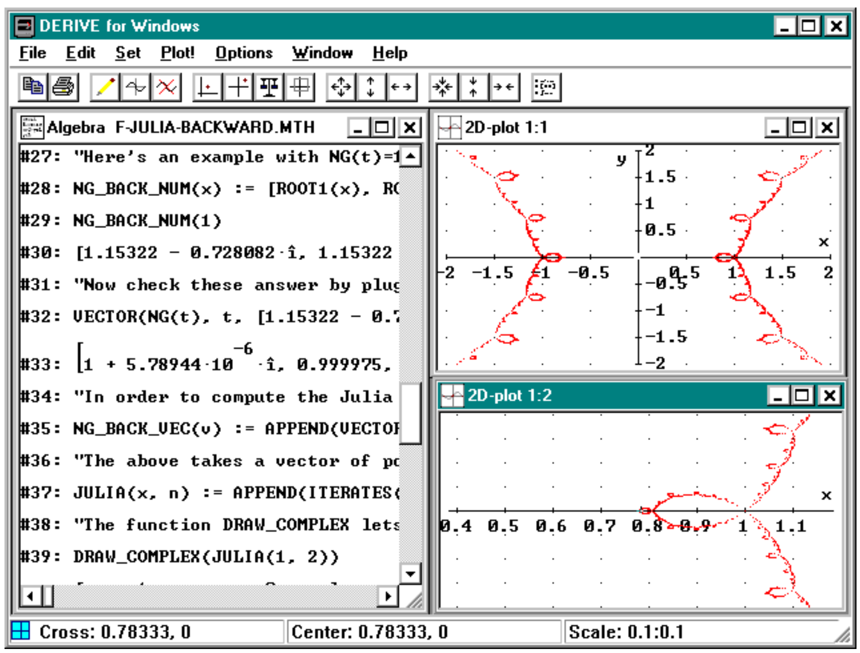


Figure 5.6: Bad Newton starting points for $x^3 - 3x = 0$ in the complex plane

The trouble with the backward method is that it uses the cubic formula for solving 3rd order equations and this formula is pretty complicated. Even worst is the fact that there is no analogous formula for degrees 5 or greater. To get around this problem there is the "forward method" which involves simply looking at the sequence:

$$\mathrm{NG}(x_0), \quad \mathrm{NG}(\mathrm{NG}(x_0)), \quad \mathrm{NG}(\mathrm{NG}(\mathrm{NG}(x_0))), \ldots$$

and checking whether it gets closer and closer to root or else just wanders around forever. Since you have to do this for each point or pixel in the graph this can be a very lengthy computation. A number of shortcuts and tricks are typically employed and you can study the file F-JULIA-FORWARD.MTH to see how we did it. Or you can just check out the pictures; see Figure 5.7.

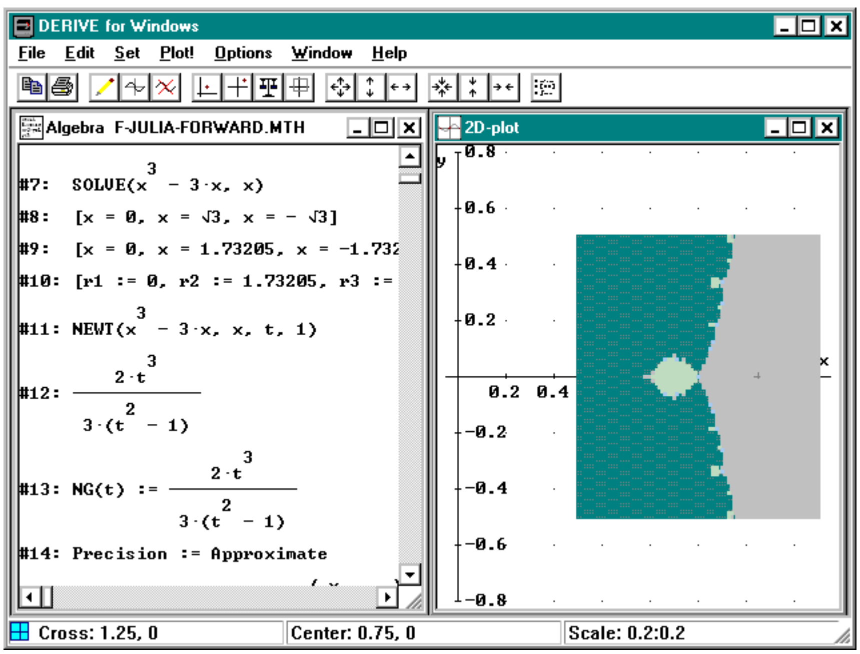


Figure 5.7: Basins of attraction for $x^3 - 3x = 0$

5.5 Bisection Method*

We now consider a very simple technique which is applicable to any continuous function $f(x)$. If $f(x)$ is continuous and $f(a) < 0$ and $f(b) > 0$, i.e., it is below the x–axis at a and above the x–axis at b, then the Intermediate Value Theorem tells us that $f(x)$ must have a zero between a and b. Assume $a < b$. The bisection method evaluates $f(x)$ at $x = \frac{a+b}{2}$, the midpoint of a and b (which is why it is called the bisection method). If $f(\frac{a+b}{2}) > 0$ then there must be a root in the interval $[a, (a+b)/2]$; otherwise there must be a root in the interval $[(a+b)/2, b]$. In the former case we take the interval $[a, (a+b)/2]$ and apply the bisection method to it; otherwise we use $[(a+b)/2, b]$. At each stage the root lies in an interval which is only half the size of the previous stage. So we can eventually find the root to any number of decimal places.

We can automate this process by authoring two functions:

```
      F(x) :=
 BIS2(a,b) := IF(f(a)f((a+b)/2)<0, [a, (a+b)/2], [(a+b)/2, b])
    BIS(v) := BIS2(v SUB 1, v SUB 2).
```

The main function is `BIS(v)` and `BIS2(a,b)` is a helper function. The argument `v` to `BIS` is a vector with two entries, e.g., `[a, b]`. The DERIVE function `SUB`, which we discussed in the previous section, returns the parts of a vector so that `[a,b] SUB 1 = a` and `[a,b] SUB 2 = b`. So `BIS` starts with a vector like `[a,b]` and calls `BIS2(a,b)`. This then uses the values $f(a)$ and $f((a+b)/2)$ to decide if there is a root in $[a, (a+b)/2]$ or in $[(a+b)/2, b]$. In the discussion above we assumed that $f(x) < 0$ and that $f(b) > 0$. The way we have defined `BIS` it will work also in the case $f(x) > 0$ and $f(b) < 0$. To do this we test if the product $f(a)f((a+b)/2)$ is negative. If it is, then one of $f(a)$ and $f((a+b)/2)$ is negative and the other is positive. In this case the points $(a, f(a))$ and $(\frac{a+b}{2}, f(\frac{a+b}{2}))$ lie on opposite sides of the x–axis and so there must be a root in the interval $[a, (a+b)/2]$. In the other case, $f(a)f((a+b)/2)$ is positive and so they have the same sign. In this case $f((a+b)/2)$ and $f(b)$ must have the opposite signs (why?) and so there is a root in $[(a+b)/2, b]$.

Let us try the equation $\ln x = 1$ which has the (unique) solution $x = e = 2.718\ldots$. Of course we are finding the root of $\ln x - 1$ so we author `f(x) := ln(x) - 1` and apply `BIS`. Graphing $f(x)$ shows that there is a root between 2 and 3 so we author `BIS([2,3])`. This returns $[2.5, 3]$, indicating that $2.5 < e < 3$.

Now we want to apply `BIS` to the answer $[2.5, 3]$. You can do this several times by choosing author, typing `BIS`, and then inserting the highlighted vector. Once again we have an iteration process and we can use the `ITERATES`

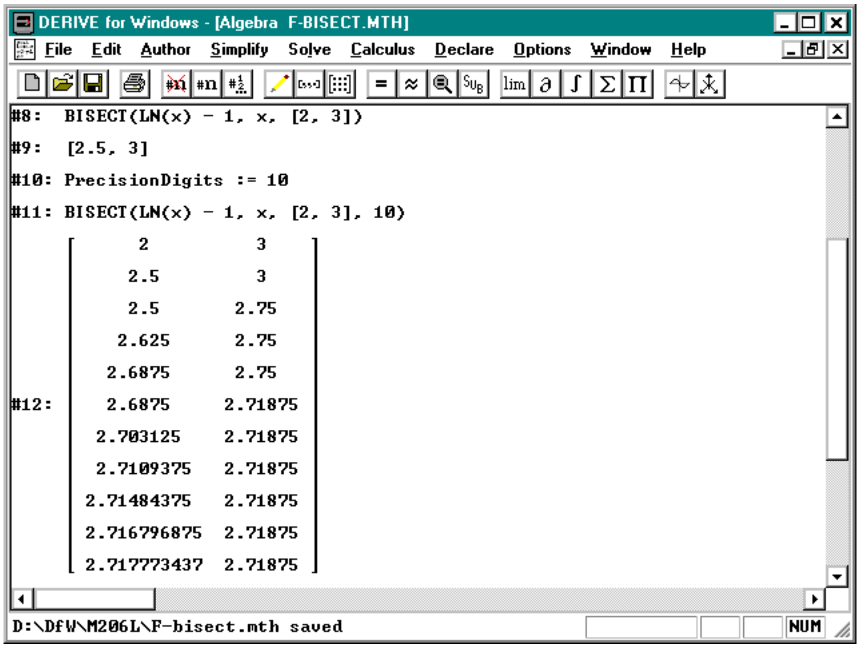


Figure 5.8: Bisection method for finding roots

function that does this for you.

Using this technique, we author

```
ITERATES(BIS(v),v,[2,3],10)
```

and then approximate it to see how well this approximates e, see Figure 5.8.

An easier way to see the bisection method in action is to use the function `BISECT(u,x,v,k)` in the utility file ADD-UTIL.MTH. To get the above results we would simply enter `BISECT(ln x-1,x,[2,3],10)` and the press the approximation $\boxed{\approx}$ button. It is interesting to compare the results of the bisection method with the Newton method of the previous section. The bisection method is fairly fast at getting a good approximation but not nearly as fast as the Newton method.

The bisection method will work for any f that is continuous on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs. It is easy to see that after n iterates the error is at most $(b - a)/2^n$. (In fact this is the width of the resulting interval. If we choose the midpoint as our estimate, the error will be at most $(b - a)/2^{n+1}$.)

5.6 Laboratory Exercises

Start off your lab by Loading the ADD-HEAD.MTH file (use File/Load/Math). Note that the syntax of the `NEWT` function is displayed on the second line of the ADD-HEAD file. There are two possibilities: `NEWT(u,x,a)` where a is the starting point in Newton's method applied to the expression u in the variable x. Alternately, the function `NEWT(u,x,a,k)` gives a vector containing the intial guess a followed by the first k approximates. The function `DRAW_NEWT(u,x,a,k)` produces the graphical demonstration of the Newton method.

1. The equation $x^2 = 2$ has solutions $x = \pm\sqrt{2}$. Use the function $x^2 - 2$ and `NEWT` to estimate $\sqrt{2}$.

a. Give the 5$^{\text{th}}$ iterate starting at $x = 10$

b. Plot the graph of x^2-2 and the output to `DRAW_NEWT(x^2-2,x,10,5)`.

c. What happens when your start at $x = -10$?

d. What's wrong with the starting point $x = 0$? Explain this both numerically and graphically.

2. In a manner similar to Problem 1, use `NEWT` to estimate $\sqrt[3]{7}$.

a. Give the 5$^{\text{th}}$ iterate starting at $x = 2$.

b. By comparing with the approximate given by Derive how many decimal places (roughly) does the Newton approximations share with the actual answer. Note that you may need to increase the number of digits you are working with (see Section 0.6).

3. Plot the graphs of x^2 and $\sin x$.

a. Determine graphically where the two graphs intersect. Give a rough estimate of the accuracy of this method? (Hint: If you use the right arrow key to change the position of the crosshair, how much does its x–coordinate change?)

b. Next use Newton's method to find all solutions to $x^2 - \sin x = 0$. Give the 5$^{\text{th}}$ iterate starting at $x = 2$.

c. Solve this equation numerically by using Solve/Numerically menu. Compare the solution you get using DfW's `SOLVE` function with your approximation above using Newton's method.

4. Let $f(x) = x^3 - 5x$. Graph $f(x)$.

a. Use Solve to find the all roots of $x^3 - 5x = 0$.

b. Plot the output to `DRAW_NEWT(x^3-5x,x,1,5)` and analyze the first 5 iterates in the Newton approximation method starting at $x = 1$. Explain in words what goes wrong when you start at $x = 1$.

c. Do the same but with $x_0 = 1.01$ and with $x_0 = 0.99$.

5. Again let $f(x) = x^3 - 5x$.

a. Find the formula for $\mathrm{NG}(x)$ for this f.

b. Find a point x_0 where is $\mathrm{NG}(x)$ undefined. (There are two such points; find either one.) Give the exact answer and then approximate it.

c. Use DERIVE's Solve/Numerical to solve $\mathrm{NG}(x) = x_0$. Call the answer x_1.

***d.** Do this once more, that is, Solve/Numerical $\mathrm{NG}(x) = x_1$. Call the answer x_2. If you continued this forever what do you think the sequence

$$x_0, x_1, x_2, x_3 \ldots$$

would look like? What are their signs? What do you think $\lim_{n\to\infty} |x_n|$ is?

e. Choose any numbers a and b which satisfy

$$|x_2| < a < |x_1| < b < |x_0|$$

To which root does Newton's method converge if we start with a? with b?

6. Let $f(x) = x^2 + 1$. Graph $f(x)$. Find 10 iterates of Newton's method starting with $x_0 = 0.5$ and $x_0 = 0.501$. Explain why you think the successive approximations don't seem to be converging to anything.

The rate of the convergence of Newton's method to a solution r of $f(x) = 0$ is determined by $|\,\mathrm{NG}'(r)|$. Since $\mathrm{NG}(r) = r$ we say that r is a *fixed point* for $\mathrm{NG}(x)$. If $0 < |\,\mathrm{NG}'(r)| < 1$ then r is said to be an attractive fixed point because nearby points are drawn to r by iterating. If $\mathrm{NG}'(r) = 0$, then r is called a super attractive fixed point. The hypotheses of Theorem 1 on page 71 imply that $\mathrm{NG}(r) = r$ and $\mathrm{NG}'(r) = 0$ which guarantees that the convergence was very fast. In the following problems you explore situations where $\mathrm{NG}'(r) \neq 0$. As long as $|\,\mathrm{NG}'(r)| < 1$ Newton's method will still converge to r if x_0 is close enough to r, but not as fast as the super attractive case.

***7.** Theorem 1 had the hypothesis that $f(r) = 0$ and $f'(r) \neq 0$. In this problem we explore what happens to a function when $f'(r) = 0$. Let $f(x) = x(x^2 - 2)^2$.

- **a.** Graph $f(x)$ and plot `DRAW_NEWT(x(x^2-2)^2,x,3,5)` (rescale to get a good picture).
- **b.** Find the first 10 Newton iterates starting with $x = 2$. How fast are they approaching $\sqrt{2}$ compared with the example shown in formulas (4) and (6)? (Use 10 digits precision.)
- **c.** Compute (exactly) $\mathrm{NG}(\sqrt{2})$ and $\mathrm{NG}'(\sqrt{2})$, where NG is defined by formula (2). Is $\sqrt{2}$ a super attractor?
- **d.** Find a and b so that $a \neq b$ and $\mathrm{NG}(a) = b$ and $\mathrm{NG}(b) = a$. (Hint: Start by visualizing this situation graphically. Then try guessing an approximate solution by looking at the graph and experimenting with the `DRAW_NEWT` function. Finally, use algebra to solve the equation: $\mathrm{NG}(\mathrm{NG}(a)) = a$ for a and then put $b = \mathrm{NG}(a)$.)
- **e.** Suppose now that $f(x) = x(x^2 - 2)^3$. Find $\mathrm{NG}'(\sqrt{2})$. What do you think $\mathrm{NG}'(\sqrt{2})$ would be for $f(x) = x(x^2 - 2)^k$? (Look up Theorem 2 on page 73 to see if this situation is a consequence of that result.)

8. The function $f(x) = x^{1/3}$ has a root at $x = 0$. Find $\mathrm{NG}(x)$, $\mathrm{NG}'(x)$, and $\mathrm{NG}'(0)$. Find 10 iterates of Newton's method starting with $x_0 = 0.1$. (Note: Make sure that the Precision Mode is set to Exact or else there may be problems with this exercise.)

9. Use the bisection method to estimate $\sqrt{2}$.

***10.** The light area in Figure 5.5 on page 77 shows the basin of attraction of the root 1 when using Newton's method on $x^3 - 1$. The origin of the complex plane is in the middle of this figure. Note that most of the negative real axis (the negative x–axis) is in the white area. This means that starting with most negative real numbers, Newton's method will converge to 1. Try this for $x_0 = -1$ and -2. If you look closely at the figure you see that black pinches down on the negative real axis at various points. Find the value of the first such point to the left of the origin.

Chapter 6

Numerical Integration Techniques

6.1 Introduction

This lab discusses numerical integration. Numerical integration is described in most calculus books and is sometimes covered in second semester calculus. You may want to look over this part of your calculus text.

A function is called *elementary* if it is made up of sums, products, powers, and compositions of the trig functions and $\ln x$ and e^x. Although the derivative of any elementary function is elementary, not all such functions have elementary antiderivatives. For example, there is no elementary function whose derivative is $\sin(x^2)$, i.e., $\int \sin(x^2)\, dx$ is not an elementary function. Consider the problem

$$\int_{-1}^{1} \sin(x^2)\, dx$$

Even though $\sin(x^2)$ has no elementary antiderivative, the area defined by the integral certainly exists. So how do we find it? We use numerical integration.

Consider the integral $\int_a^b f(x)\, dx$, and for simplicity assume $f(x) \geq 0$ and that $a < b$. The idea of numerical integration is to choose intermediate points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ and estimate the area in the strip below $f(x)$ for $x_i \leq x \leq x_{i+1}$ and then add up these estimates; see Figure 6.2 on page 92. Of course the width of this strip is $x_{i+1} - x_i$. The height varies with x. Some of the most common ways of estimating the area of the strip are:

- Left endpoint: $f(x_i)\,(x_{i+1} - x_i)$
- Right endpoint: $f(x_{i+1})\,(x_{i+1} - x_i)$
- Midpoint: $f(\frac{x_{i+1} + x_i}{2})\,(x_{i+1} - x_i)$
- Trapezoid: $\frac{1}{2}\,[f(x_{i+1}) + f(x_i)]\,(x_{i+1} - x_i)$
- Simpson's Rule: $\frac{1}{6}\left[f(x_{i+1}) + 4f(\frac{x_{i+1} + x_i}{2}) + f(x_i)\right](x_{i+1} - x_i)$

The last one, Simpson's Rule, is based on the best quadratic approximation to $f(x)$. This basic idea was derived in Exercise 4 on page 63 in Chapter 4. Section 6.5 on page 96 has a detailed explanation.

Usually we choose the x_i's equally spaced, so that

$$x_i = a + \frac{b - a}{n}\, i \tag{1}$$

Of course, in this case, $x_{i+1} - x_i = \dfrac{b - a}{n}$. Thus, if we use the left endpoint approximation, we get

$$\int_a^b f(x)\,dx \approx \frac{b - a}{n} \sum_{i=0}^{n-1} f(x_i) \tag{2}$$

Notice that we factor out the term $\dfrac{b - a}{n}$ and multiply by the sum rather than multiplying every term.

6.2 An Example

Formula (2) suggests how we might do numerical integration with Derive. Let u be the expression for $f(x)$. We can define a Derive function for the left endpoint method by

```
LEFT(u,x,n,a,b) :=
        (b-a)/n * SUM(SUBST(u, x, a + k*(b-a)/n), k, 0, n-1)
```

(Recall that `SUBST(u, x, a)` substitutes a for x in u so `SUBST(u, x, a+k(b-a)/n)` really evaluates u at $x = a + k(b-a)/n$.) `LEFT` is already defined for you in the file ADD-UTIL.MTH. All of the other methods mentioned above are also defined in that file with the names: `RIGHT`, `MID`, `TRAP`, and `SIMP`.

Now let's try an example. Although we would normally use these approximations for integrating expressions without an elementary antiderivative, we can test how good they are by applying them to something we do know how to integrate:

$$\int_1^2 \frac{1}{x}\,dx = \ln 2 \approx 0.693147180559$$

To use the left endpoint method with $n = 10$ intervals, we would just author and then approximate

```
LEFT(1/x, x, 10, 1, 2)
```

Doing this gives the answer 0.718771. Similarly if we wanted to use the trapezoid method we would author and approximate `TRAP(1/x, x, 10, 1, 2)` which gives 0.693771.

We want to compare the accuracy of these methods of approximation and also see how much the accuracy is improved by increasing n. We will try them for $n = 10$, 100, 1,000 and 10,000. A fancy way to see and compare approximation values, using the left endpoint rule for a range of n is to start by authoring the vector

```
[10^n,LN(2),LEFT(1/x,x,10^n,1,2),LEFT(1/x,x,10^n,1,2)-LN(2)].
```

Then, use the Calculus/Vector menu to produce

```
vector([10^n, LN(2), LEFT(1/x,x,10^n,1,2),
                LEFT(1/x,x,10^n,1,2) - LN(2)], n, 1, 4)
```

where the Variable n varies from a Starting value of 1 to an End value of 4. Approximating this expression yields a 4×4 matrix with the first column being the number of partitions, the second column being the exact value, the third column being the approximate value obtained from the left endpoint method and the fourth column being the error. See Section 0.14 on page 22 for more discussion on the `vector` function.

Notice from Figure 6.1 that the accuracy in this method seems to be roughly 1, 2, 3 and 4 digits respectively. This is in fact the case and it can

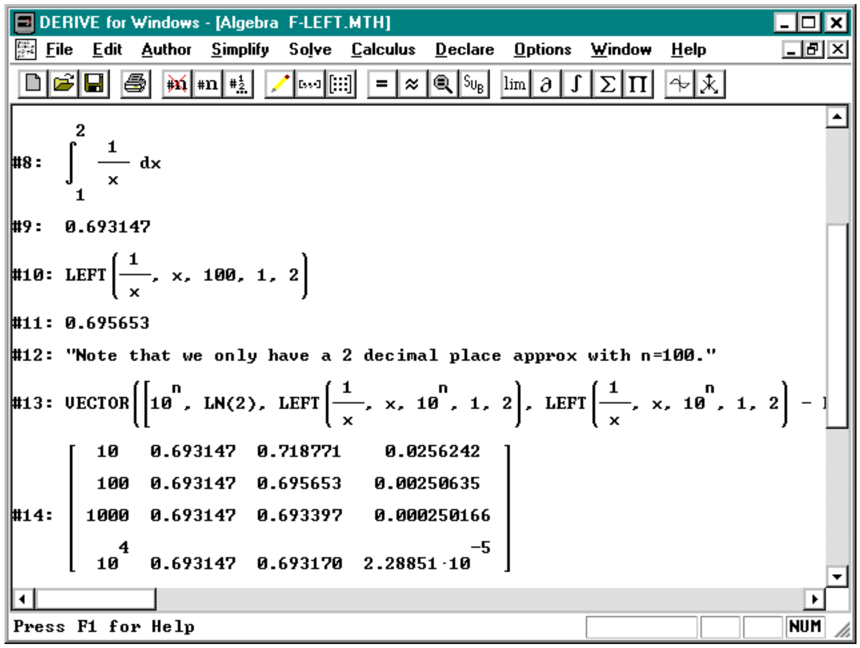


Figure 6.1: Approximating $\ln 2$ with left endpoint method

be proved that using 10^n subdivisions yields an accuracy of n decimal places. This is not very efficient since it requires a billion computations (10^9) to achieve calculator accuracy of 9 digits. Try comparing computation times for various powers of 10 to see how this rapidly becomes impractical. If we try to obtain simple calculator accuracy of 8-12 decimal places, then this can take hours on a PC which is impractical. It is for this reason that we investigate the other methods for computational purposes.

By replacing the left endpoint method with the trapezoid method in the computation in Figure 6.1 we see a remarkable difference. The accuracy now appears to be approximately 2, 4, 6 and 8 digits respectively. Thus, the 4 decimal place accuracy achieved by the left endpoint method using 10,000 rectangles is equivalent to the trapezoid method using only 100 trapezoids.

We can summarize the theoretical error for these methods as follows. It can be shown that error in using the left endpoint method is no greater than

$$\left[\frac{(b-a)^2}{2} \max_{x\in[a,b]} |f'(x)|\right] \frac{1}{n}. \tag{3}$$

On the other hand, the error in using the trapezoid method is no greater than

$$\left[\frac{(b-a)^3}{12}\max_{x\in[a,b]}|f''(x)|\right]\frac{1}{n^2}. \tag{4}$$

In our example (with $f(x) = 1/x$, $a = 1$ and $b = 2$) we have the bracketed quantity in (4) is equal to $1/6$ so that the error is no greater than $n^{-2}/6$. Thus, $n = 100$ indeed yields an error of less than .00002 or approximately 4 decimal digits. You might want to modify the previous table we did in Derive to add another column displaying this theoretical error estimate (3) (and (4) for the trapezoid method) and compare it to the actual error. Although the trapezoid method is quite accurate and fairly efficient, the Simpson's Rule is vastly more efficient. The error in using the Simpson method is no greater than

$$\left[\frac{(b-a)^5}{180}\max_{x\in[a,b]}|f^{(4)}(x)|\right]\frac{1}{n^4}. \tag{5}$$

Notice the main difference between (4) and (5) is that we now have an error which is roughly $1/n^4$ (the bracketed quantity in our example is $24/180$). Thus, with $n = 10$ we obtain the same accuracy as $n = 100$ in the trapezoid method or $n = 10,000$ in the left endpoint method. A table illustrating these differences can be obtain by approximating

```
vector([LEFT(1/x,x,10^n,1,2) - LN(2), TRAP(1/x,x,10^n,1,2)
                 - LN(2),SIMP(1/x,x,10^n,1,2) - LN(2)], n,1,4).
```

These functions are available by doing Load/Utility with the file ADD-UTIL.MTH. Seeing the accuracy of SIMP$(1/x, x, 10^4, 1, 2)$ requires 16 digits of accuracy. Recall from Section 0.6 how to increase the accuracy of a calculation.

To get a geometric feeling for why the trapezoid method is so much better than the left endpoint method one need only draw a sketch comparing the two methods. It's possible to graphically represent these approximations using Derive. Recall from Chapter 4 that one can plot a collection of points, (x_1, y_1), (x_2, y_2), …, (x_n, y_n), by plotting an $n\times 2$ matrix. Thus, a rectangle can be drawn by plotting a 5×2 matrix. (Note: The 5th point is the same as the first point so that the figure is closed.) In order to draw n rectangles, one plots an n-vector with entries corresponding to each of the rectangles. This vector resembles a $5 \times 2n$ matrix but in facts its a vector with matrix entries. To generate this vector use the function `DRAW_LEFT(u,x,n,a,b)` for the left endpoint method and `DRAW_TRAP(u,x,n,a,b)` for the trapezoid method. Both of these are defined in ADD-UTIL.MTH.

Figure 6.2 illustrates both of thes. One must zoom in a bit to see that the trapezoid is actually different from the original curve (even for $n = 4$).

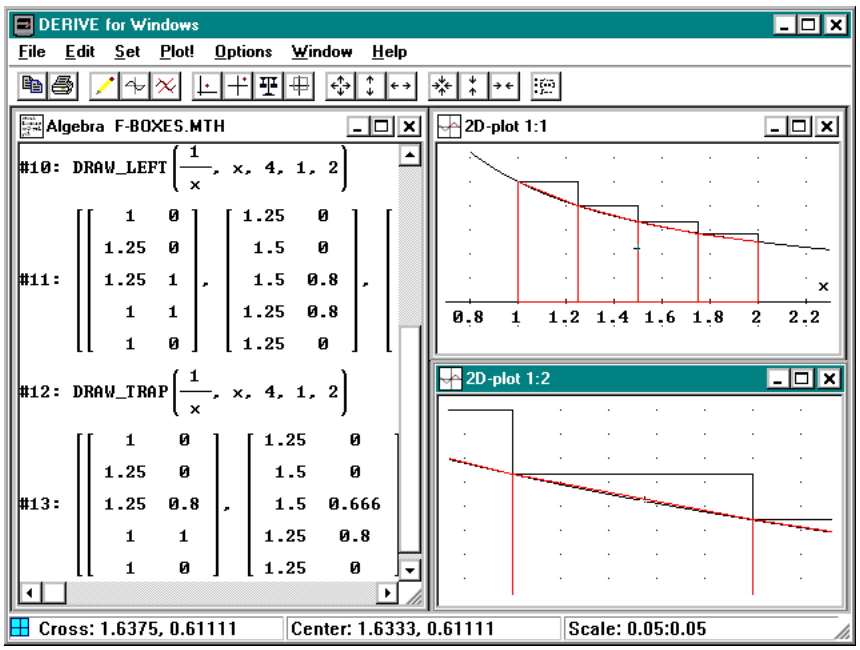


Figure 6.2: Rectangular vs trapezoidal approximation

6.3 Theorem on Error Estimates*

Let us indicate how one obtains some of these error estimates by proving the following theorem:

Theorem 1. *Suppose that $f(x)$ is a continuous function on the interval $[a, b]$. The following hold:*

(a) *If $f'(x)$ is bounded on the interval $[a,b]$, then the error in approximating $\int_a^b f(x)\,dx$ with* $\mathrm{LEFT}(f(x),x,n,a,b)$ *is proportional to $1/n$.*

(b) *If $f''(x)$ is bounded on the interval $[a,b]$, then the error in approximating $\int_a^b f(x)\,dx$ with* $\mathrm{TRAP}(f(x),x,n,a,b)$ *is is proportional to $1/n^2$.*

(c) *Finally, if $f^{(4)}(x)$ is bounded then the error in approximating the integral using Simpson's Rule* $\mathrm{SIMP}(f(x),x,n,a,b)$ *is proportional to $1/n^4$.*

Proof. We'll prove parts (a),(b) and leave (c) to a more advanced text. We first show that the error obtained by approximating a function $f(x)$, over the k^{th} sub-interval $[x_{k-1},x_k]$, by the constant $f(x_{k-1})$ is proportional to $1/n$. (x_k is defined by (1).) This estimate uses the Mean Value Theorem as follows: for $x_{k-1}\le x\le x_k$ we have

$$|f(x)-f(x_{k-1})| = |f'(c_x)(x-x_{k-1})| \le \max_{x\in[a,b]}|f'(x)|\frac{(b-a)}{n}.$$

This bounds how much $f(x)$ and $f(x_{k-1})$ can differ for x between x_{k-1} and x_k; and this means the error in using the left endpoint estimate for the strip between x_{k-1} and x_k is at most the width of the strip, $(b-a)/n$, times this bound. Adding this over all n strips gives

$$\left|\int_a^b f(x)\,dx - \mathrm{LEFT}(f(x),x,n,a,b)\right| \le \left[(b-a)^2\max_{x\in[a,b]}|f'(x)|\right]\frac{1}{n}$$

which is the desired result. This completes the proof of part (a).

The proof of part (b) is similar except it uses the Mean Value Theorem three times. We estimate the error from approximating $f(x)$ by the linear function obtained from the endpoints values $f(x_{k-1})$ and $f(x_k)$. Thus, for $x_{k-1}\le x\le x_k$ we have

$$\begin{aligned}
&\left|f(x)-\left[\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}\cdot(x-x_{k-1})+f(x_{k-1})\right]\right| \\
&\quad= \left|(f(x)-f(x_{k-1}))-\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}\cdot(x-x_{k-1})\right| \\
&\quad= |f'(c_1)(x-x_{k-1})-f'(c_2)(x-x_{k-1})| = |f''(c_3)||c_1-c_2||x-x_{k-1}| \\
&\quad\le \max_{x\in[a,b]}|f''(x)|\left(\frac{b-a}{n}\right)^2
\end{aligned}$$

and thus at each point the error is proportional to $1/n^2$ and so is the integral over $[a, b]$. More precisely,

$$\left| \int_a^b f(x)\, dx - \text{TRAP}(f(x), x, n, a, b) \right| \leq \left[(b-a)^3 \max_{x \in [a,b]} |f''(x)| \right] \frac{1}{n^2}$$

□

We note that the error estimates above differ from (3) and (4) only in the constant term and not the power of n. To obtain the better constant more careful estimation needs to done in the above argument. On the other hand, the constants obtained above suffice for most applications.

6.4 More on Error Estimates*

In order for any method of approximation to be useful we must know something about the error. The error estimates given in equations (4) and (5) usually work quite well. But they do require certain boundedness assumptions which are not always true. Consider

$$\int_0^1 \frac{dx}{1 + x^{3/2}} \tag{6}$$

Use DERIVE to graph $g(x) = 1/(1 + x^{3/2})$. Notice that the graph is pretty tame; there are no wild oscillations and it would appear that the trapezoid method could be used to obtain a good approximation of (6). In fact it does give a good approximation.

In order to use (4) to estimate the error in using the trapezoid rule to evaluate (6) we need to find g''. Use DERIVE to do this. Note that $g''(0)$ is undefined; but that $\lim_{x \to 0^+} \sqrt{x} g''(x) = -\frac{3}{4}$. This means that $g''(x) \approx -\frac{3}{4} x^{-1/2}$ and hence is not bounded on $[0, 1]$ so that (4) gives us no information about the error.

We can work around this problem by noticing that for each n we can apply (4) to the interval $[\frac{1}{n}, 1]$ instead and use a different technique for that first interval. Thus, using $|g''(1/n)|$ for the maximum on $[1/n, 1]$ (check that this is valid for all large n), we obtain from (4) that

$$\left| \int_{1/n}^1 g(x)\, dx - \text{TRAP}(g(x), x, n-1, 1/n, 1) \right| \leq c\sqrt{n} \cdot \frac{1}{(n-1)^2} \approx \frac{c}{n^{3/2}}.$$

On the small interval we observe that $g(x)$ is decreasing for $x > 0$ and that $g(0) - g(x) = x^{3/2}/(1 + x^{3/2}) \leq x^{3/2}$. Thus, by comparing areas we see that

$$\left| \int_0^{1/n} g(x)\,dx - \mathrm{TRAP}(g(x), x, 1, 0, 1/n) \right| \leq \frac{1}{n}(g(0) - g(\frac{1}{n})) \leq \frac{1}{n^{5/2}}.$$

Combining these estimates shows that the error obtained using the trapezoid method is proportional to $n^{-3/2}$ (which is the larger of the two errors). This is a better result than $1/n$ but not as good as $1/n^2$. Actually, one can improve the 3/2-power a little by refining these estimates.

The next question is what can you do without explicit estimates like the above but only using monotonicity or convexity of the graph. If f is increasing on $[a, b]$ notice that the left endpoint method of estimating $\int_a^b f(x)\,dx$ always underestimates the integral while the right endpoint method overestimates it. Similarly, if f is decreasing the opposite inequalities hold. If we let $\mathrm{LEFT}(f(x), x, n, a, b)$ and $\mathrm{RIGHT}(f(x), x, n, a, b)$ be the left and right endpoint estimates then:

$$\mathrm{LEFT}(f(x), x, n, a, b) \leq \int_a^b f(x)\,dx \leq \mathrm{RIGHT}(f(x), x, n, a, b) \quad \text{if } f'(x) \geq 0 \text{ on } [a, b] \tag{7}$$

and

$$\mathrm{RIGHT}(f(x), x, n, a, b) \leq \int_a^b f(x)\,dx \leq \mathrm{LEFT}(f(x), x, n, a, b) \quad \text{if } f'(x) \leq 0 \text{ on } [a, b] \tag{8}$$

See Figure 9.2 on page 141 which makes these relations quite obvious.

A similar relation holds between the trapezoid and midpoint methods but depends on the concavity, i.e., the second derivative of f rather than the slope, i.e., the first derivative of f. If we let $\mathrm{TRAP}(f(x), x, n, a, b)$ and $\mathrm{MID}(f(x), x, n, a, b)$ be the trapezoid and midpoint estimates then

Theorem 2. *If f is concave up on $[a, b]$, i.e., $f''(x) \geq 0$, then*

$$\mathrm{MID}(f(x), x, n, a, b) \leq \int_a^b f(x)\,dx \leq \mathrm{TRAP}(f(x), x, n, a, b)$$

If f is concave down on $[a,b]$, i.e., $f''(x) \le 0$, then

$$\mathrm{TRAP}(f(x), x, n, a, b) \le \int_a^b f(x)\,dx \le \mathrm{MID}(f(x), x, n, a, b)$$

Figure 6.3 shows why this is true. It has two graphs of the same function which is concave up. the line in the left part shows the trapezoid used in the trapezoid rule. Clearly it overestimates the integral. The midpoint rule is illustrated in the right graph. The midpoint rule gives the area under the line AB. The line CD is the tangent line through the midpoint. The area below AB is the same as the area below CD (why?). So both are the midpoint estimate. But clearly the area under CD is less than the area under the curve.

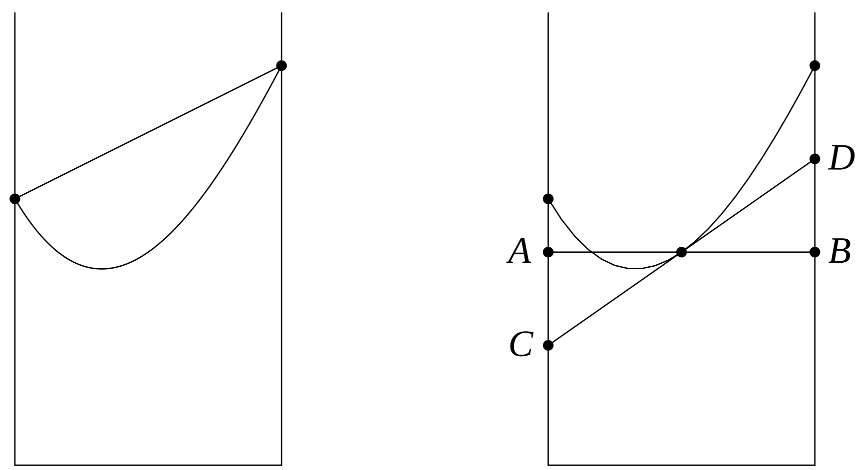


Figure 6.3: Trapezoid and midpoint rule for concave functions

6.5 Deriving Simpson's Rule*

Simpson's Rule uses the quantity

$$\frac{1}{6}\left[f(x_1) + 4f(\frac{x_1+x_2}{2}) + f(x_2)\right](x_2 - x_1) \qquad (9)$$

to approximate $\int_{x_1}^{x_2} f(x)\,dx$. The can be derived by solving for the quadratic $g(x) = ax^2 + bx + c$ which passes through the 3 points (x_1, y_1), (x_2, y_2) and (x_3, y_3); where $y_i = f(x_i)$ and $x_3 = (x_1 + x_2)/2$ which is simply the midpoint or average of x_1 and x_2. One then computes $\int_{x_1}^{x_2} g(x)\,dx$ and uses this for our approximation. Now the algebra involved in this computation is fairly formidable and yet the beauty of it is that the answer (given in (9)) is so simple. That's why the formula for Simpson's Rule looks hardly any different from the formula for the left endpoint rule and as a result the computation times are approximately the same.

Now the algebra involved is the same as that of Chapter 4. We solve 3 equations for the unknowns a, b and c, then we integrate the result. Alternately, we can make the derivation into a two step process by using the function `CURVEFIT(x,data)` where the data matrix is

$$\text{data} := \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \frac{x_1+x_2}{2} & y_3 \end{bmatrix}$$

The resulting quadratic polynomial contains some pretty large expressions involving x_i and y_i. Nevertheless, one need only integrate this expression over the interval $x_1 \leq x \leq x_2$ to get the desired result.

6.6 Laboratory Exercises

Start off your lab by Loading the ADD-HEAD.MTH file and simplifying the `LOAD("add-util")` expression[1]. After you have done this the functions described in Section 6.2: LEFT, MID, TRAP, and SIMP, which compute the integral approximations using respectively the left endpoint method, the midpoint method, the trapezoid method and Simpson's rule, will all be defined. In addition the functions DRAW_LEFT and DRAW_TRAP, which draw the rectangles and trapezoids used in the graphical demonstration of Figure 6.2 on page 92, will be defined.

1. Evaluate TRAP$(1/x, x, n, 1, 2)$ and SIMP$(1/x, x, n, 1, 2)$ for $n = 10$, 100, and $1,000$. Also use DERIVE to find $\ln 2$ using 15 decimal place

[1]This expression must evaluate to TRUE or else something is wrong. Typically, if it evaluates to FALSE then the file is not in the default directory.

precision. Which of the 6 approximations above (if any) gives $\ln 2$ to 10 or more decimal places?

2. Use the trapezoid method and Simpson's Rule to approximate each of the following integrals. Use $n = 10$, 20, and 30. DERIVE has its own method of doing approximate integration. Find the answer it gets. You can do this by authoring the integral and choosing approximate. Compare the decimal accuracy of the Simpson approximates with the one computed by DERIVE above.

a. $\displaystyle\int_{-1}^{1} \sin(x^2)\,dx$ **b.** $\displaystyle\int_{0}^{3} \frac{1}{1+x^3}\,dx$

3. Using the midpoint method $\mathrm{MID}(1/x, x, n, a, b)$, approximate $\ln 10$ ($= \int_1^{10}(1/x)dx$) using $n = 10$, 100, 1000 and compare your answers with DERIVE's approximation.

4. For the following integrals use the error estimate (4) described above to find an n large enough so that the trapezoid method will give an approximation of the integral with error at most 0.005. Give both the approximate value of the integral and the smallest n which guarantees (using formula (4)) that you will be within this error, and also give $M_2 = \max\{|f''(x)| : a \le x \le b\}$.
Hints: Use DERIVE to find f', f'', and f'''. For the first integral below, you can easily see that the maximum for $|f''|$ occurs when $x = 1$. For the second, solve $f'''(x) = 0$; this tells you where the maximums of $|f''(x)|$ can occur, and, using this (and maybe some plotting), you can find M_2. For the third integral don't forget that M_2 if the maximum of the **absolute value** of $f''(x)$ on $[0, 2]$. Once you have M_2, find n large enough so that the error given in (4) is at most 0.005.

a. $\displaystyle\int_{1}^{e^2} \ln x\,dx$ **b.** $\displaystyle\int_{\frac{1}{2}}^{2} \frac{1}{1+x^2}\,dx$

c. $\displaystyle\int_{0}^{2} \frac{1}{1+x^2}\,dx$

5. Do the same as the last problem, but use Simpson's Rule this time and of course use formula (5) instead of (4).

***6.** Explain why the area below AB is the same as the area below CD in Figure 6.3.

7. Find where $1/(1+x^{3/2})$ is concave up and where it is concave down in the interval $[0,1]$. Use Theorem 2 to give lower and upper estimates for

$$\int_0^1 \frac{dx}{1+x^{3/2}}.$$

Use $n = 20$.

8. Prove the following simple relationship between the trapezoid, midpoint, and Simpson's rules:

$$\mathrm{SIMP}(f(x), x, n, a, b) = \frac{1}{3}\,\mathrm{TRAP}(f(x), x, n, a, b) + \frac{2}{3}\,\mathrm{MID}(f(x), x, n, a, b)$$

Hint: First define $f(x)$ to an unspecified function by Authoring `f(x) :=`. Now Author and Simplify the two expressions

```
SIMP(f(x),x,n,a,b)
(1/3) TRAP(f(x),x,n,a,b) + (2/3) MID(f(x),x,n,a,b)
```

Finally simplify the difference of the two resulting expressions.

9. Do the calculations needed to verify Simpson's rule as outlined in Section 6.5. This is the same problem as Exercise 4 in Chapter 4.

Chapter 7

Exponential Growth and Differential Equations

7.1 Introduction

Suppose that y is a function of x. A first order differential equation is an equation which involves x, y and its derivative y'. An n^{th} order differential equation involves $x, y, y', \ldots, y^{(n)}$. For example, $y'' + xy' = x^2 + 1$ is a second order differential equation.

Differential equations occur frequently in every field of science and engineering, especially biology. Libraries have many volumes devoted to solving differential equations (even for first order differential equations). In this chapter we study first order differential equations and show some of the applications. One of the most important examples is *population growth* (of humans, cells, radioactive substances, savings account balances, etc.)

We will show you how to get an exact solution to what are known as linear first order differential equations and we will introduce slope fields and Euler's method for obtaining approximate solutions to more general first order differential equations.

7.2 Examples

Population Growth. The standard model for population growth states that the rate of change $y'(x)$, of the population size $y(x)$, with respect to time x is proportional to the population size at any given time. This means that $y'(x) = ky(x)$ for some fixed constant k and all x. Now it is easy to check that $y(x) = y_0 e^{kx}$ satisfies the relations

$$y' = ky \qquad y(0) = y_0 \tag{1}$$

where we simplify our notation by dropping the explicit reference to the variable x. Thus, the exponential function provides a model for population growth. Recall from Problem **7** on page 64 that we compared a linear model versus the exponential model above for the population of Mexico and found a significant difference in the long run behavior with the exponential model giving a much larger growth. This comparison was also made in Section 3.3 where it was shown that exponential growth eventually exceeds the growth of any polynomial.

Now it turns out that the exponential solution to equation (1) is the *only* solution to that equation. To prove this we suppose that $u(x)$ is any solution to (1). We need to show that $u(x)/e^{kx} = e^{-kx}u(x)$ is a constant, so we compute it's derivative and observe that

$$(e^{-kx}u)' = e^{-kx}u' - ke^{-kx}u = e^{-kx}(u' - ku) = 0 \tag{2}$$

holds for all x. Hence integrating gives

$$e^{-kx}u(x) = c.$$

We solve for the constant c by substituting $x = 0$ in the above to get $c = u(0) = y_0$ and then multiply both sides by e^{kx} to obtain

$$\boxed{u(x) = y_0 e^{kx}}$$

as we claimed[1].

Equation (1) is a special case of the general equation

$$\boxed{y' + p(x)y = q(x), \qquad y(x_0) = y_0} \tag{3}$$

since (1) can be written as $y' - ky = 0$. Thus, in (3) the functions $p(x) = -k$, $q(x) = 0$ and initial time $x_0 = 0$. Any differential equation with the form of (3) is called a *linear first order differential equation*. In Section 7.5 we

[1]In DERIVE multiply the *equation* $e^{-kx}u(x) = c$ by e^{kx} by right clicking and inserting (or press the F4-key). Then, simplify.

prove that any such equation has a unique solution which is obtained in manner a similar to the above. See Theorem 1 in that section for the formula for the solution.

The formula for the solution to (3) can be made into a DERIVE function quite easily. This has been done in DERIVE's utility file ODE1.MTH with the name `LINEAR1`. For convenience we have added this function to our utility file ADD-HEAD.MTH but we use the shorter name `DE`. It has the form

```
DE(p, q, x, y, x0, y0)
```

where p and q are expressions in the variable x. The initial conditions are $y = \mathrm{y0}$ when $x = \mathrm{x0}$. For example, simplifying `DE(-2,0,x,y,0,5)` would yield the expression $y = 5e^{2x}$. This is the solution to $y' = 2y$ where $y(0) = 5$; see Figure 7.1 on the next page.

Newton's Law of Cooling. Another important example of differential equations is *Newton's Law of Cooling*. According to this law a hot pan of temperature y_{hot} will have a temperature of $y(t)$ at time t which decreases, i.e., will cool down, when placed in a vat of cool water of temperature $y_{\text{cool}} < y_{\text{hot}}$. The key point of the law is that the rate of change in the temperature, y', is proportional to $y(t) - y_{\text{cool}}$, which is the difference in the current temperature of the (hot) pan and the (cool) water. This says that

$$y' = -k(y - y_{\text{cool}}) \qquad \text{where} \qquad y(0) = y_{\text{hot}} > y_{\text{cool}} \tag{4}$$

and $k > 0$ is a constant which depends on the physical properties of the pan, for example, copper cools faster than iron so the corresponding k-value would be larger. Notice that the derivative y' above is negative since the temperature is decreasing.

We rewrite equation (4) so that it has the form of the general first order linear differential equation in (3):

$$y' + ky = ky_{\text{cool}} \qquad y(0) = y_{\text{hot}} \tag{5}$$

and thus we can solve this equation with DERIVE by using the `DE` function. We will use the variables `yh` and `yc` in place of y_{hot} and y_{cool}. So that these are treated as single variables (and not as $y \cdot h$) we first Author the vector `[yh:=,yc:=]`. Then we Author

```
de(k,k*yc,t,y,0,yh)
```

Simplifying the expression gives the solution

$$\boxed{y(t) = (yh - yc)e^{-kt} + yc.} \tag{6}$$

See Figure 7.1 for a demonstration of these functions and observe how rapidly the temperature $y(t)$ tends to the water temperature yc. Use DERIVE to calculate $\lim_{t\to\infty} y(t)$.

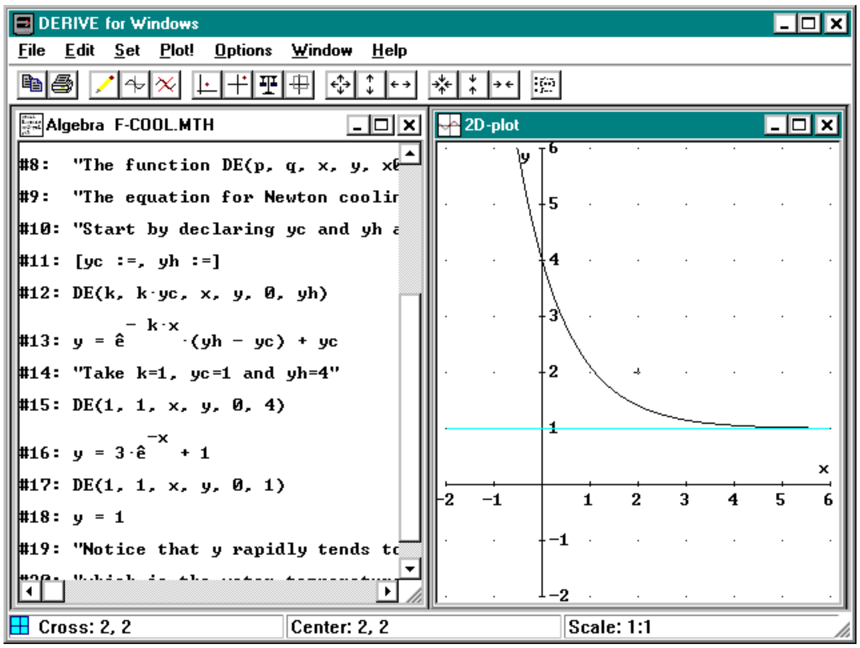


Figure 7.1: Solving Newton's cooling equation

By looking back at (2) on page 102 and making a small modification of that argument we see how the above solution is derived; namely, by (5),

$$(e^{kx}y)' = e^{kx}y' + ke^{kx}y = e^{kx}(y' + ky) = y_{\text{cool}}ke^{kx} \tag{7}$$

and hence integrating gives

$$e^{kx}y(x) = \int y_{\text{cool}}ke^{kx}\,dx = y_{\text{cool}}e^{kx} + c.$$

Now solving for c by substituting $x = 0$ in the above yields $c = y_{\text{hot}} - y_{\text{cool}}$ and then multiplying both sides by e^{-kx} gives the desired result above.

The argument above gives us a pretty good idea how to solve the general differential equation $y' + p(x)y = q(x)$. One multiplies y by an appropriate exponential μ, differentiates and then replaces the quantity $y' + p(x)y$ by $q(x)$. Integrating the result essentially solves the problem. That critical multiplying exponential turns out to be

$$\mu(x) = e^{\int p(x)\,dx} \qquad \text{since} \qquad \mu'(x) = p(x)\mu(x).$$

See the proof of Theorem 1 on page 110 for the details.

Radioactive Decay. In certain radioactive materials some particles change from one form to another. The number of particles decaying in this way in a small time period is proportional to the size of the material. So that, for example, if you have twice as much radioactive material the number of particles decaying is twice as great. If $A(t)$ is the amount of radioactive material at time t, then A satisfies the differential equation $A'(t) = -kA$. Here we have $k > 0$ and have written it in this way to emphasize that the derivative is negative since the amount of material decreases with time. Except for this minus sign this is the same as the population model above. It is easy to see the solution is

$$\boxed{A(t) = A_0 e^{-kt}} \tag{8}$$

The *half-life* of a radioactive substance is the time it takes for half of it to decay. We can find this by solving $A(t) = A_0/2$ for t. By (8) this gives the equation $A_0/2 = A_0 e^{-kt}$. Cancelling the A_0, we get the equation $1/2 = e^{-kt}$, so that $-kt = \ln(1/2) = -\ln 2$ or $t = \ln 2/k$. Notice that the half-life is independent of A_0.

Observe that we can compute the solution above with DERIVE using the variables t and a. We Author `a0:=` (to declare it as a multi-letter variable) then Author `DE(k,0,t,a,0,a0)` and simplify to get $a = a_0 e^{-kt}$.

7.3 Approximation of Solutions

The general first order differential equation has the form $y' = f(x, y)$ with initial conditions (x_0, y_0), i.e., $y = y_0$ when $x = x_0$. The techniques for solving differential equations that we discussed in the previous sections and which are used to prove Theorem 1 on page 110 do not extend to all differential equations. In fact, many important differential equations cannot be *solved* explicitly. We encountered this situation earlier with integrals[2] and this suggests trying to find numerical approximations to the solution. The critical observation to make is that the equation $y' = f(x, y)$ tells us the *slope* of the tangent line to the solution $y(x)$. Thus, by drawing many small line segments of slope $f(x, y)$, through the point (x, y) in the plane, we obtain an approximate picture of the solution whose graph contains the point (x, y). By drawing several of these partial tangent lines we get an approximate picture of $y(x)$ by drawing a curve which conforms to these slopes. These diagrams are called *slope* or `direction fields`.

The file ADD-UTIL.MTH has the function `DF` (for direction field) which will make a matrix. When this matrix is plotted it draws the 'slope field.' The form of `DF` is

```
DF(r,x,x0,xm,m,y,y0,yn,n)
```

where the first argument `r` is $f(x, y)$ and `x0, xm, m` represent the initial and final x-values in a rectangular grid with m x-values plotted. Similarly, `y0, yn, n` represent the initial and final y-values in a rectangular grid with n y-values. Hence, the total number of line segments plotted will be $m \cdot n$. In order that line segments are plotted, not just the endpoints, we put the plotting window into connected mode by choosing Options/Points and setting Connect to 'Yes.'

As an example, we can take the cooling problem above, namely,

$$y' + y = 1$$

so that $f(x, y) = -(y - 1) = 1 - y$. We simplify the expression

```
df(1-y,x,0,4,8,y,0,4,8)
```

to get the slope field. In the plot window select Option/Plot Color and set it to 'Off' so that all slope lines will be in one color. Of course, if you like colorful diagrams then you can skip that last step. Also choose Option/Points to set the Connected Mode and to set Size to Small. Make sure to delete all existing graphs and then plot the slope field. Try to picture the solution though a given initial point $(0, y_0)$ by following the slope field. Finally, plot some actual solutions that we obtained above using the `DE` function and see

[2]Notice that the simple differential equation $y' = f(x)$ has solution $y = \int f(x)\,dx$ so that the class of differential equations contains all integration problems.

how it conforms to the slope field. See Figure 7.2 where we have graphed the solution $y = 3e^{-x} + 1$, which corresponds to the initial condition $y(0) = 4$. Try several other initial conditions to see how the slope lines approximate the solution.

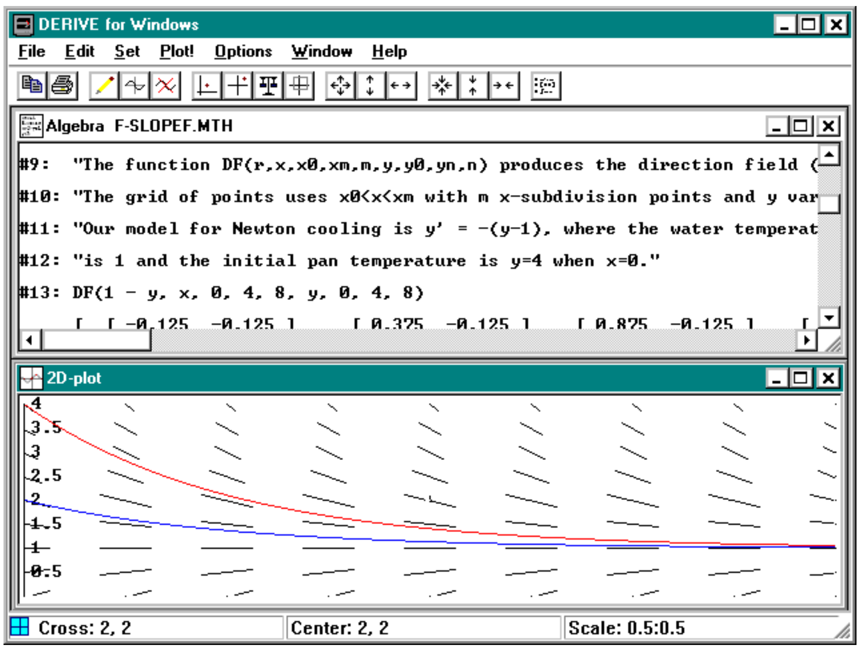


Figure 7.2: Slope field for the Newton cooling problem

Another Population Model.* In the model we used for population growth we had

$$\frac{dP}{dt} = kP.$$

This works well for many populations. But the population cannot continue to grow forever. When a country no longer has room for expansion the rate of growth slows. For example, a bacteria culture in a petri dish will satisfy the above differential equation for awhile, but as the dish fills the above equation becomes invalid. Verhulst, a Belgian mathematician, proposed a model using the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{P_1}\right). \tag{9}$$

Notice that when P is small compared to P_1, the derivative is approximately kP, as before. But as P approaches P_1, P' approaches 0.

Unfortunately this equation is not of the form of (3) so no exact solution is apparent. But we can always look at the slope field to get an approximate idea as to what the solution looks like. To see a demonstration we Load the file F-VERHUL.MTH and look it over line by line. The given example

$$\frac{dP}{dt} = P\left(1 - \frac{P}{5}\right) \qquad \text{where} \qquad P(0) = 1.$$

has the slope field function entered on line 6 with the above equation entered. You should highlight this expression, press $\boxed{\approx}$ and then plot the resulting data matrix. Now starting at the initial point $(0, 1)$ *follow the slopes field* with your finger to get an approximate solution. See Figure 7.3 for a graph of the exact solution to the above along with some of the derivation below.

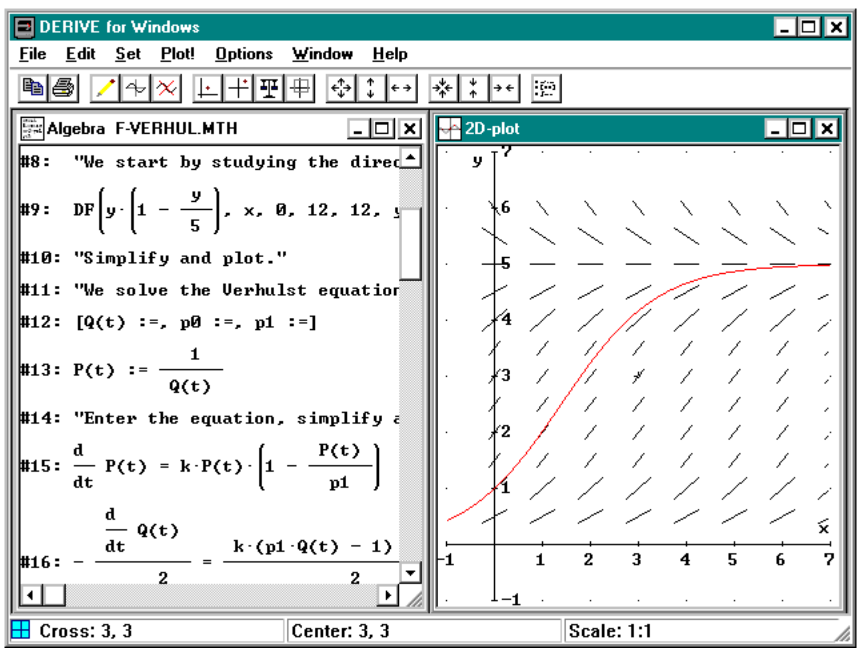


Figure 7.3: A graph of a Verhulst population curve

Even though the Verhulst equation is not of the form of (3) we can still solve the equation exactly provided we solve for $1/P$ instead P. If we let

$R = 1/P$ (R stands for reciprocal) then

$$\frac{dP}{dt} = \frac{d(1/R)}{dt} = -\frac{1}{R^2}\frac{dR}{dt}$$

and (9) becomes

$$\begin{aligned}\frac{dR}{dt} &= -R^2\frac{k}{R}\left(1 - \frac{1}{P_1 R}\right)\\ &= \frac{k}{P_1} - kR\end{aligned}$$

This is of the form of (3) with $p(x) = k$ and $q(x) = k/P_1$, so we can solve it by Authoring `DE(k,k/P1,t,R,0,1/P0)`. ($P_0 = P(0)$ is the initial population so $R_0 = 1/P_0$.) This gives the solution

$$R = e^{-kt}\left(\frac{1}{P_0} - \frac{1}{P_1}\right) + \frac{1}{P_1}$$

Inverting R gives

$$P = \frac{1}{R} = \frac{P_0 P_1 e^{kt}}{P_0 e^{kt} - P_0 + P_1}$$

or

$$\boxed{P = \frac{P_0 P_1}{P_0 + (P_1 - P_0)e^{-kt}}} \tag{10}$$

Notice that $P(0) = P_0$, as we would expect, and that $\lim_{t\to\infty} P(t) = P_1$.

7.4 Euler's Approximation Method*

The method of slope fields suggest an approximation technique known as *Euler's method*. The idea is to approximate the solution to

$$y' = f(x, y) \quad \text{where} \quad y(x_0) = y_0$$

by a piecewise linear function passing through a sequence of points

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$$

obtained by using the slope at (x_{i-1}, y_{i-1}), which is $f(x_{i-1}, y_{i-1})$, to construct the next point (x_i, y_i), where the increment in x is a fixed amount, say $x_i = x_{i-1} + h$. The DERIVE function is deceptively simple:

```
EULER(r,x,y,x0,y0,xn,n):=
   ITERATES(v+[1,LIM(r,[x,y],v)]*(xn-x0)/n,v,[x0,y0],n)
```

where x varies between $x_0 \le x \le x_n$ and we use n points in the approximation scheme. This function is also in the file ADD-UTIL.MTH. It is a slight variant of the `EULER` function in the utility file ODE_APPR.MTH that comes with DERIVE. Try the Newton cooling problem

```
EULER(1-y,x,y,0,4,4,16)
```

to see how this works. Again, you must be sure that your graphics window is in single color, connected mode for this to plot properly. See the Figure 7.4 on the next page for a demonstration of this technique. You should try larger and larger n to see that the approximations converge, as $n \to \infty$, to the solution for $0 \le x \le 4$.

7.5 Linear First Order Differential Equations

In this section we solve the linear first order differential equation

$$y' + p(x)y = q(x) \qquad \text{with} \qquad y(x_0) = y_0 \tag{11}$$

by proving the following theorem:

Theorem 1. *Suppose that $y(x)$ satisfies (11) where $p(x)$ and $q(x)$ are continuous functions of x. If y satisfies the initial condition $y(x_0) = y_0$ then*

$$y = e^{-\int_{x_0}^{x} p(u)\,du} \cdot \left(\int_{x_0}^{x} q(u) e^{\int_{x_0}^{u} p(v)\,dv}\,du + y_0 \right). \tag{12}$$

Proof. Let $h(x) = e^{\int_{x_0}^{x} p(u)\,du}$. By the fundamental theorem of calculus, $\frac{d}{dx}\int_{x_0}^{x} p(u)\,du = p(x)$. So $h'(x) = \frac{d}{dx}\left[e^{\int_{x_0}^{x} p(u)\,du}\right] = p(x)h(x)$. Thus

$$(h(x)y)' = h(x)y' + h'(x)y = h(x)y' + p(x)h(x)y$$

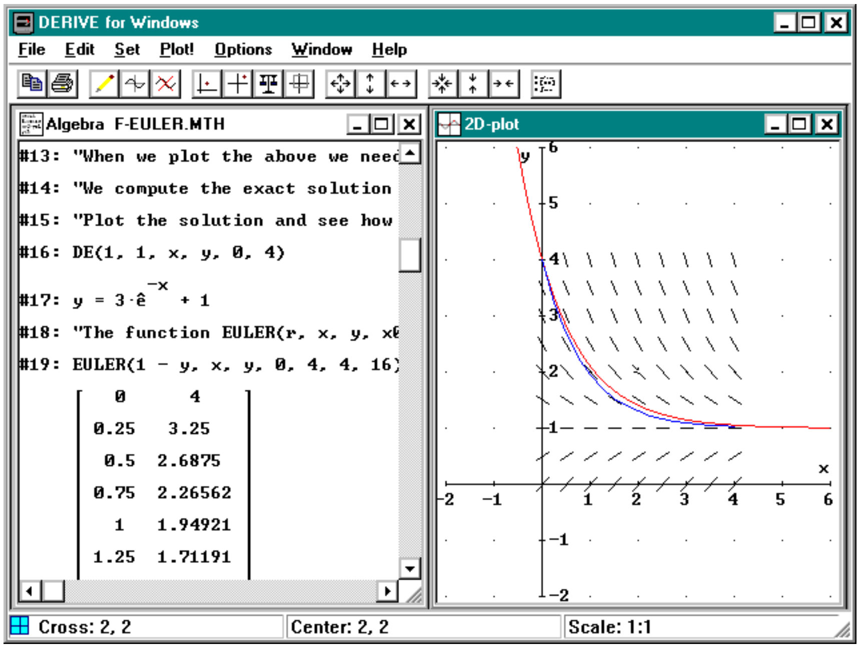


Figure 7.4: Euler's method for approximating solutions

If we multiply equation (11) by $h(x)$ and use the above, we see that $(h(x)y)' = h(x)q(x)$. If we integrate both sides of this from x_0 to x and use the fact that $h(x_0) = 0$, we get $h(x)y = \int_{x_0}^{x} h(u)q(u)\,du + C$, or

$$y = e^{-\int_{x_0}^{x} p(u)\,du} \cdot \left(\int_{x_0}^{x} q(u) e^{\int_{x_0}^{u} p(v)\,dv}\,du + C \right).$$

Since $y(x_0) = y_0$, we see that $C = y_0$ and thus (12) holds. $\square$

As we said, the solution (12) to the differential equation can be made into a Derive function quite easily. You should look at the formula above and see if you can write a Derive function that will produce the solution. Then compare your answer with the following definition of the function DE.[3]

```
(13) de(p,q,x,y,x0,y0):= y =
       ê^(-int(p,x,x0,x)) * (int(q*ê^int(p,x,x0,x),x,x0,x)+y0)
```

7.6 Laboratory Exercises

The functions discussed in this chapter, DE, DF, and EULER, are all defined in the file ADD-UTIL.MTH. Be sure to File/Load/Math this file.

1. If money earns interest compounded continuously and $y(t)$ is the amount of money at time t, then y satisfies the differential equation $y' = ry$, where r is the interest rate.

   a. What is the solution to the differential equation $y' = ry$?

   b. Find how long it takes for your money to double for $r = 3\%$, 5%, and 10%? (This means that $r = 0.03$, 0.05, and 0.1 in the above equation.)

2. Normal body temperature is 98.6°F. If someone dies, then the body cools according to Newton's law of cooling. It is known that, if the surrounding temperature is a constant 64°, then the body will cool to 92° in 3 hours.

   a. Use this information to compute the constant k in (6) on page 104.

---

[3] You might notice that the formula for the solution to the differential equation in Theorem 1 is careful about the "dummy variables" in the integrals. This is because in calculus we avoid integrals of the form $\int_a^x f(x)\,dx$ because the integration variable x might be mistaken for the upper limit x. Since the integration variable is completely arbitrary we usually take it to be t or u in such a situation. On the other hand, for the DERIVE function `de` above we used expressions like `int(f(x),x,0,x)` because the integration is done before the limits of integration are substituted. The computer does this correctly but it is usually foolhardy for students to try this since it is so easy to make mistakes such as

$$\int_0^x x\,dx = x\int_0^x dx = x^2$$

when then answer should be $x^2/2$.

b. Now suppose that a murder victim's body is found at 12AM with a temperature of 86°. Assuming an air temperature of 64°, determine when the murder was committed?

3. Consider the differential equation

$$y' + y = \sin x \qquad \text{with} \qquad y(-1) = -1.$$

a. Use the `DF` function to draw a 10×10 grid of slope lines using $-1 \le x \le 3$ and $-1 \le y \le 1$. (You need to have the graphics window in the connected state; see the instructions for this on page 106.)

b. Now use the `DE` function to find the solution to the differential equation and plot the answer to see how it conforms to the slope lines.

c. Double check that the answer you get from the `DE` function is indeed the solution by verifying that it solves the differential equation and the initial conditions.

4. Suppose a body of mass m is dropped from high in the atmosphere. Let v be its downward velocity as a function of time t. There are two forces acting on the body: gravity and wind resistance. The force due to gravity is mg, where g is a constant; the force due to wind resistance is $-kv$ (the minus since it is upward). Newton's law says $F = ma$, where $a = v'$ is the body's acceleration. This leads to the differential equation

$$ma = mv' = mg - kv.$$

Solve this equation for v with $v(0) = 0$. Find $\lim_{t\to\infty} v(t)$ (don't include the `v=` part from above). DERIVE returns an expression containing `SIGN(km)` because it does not know that k and m are positive. Use Declare/Variable Domain to declare that k is a positive real number, and do the same for m. Now reevaluate the limit. Note that v never exceeds this value, which is called the *terminal velocity*. No wind resistance corresponds to $k = 0$. Find v in this case both by solving the differential equation with $k = 0$, and by taking the limit of the general solution for v found above as $k \to 0$.

***5.** Suppose the population growth of a small country satisfies (9) with $P_1 = 10$ and $k = 0.05$ (with population in millions). Plot the direction field for this. (There are instructions for doing this in Section 7.3.) Suppose $P(0) = 2$. Find $P(20)$, $P(50)$, and $P(100)$. Graph $P(t)$. Adjust the scale of the graph so that you get a clear picture of the nature of the population growth.

6. Carbon-14, ^{14}C, is an unstable isotope of carbon that slowly decays to the more stable ^{12}C. While an organism is alive it has a constant amount of ^{14}C, but after it dies, the amount decreases according to (8). If 200 years after the organism dies, the amount of ^{14}C is 97.6% of the original amount, what is the half-life of ^{14}C? If the burnt wood from a prehistoric campsite contains 29% of the original amount of ^{14}C, how old is the campsite?

Chapter 8

Polar and Parametric Graphs

8.1 Introduction

Graphs of the form $y = f(x)$ or $x = g(y)$ can be used to represent a wide variety of curves in the plane, there are many important curves, such as circles or ellipses, that cannot be represented by a single graph of this type. More generally, imagine the curve traced out by an ant walking on a flat surface. In this chapter we will introduce two techniques for plotting general curves. One is the method of polar coordinates, which is a coordinate system based on angles and distance from the origin. The other is the method of parametric representation, which allows one to specify completely arbitrary curves like the motion of a particles (or the ant).

8.2 Polar Coordinates

We can specify a point in the plane by how far it is from the origin and what angle the line from the point to the origin makes with the x–axis. If r is the distance from the origin and θ is the angle, we say that $[r, \theta]$ are the *polar coordinates* of the point; see Figure 8.1 on the following page. Thus, for example, the point with rectangular coordinates $(1, 1)$ would have polar coordinates $[\sqrt{2}, \pi/4]$. The way to envision plotting a polar point $[r, \theta]$ is to stand at the origin facing out towards the positive x–axis and then turn counter-clockwise by the angle θ and then move r unit in the direction you are now facing. We usually think of r as being nonnegative, but if r is negative, we simply go backwards $|r|$ units. Similarly, we plot negative angles

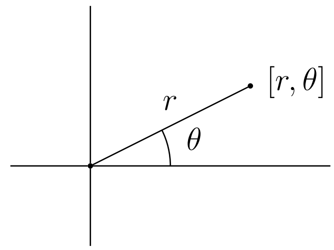


Figure 8.1: Polar Coordinates

by turning clockwise instead of counter-clockwise. This leads to a surprising difference compared to rectangular coordinates; namely, two different polar coordinates can represent the same point. Thus, $[-2, \pi/4] = [2, 5\pi/4] = [2, -3\pi/4]$. Note that $[0, \theta]$ is the origin regardless of what θ is. Your calculus text has a more detailed description of polar coordinates.

A basic problem in polar graphing is to plot a function such as $r = \rho(\theta)$, i.e., plot all points $[r, \theta]$ where r is given by the function $\rho(\theta)$. For example the circle of radius a, centered at the origin, is the graph of $r = a$. Thus, $\rho(\theta) = a$ is a constant functions. Note that to draw this circle in rectangular coordinates you must think of this curve as two graphs, namely, $y = \sqrt{a^2 - x^2}$ *and* $y = -\sqrt{a^2 - x^2}$. This simple example already shows that some curves are more easily represented with polar coordinates.

Let us now try something harder such as $r = 1 + \cos\theta$. One then graphs the curve by computing r for lots of θ's by thinking about the geometry of the angle θ and the value of r. This is usually done with angles such as $\theta = 0$, $\pi/4$, $\pi/2$, $3\pi/4$ and π which corresponds to 45° increments in the angle. By authoring `vector([1+cos θ, θ], θ, 0, π, π/4)` and simplifying this expression gives a table of polar points which can be plotted by hand or as a set of points in DERIVE. We'll need to plot more θ's but this is a start. A nice technique for viewing the data is to use the `APPROX` function to get decimals for the r–values. We then get r as a decimal and θ expressed in the usual radian notation for the angles. See Figure 8.2 on the next page.

DERIVE can plot these points in polar coordinates by selecting the Option menu and then selecting the Polar option on the Coordinates menu. Then, plot the points just as we did in rectangular coordinates by highlighting the matrix of points and clicking the plot button in the graphics window. After plotting these 5 points we try to imagine the rest of the graph by interpolating

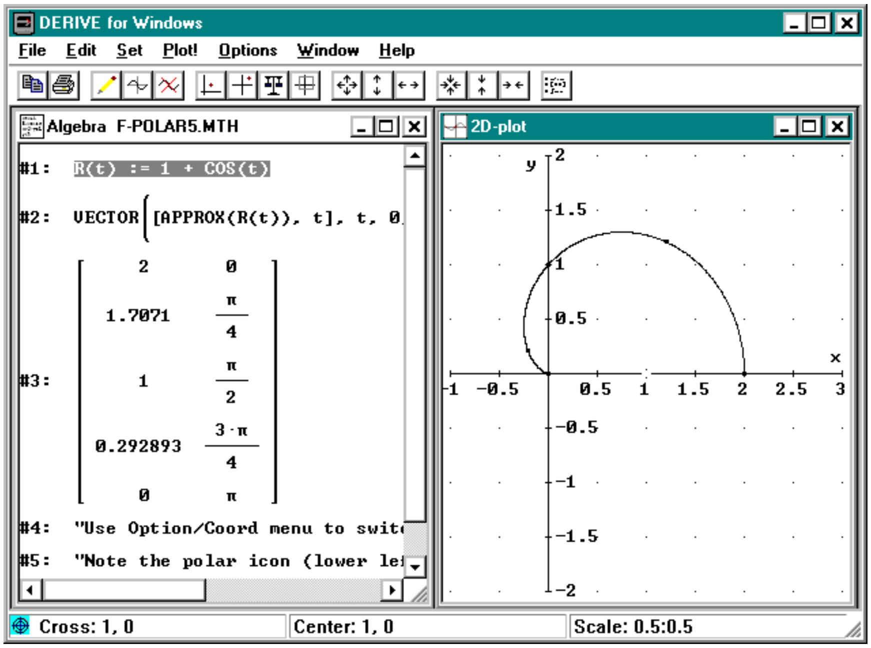


Figure 8.2: Plotting points in polar coordinates

other values of θ. Of course, DERIVE will plot the curve for us. We enter either `1 + cos t` or just highlight expression `#1` in Figure 8.2, then click the Plot button in the graphics window. So far, this is just like rectangular plots except for the change in the coordinate mode. But now DERIVE will prompt you for the parameter interval (interval of θ's to use) and suggest the default range of $-\pi \leq \theta \leq \pi$. Since many of the standard examples of polar curves involve the θ-variable only in the form of either $\cos\theta$ or $\sin\theta$, it usually suffices to only consider $0 \leq \theta \leq 2\pi$ (or as DERIVE prefers $-\pi \leq \theta \leq \pi$). Of course, you can change it to whatever interval you want. For example, in Figure 8.2 the range $0 \leq \theta \leq \pi$ was used. You might want to plot the full graph at this point by using the default range. The resulting graph heart shaped curve is called a *cardioid*.

Tracing. It is important to actually see the curve being plotted but the computer plots so quickly that it is nearly impossible to see it happen. DERIVE has an approach for "driving" around a curve called *tracing*. After plotting the polar curves above select the Trace Mode option on the Options menu (or just press F3 to toggle the Trace mode) and the cross will turn into a box and it will be moved onto the last curve plotted. Now press and hold down the right arrow key and watch the little car drive around the curve. You can see the value of θ, which we can also interpret as time, as it increases, as well as the r and θ coordinates, on the lower part of the screen. If you have more than one graph you can switch between curves by using the up or down arrow keys.

When plotting the cardioid $a = 1$ pay particular attention to the way the plotting slows down as we approach the cusp. It turns out that the only way for cusps or corners to occur in the graph, when $r(\theta)$ is differentiable, is for the plotting to slow to a stop and then to start up again. This notion of speed will be discussed in Section 8.4.

8.3 Rotating Polar Curves

A nice feature of polar coordinates is the ease with which we can rotate a figure. For example, if we plot $r = \rho(\theta)$ and we want to rotate the picture clockwise by an angle α we simply plot $r = \rho(\theta + \alpha)$ instead. Try this out in for yourself using DERIVE.

Here is an interesting application of this idea. Did you know that the curve $y = 1/x$, which is used to define the natural logarithm, is a hyperbola. The equation does not make this apparent since using the usual convention; namely, the axes should be chosen parallel and perpendicular to the axes of symmetry, we are supposed to have the equation of the form:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1.$$

We need to discuss converting polar graphs to rectangular graphs and vice versa. Figure 8.1 on page 116 makes it clear how to do this. The algebraic relationship between the polar coordinates $[r, \theta]$ and the rectangular coordinates (x, y) is given by the right triangle formed from the 3 points: $(0, 0)$, (x, y) and $(x, 0)$. The equations are:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta. \tag{1}$$

In Figure 8.3 on the next page we enter $xy = 1$, convert to polar coordinates by using the above equations and then rotate by $\alpha = \pi/4$ in the clockwise direction by substituting $\theta + \pi/4$ for θ. We now what to apply the trigonometric formulas:

$$\sin(A+B) = \sin A \cos B + \sin B \cos A$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

but DERIVE does not simplify these by default. Instead we need to choose Declare/Algebra State/Simplification and on the Trigonometry box we select Expand. Simplifying these standard formulas will now yield the above.

Simplifying our rotated curve now yields: $r^2 \cos^2 t - r^2/2 = 1$. Converting back to rectangular coordinates we use (1) to replace $r^2 \cos^2 t$ with x^2 and r^2 with $x^2 + y^2$. This yields the desired result; namely, rotating the graph $y = 1/x$ by 45° results in an equation $x^2 - y^2 = 2$ which is a hyperbola.

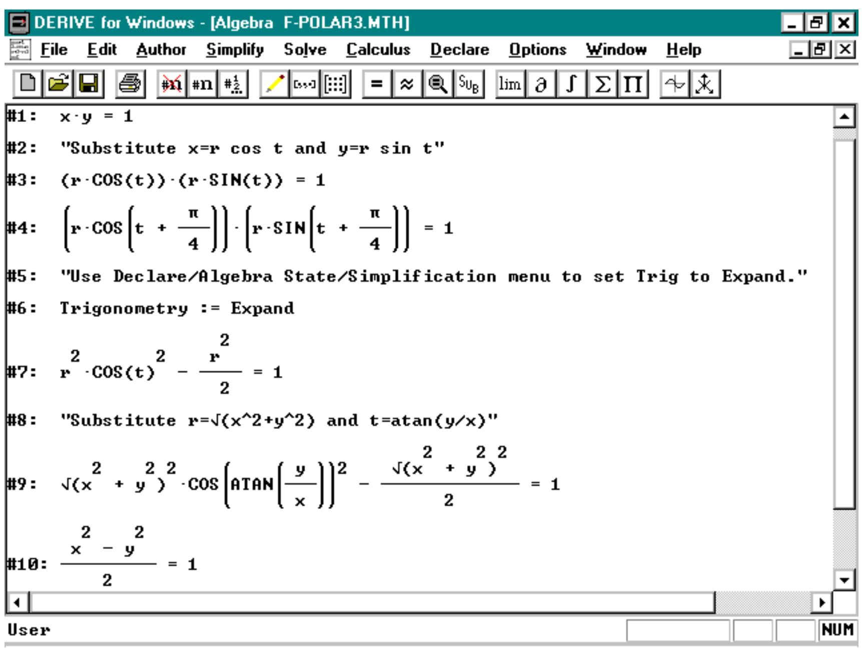


Figure 8.3: Showing that $y = 1/x$ is a hyperbola

Actually, the method used in Figure 8.3 to convert back to rectangular coordinates is to substitute $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ (DERIVE denotes the inverse tangent function by `ATAN`). But there can be problems with this approach. For example, consider the polar point $[\sqrt{2}, 3\pi/4]$ which clearly corresponds to the point $(-1, 1)$ in rectangular coordinates. But $\tan^{-1}(-1) = -\pi/4$ instead of $3\pi/4$ because the definition of the inverse tangent uses the *principle* angles $-\pi/2 < \theta < \pi/2$. Thus, for points with $x < 0$ we should substitute $\theta = \tan^{-1}(y/x) + \pi$ instead.[1] Of course, in our example it doesn't cause a problem thanks to the fact that

$$\cos^2(\theta + \pi) = \cos^2\theta$$

for all θ (check this using DERIVE).

8.4 Parametric Curves

As we saw in the last section we obtain many interesting curves by plotting $r = \rho(\theta)$ with $\alpha \le \theta \le \beta$ in polar coordinates. However, there are still limitations on the shape of a polar curve (just as there are limitations on the shape of a rectangular graph) although these limitations are not as transparent since we have seen examples of looping in the limaçon curves.

To study general curves we need the idea of *parametric curves.* To specify the motion of a particle in the plane; for example, the position of the ant crawling around on the plane, we return to rectangular coordinates and give the x–coordinate as a function, $x = x(t)$, of a parameter t (which is usually thought of as time) and similarly for $y = y(t)$. This means that at time t_0 the particle is at the point $(x(t_0), y(t_0))$.

As an example, the equations (1) on page 118 show that the polar graph $r = \rho(\theta)$ for $\alpha \le \theta \le \beta$ can be thought of as a parametric graph if we set

$$x(t) = \rho(t)\cos t, \qquad y(t) = \rho(t)\sin t \qquad \text{where} \qquad \alpha \le t \le \beta.$$

Of course, this makes the plotting problem harder since we probably wouldn't use the geometry of polar coordinates to plot points. The computer on the other hand doesn't use geometric consideration since it just plots lots of points and connects them with line segments.

Let us consider the non-polar example

$$x(t) = 4\cos t, \qquad y(t) = \sin t, \qquad \text{where } 0 \le t \le \pi.$$

[1] DERIVE's function `ATAN` has a two-variable form `ATAN(y, x)` which does the right thing.

We can plot n points in order by taking $t_i = t_{i-1} + \Delta t$ where $\Delta t = (\beta - \alpha)/n$ and making a $n \times 2$–matrix using the `vector` function. Enter `[4*cos t,sin t]` and then use Calculus/Vector with Start: 0, End: π and Step: .2 (this gives 16 points). Now we can plot this as usual in rectangular coordinates (you will need to switch back to rectangular coordinates). To draw a curve, select Options/Points again and set plotting mode to connected. Then, re-plot the points. See Figure 8.4 and Load the file F-PARAM1.MTH for a demonstration.

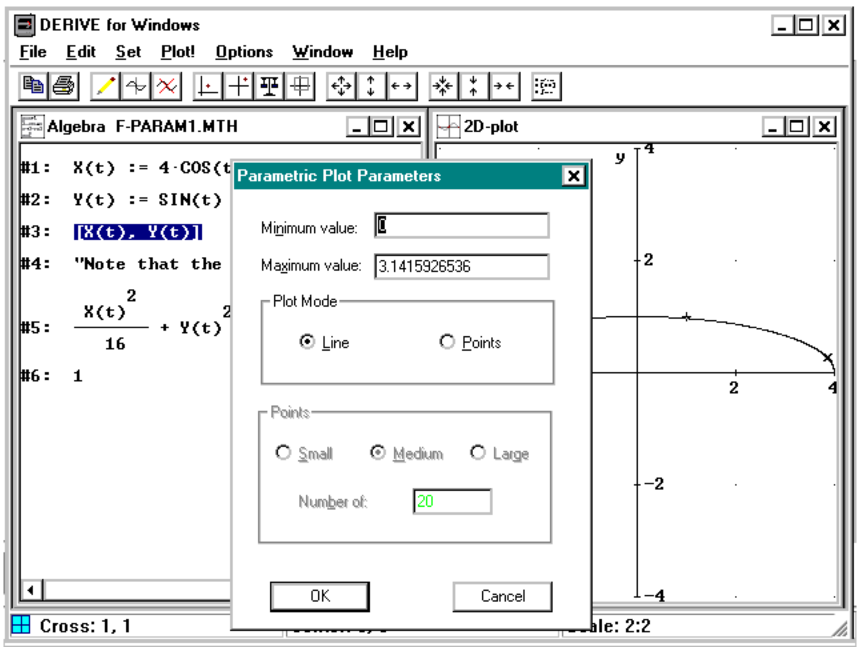


Figure 8.4: Parametric plot of a semi-ellipse

As with polar curves DERIVE has a simplified way to plot parametric curves. You simply plot the vector `[4*cos t,sin t]`. DERIVE will ask for the parameter interval and then plot the curve. You might have thought that DERIVE would plot the two functions $4\cos t$ and $\sin t$ since we know that this happens for 3 or more functions in a vector. But when a vector contains only two functions, it is treated as a parametric curve.

Looking at the picture you might have guessed that the curve in Figure 8.4 was an ellipse (even if you didn't read the caption) because of it's oval shape. Of course, not all oval shaped curves are ellipses but indeed this example is one since one easily checks that

$$\frac{x(t)^2}{4^2} + \frac{y(t)^2}{1^2} = \sin^2 t + \cos^2 t = 1 \quad \text{for all } t$$

and hence the particle travels along the ellipse $x^2/4 + y^2 = 1$ centered at the origin with semi-major axis 4 and semi-minor axis 1. Observe that this information does not tell you how the particle travels around on this ellipse. For instance, is it going clockwise or counterclockwise? Does it ever stop?

See Figure 8.4 again and try the slow down technique to understand how the parametric curve can be thought of as a particle moving along a curve (like a car traveling over a roadway). By using this technique it is apparent that the motion is counterclockwise (as time t increases) and it never stops. This interpretation will be extremely important in later courses when Newton famous $F = ma$ law is used to analyze the forces acting on a moving particle.

Tracing parametric curves. Let us recall the tracing technique from Section 8.2, which we used for polar curves. We now want to "drive" around a parametric curve and observe its speed. After plotting the parametric curve above press F3. The cross will turn into a box on the curve and pressing and holding down the right arrow key will move the little car drive around the curve. You can see the time parameter as it increases, as well as the x and y coordinates, on the lower part of the screen.

By watching the particle move while you press and hold down the right arrow key, you can see that the particle is traveling in the counterclockwise direction and a careful inspection will reveal that the speed (rate of change of distance with respect to time) is slower on the sides than the top and bottom parts of the curve. This is actually a consequence of one of Kepler's laws of planetary motion. This law states that certain moving bodies revolving about a central point (such as the origin in this example) sweep out equal area in equal time. Assuming this fact, then the particle needs to be faster near the top and bottom since these points are closer to the origin and hence sweep out less area. Whereas the left and right portions of the curve are further from the origin and hence require less time to sweep out an equal amount of area.

One can calculate the speed directly as follows: Over a small time interval Δt the x-position changes by Δx $(= x(t + \Delta t) - x(t))$ and the y-position changes by Δy. Thus, the distance traveled during that time interval is approximately $\sqrt{\Delta x^2 + \Delta y^2}$ and hence the average speed is given by

$$\frac{\sqrt{\Delta x^2 + \Delta y^2}}{\Delta t} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta x}{\Delta t}\right)^2} \approx \sqrt{x'(t)^2 + y'(t)^2}$$

Taking limits as $\Delta t \to 0$ leads to the formula:

$$\text{Speed at time } t = \sqrt{x'(t)^2 + y'(t)^2}. \tag{2}$$

DERIVE has an alternate approach for curves described by `[x(t),y(t)]`; namely, one uses Calculus/Differentiate on the vector and then applies `ABS` to the result. This works because `ABS([a,b])` simplifies to $\sqrt{a^2 + b^2}$.

Another use of DERIVE's tracing feature is for curves that retrace themselves and hence make motion on the curve difficult to see. Try the example, $x = \sin t \cos t$ and $y = \sin 2t$ for $0 \le t \le 2\pi$. That is, plot the vector `[sin t cos t, sin(2t)]`. Surprisingly, the picture is simply a line segment with endpoints $(-1/2, -1)$ and $(1/2, 1)$. But how does the particle travel around this curve? By pressing `F3` and tracing the curve we see a back and forth motion which reminds us of a swinging pendulum. In fact, by carefully observing the motion near the endpoints we see the particle slow down and stop. Then, it turns around and goes back in the opposite direction gaining speed as it approaches the center of the line segment and then slowing down as it approaches the other endpoint. A point where the speed is zero is actually the only way a smoothly parametrized curve, i.e., one for which $x(t)$ and $y(t)$ are continuously differentiable, can have cusps (like the cardioid) or corners (as in this example) or otherwise exhibit nonsmooth behavior. Check directly the speed at the endpoints.

As a last example, enter $x = 2\cos^2 t$ and $y = 2 \sin t \cos t$ for $0 \le t \le \pi$. In this case we have another surprising picture of a circle, which we can verify by showing

$$(x(t) - 1)^2 + y(t)^2 = 1 \quad \text{for all } t.$$

Two interesting features are that the complete circle is plotted with t in the $[0, \pi]$ (instead of requiring $0 \le t \le 2\pi$) and also that a particle travels around the curve with uniform speed. Observe this with the tracing technique and then verify it directly using (2).

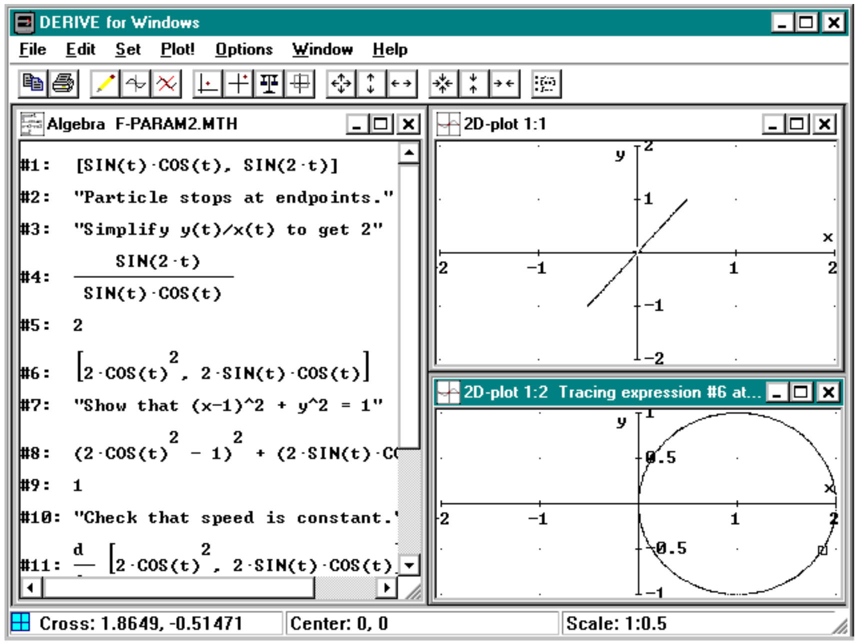


Figure 8.5: More parametric plots

8.5 Laboratory Exercises

1. Consider the polar curve $r = 2\cos\theta$.

- **a.** Plot the curve using polar coordinates.
- **b.** Describe how the curve is traced for $0 \le \theta \le 2\pi$.
- **c.** Use equations (1) on page 118 to convert the polar equation to rectangular coordinates. Use this to show that the curve is a circle of radius 1 with center at $(1, 0)$.

2. Let $r = 2\sin\theta$ be a polar curve.

- **a.** Plot the curve using polar coordinates.
- **b.** Show that the graph is a rotation of the graph in Problem 1.

3. Let $r = \sec(\theta - \pi/4)$ be a polar curve.

a. Plot the curve using polar coordinates.

b. Show that in rectangular coordinates the curve satisfies the equation: $x + y = \sqrt{2}$. (Hint: Use the Trigonometry Expand mode to simplify the equation.)

4. Plot several petal curves, $r = 2\cos(n\theta)$ for different integer choices of n. How many petals are there as a function of n?

5. Choose positive number d and e, then the family of polar curves

$$r = \frac{ed}{1 + e\cos\theta}$$

turns out to be conic sections (see your calculus text as a reference). We will examine this phenomenon with $d = 2$ and e set to 4 different positive values: $e = .5$, $e = .75$, $e = 1$ and $e = 2$.

a. Plot the first two curves ($e = .5$ and $e = .75$) with $-\pi < \theta < \pi$ and identify the conics.

b. Plot the curve with $e = 1$ with $-3.10 < \theta < 3.10$. Can you identify this conic?

c. Plot the curve with $e = 2$ with $-2.09 < \theta < 2.09$. Can you identify this conic?

d. In the last plot, what is the significance of the number $\theta = 2.09$? What curves do you get when $-\pi < \theta < -2.10$ or $2.10 < \theta < \pi$? (Warning: If you try plotting with the default range $[-\pi, \pi]$ it will *eventually* graph the complete conic but it takes a very long time! Press Esc if you can't wait.)

6. Let $x = (\cos t)^3$ and $y = (\sin t)^3$ for $-\pi \le t \le \pi$.

a. Plot the parametric curve.

b. Use DERIVE's tracing method described on page 122 to find where the speed is 0 on the graph.

c. Switch to the algebra window and verify your empirical observations by using (2) on page 123 to determine exactly where the speed is zero.

7. Let $x = t \sin t$ and $y = t \cos t$ for $-3\pi/2 \leq t \leq 3\pi/2$

a. Plot the parametric curve.

b. Use tracing to determine how a particle (ant) traverses the curve over the given t interval. Sketch arrows on the graph to indicate the motion.

c. What will happen if t is allowed to exceed $3\pi/2$? Does it go around the curve again?

***8.** Start by authoring

$$r(\theta, a) := a(e^{\sin\theta} - 2\cos(4\theta))$$

where we think of this function as a polar curve in θ with a paramenter a. Use DERIVE's `vector` function to make a vector of the function $r(\theta, a)$ where the parameter a goes from 1 to 2 in increments of size 0.25. (So after you simplify it, the vector will contain 5 functions.) Plot this vector of 5 functions using polar coordinates. Does it look like a butterfly? (This curve is similar to one described by T. H. Fay, *The butterfly curve*, Amer. Math. Monthly, vol. 96, May 1989, p. 442.) It can be viewed on the Web as Figure 3 on our home page

http://www.math.hawaii.edu/206L/

9. Let $x = t - \sin t$ and $y = 1 - \cos t$ for $t \geq 0$.

a. Plot the parametric curve.

b. Use tracing to verify that the motion stops briefly each time it touches the x–axis.

c. Verify your observations in part **b** by using the formula for speed given in (2) on page 123.

10. Imagine a circle (or wheel) of radius one rolling along the x–axis at unit speed. Now try to picture the path followed by a fixed point on this circle as its rolls. This is the parametric curve in problem **9**, it is called a *cycloid curve*. It may seem a little surprising that the speed of the point on the wheel is 0 once every time the wheel revolves even as the center of the wheel travels at a constant speed.

a. Make a graph of the speed function (2) and determine how fast the point on the wheel going when it is at its highest point? (Hint: Plot the speed function and cycloid curve together on the same graph.)

b. Load the file F-CYCL.MTH and plot expression #8 which contains the parametric curves for 5 positions of the rolling wheel along with a dot marking the particle's position on the wheel. Then plot the cycloid expression #3, see Figure 8.6.

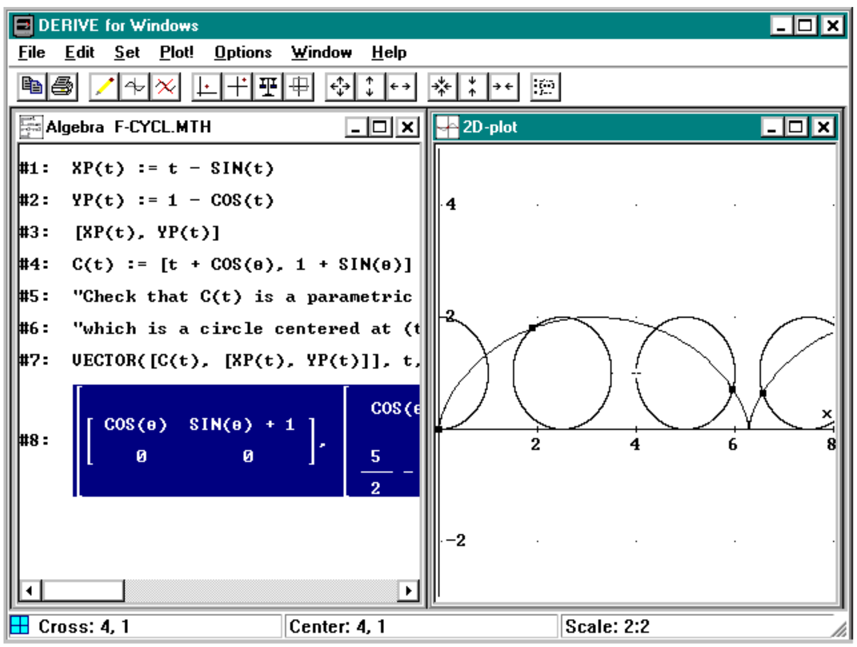


Figure 8.6: The cycloid curve and the rolling wheel

11. Plot the parametric curve $x = \sin(\pi \sin t)$ and $x = \cos(\pi \sin t)$ for $-\pi \leq t \leq \pi$.

a. What geometric object does this look like? Prove that your answer is correct.

b. Using the trace feature to see how a particle following these parametric equations moves along this geometric object. Describe this motion in words.

c. Are there places where the point seems to have speed 0? Find a formula for the speed of the particle at time t. At what times does the particle have speed 0 and what is the position of the particle at these times?

12. Two particles move in the plane. The motion of the first is described by the parametric equations

$$x(t) = 16/3 - 8t/3, \qquad y(t) = 4t - 5, \qquad t \geq 0$$

and the second one by

$$x(t) = 2\sin(\pi t/2), \qquad y(t) = -3\cos(\pi t/2), \qquad t \geq 0$$

Plot both of these curves. Find where the curves intersect. But just because the curves cross does not mean the particles collide; they might arrive at the intersection point at different times. Where *do* the particles collide?

Chapter 9

Series

9.1 Introduction

An infinite series is a sum with infinitely many terms:

$$\sum_{i=0}^{\infty} a_i = a_0 + a_1 + a_2 + \cdots$$

We define $\sum_{i=0}^{\infty} a_i = s$ to mean that

$$\lim_{n\to\infty} \sum_{i=0}^{n} a_i = s,$$

if this limit exists. If the limit does exist we say the series *converges*; otherwise we say it *diverges*. There are two basic techniques for showing that a series is convergent. One method is to show directly that the above limit exists. There are not many examples when we can do this but a particularly important one is *geometric series* which will be discussed in the next section.

The second method for showing convergence applies to series with *non-negative terms*, i.e., the case that $a_i \geq 0$ for all $i = 1, 2, \ldots$. In this case the partial sums,

$$s_n = a_0 + a_1 + \cdots + a_n = \sum_{i=0}^{n} a_i, \qquad n = 0, 1, \ldots$$

form an increasing sequence, $s_0 \leq s_1 \leq s_2 \leq \ldots$. Hence, by a fundamental property of real numbers, the limit $\lim_{n\to\infty} s_n$ exists if and only if the sequence $\{s_n\}$ is bounded. This second technique is used extensively for proving convergence and obtaining estimates on the answer. Particular examples are the *ratio test* and the *integral test* which we will discuss in this chapter.

9.2 Geometric Series

A *geometric series* is one in which the ratio of consecutive terms is constant, i.e., series of the form $\sum ax^i$. To evaluate this series let $s_n = \sum_{i=0}^n x^i = 1 + x + x^2 + \cdots + x^n$. Then

$$\begin{aligned} s_n - xs_n &= (1 + x + x^2 + \cdots + x^n) \\ &\quad - (x + x^2 + \cdots + x^n + x^{n+1}) \\ &= 1 - x^{n+1} \end{aligned}$$

Factoring out s_n and solving, we get

$$\boxed{s_n = \sum_{i=0}^{n} x^i = 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}, \qquad \text{if } x \neq 1} \tag{1}$$

It's instructive to verify this formula in DfW. You start by clicking the sum $\boxed{\Sigma}$ button and enter `x^k`. Make sure the variable is k (not x) and set the Start value to `0` and the End value to `n`. Click OK and edit the resulting expression `SUM( x^k, k, 0, n)` by multiplying it by the factor $(1 - x)$. Lastly, use Simplify/Expand to get the desired $1 - x^{n+1}$.

If $|x| < 1$, then $\lim_{n\to\infty} x^{n+1} = 0$. Thus, we get that $\lim_{n\to\infty} s_n$ *exists* and so the series is convergent. In addition, the following simple formula for evaluating geometric series holds:

$$\boxed{\sum_{i=0}^{\infty} ax^i = a + ax + ax^2 + \cdots = \frac{a}{1 - x}, \qquad \text{if } |x| < 1} \tag{2}$$

If $|x| \geq 1$ then the series diverges because $\lim_{n\to\infty} s_n$ does not exist.

We can verify this formula in DERIVE by entering, as we did above or directly, the expression `SUM( ax^k, k, 0, inf)` which displays as the left hand side of (2). Now we must declare that $-1 < x < 1$. We do this using the Declare/Variable Domain menu for the variable `x` and the open interval $(-1, 1)$. If you do this right the expression `x :∈ Real (-1,1)` will be the result. If not you should be able to edit the expression until it is right. Finally simplify the infinite series to get $a/(1-x)$ which is the desired result.

9.3 Applications

Geometric series are useful in several areas, for example, business and finance. We will give some of the important examples.

Interest. If you start with p dollars (p for *principal*) in a bank account which earns 6% per year how much money will you have after n years? Assuming the interest is compounded yearly, you will be given an interest payment of $0.06\,p$ after one year. You will still have the original p dollars so that the amount of money in the account after one year will be $p(1.06)$. Notice that this is saying that each year the amount of money in the account gets multiplied by 1.06. Thus after n years the account will have $p(1.06)^n$ dollars. If we let r denote the interest rate, the amount after n years is $p(1+r)^n$.

An interesting alternative to this formula is obtained by focusing on the year to year change in the savings account balance. Let `s_bal(k,p,r)` denote the balance after k years, starting with an amount p which is compounded annually at a rate r. This function can be defined in DfW by

```
s_bal(k,p,r):=IF(k=0,p,(1+r)*s_bal(k-1,p,r)).
```

Notice how we use the function `IF(test,true,false)`. To compute say `s_bal(2,p,r)` the first thing that happens is the test $k=0$ fails and hence we get `(1+r)*s_bal(1,p,r)`. But then `s_bal(1,p,r)` is computed in a similar manner, i.e., the test $k=0$ fails again so now

```
s_bal(2,p,r)=(1+r)*s_bal(1,p,r)=(1+r)*((1+r)*s_bal(0,p,r)).
```

Finally, `s_bal(0,p,r)` is evaluated but this time the test $k=0$ succeeds and so the answer is p. Combining the answers we get the same result as before $(1+r)^2p$. This type of computation has a fancy name; it's called *recursive programming* and it is particularly useful in situations where you have a sequence of numbers which change one to the next by a fixed rule.

Now suppose that the bank compounds your money quarterly instead of annually. This means that they give you $6/4 = 1.5\%$ interest four times a year. So the amount of money in your account after n years is $p(1.015)^{4n}$. For a general rate r compounded k times a year, the amount of money after n years is

$$\boxed{p\big(1+\frac{r}{k}\big)^{kn}} \tag{3}$$

This can be also be expressed as `s_bal(kn,p,r/k)`.

Now suppose that you deposit a dollars each year into a bank account paying a rate r in interest, compounded annually. Suppose that you opened the account with an amount p dollars. How much money will the account have after n years? This is easy to do using a small modification in the `s_bal` function as follows:

```
s_bal(k,p,r,a):=IF(k=0,p,(1+r)*s_bal(k-1,p,r,a)+a).
```

In other words we need only account for the extra a dollars which are deposited each year. We can get a nice table of values by numerically approximating

```
VECTOR([k,s_bal(k,1000,.06,100)], k, 0, 10)
```

to see how an initial balance of \$1000 will grow over a ten year period, at 6% annual interest, if we add an extra \$100 each year.

Now if you make the same table as above using the symbolic values for p, r and a you get a sequence of expressions which don't appear to follow any clear pattern. On the other hand, if we substitute $r_1 = 1 + r$ everything is much clearer. In DfW you would declare a variable `r1` and use `r1-1` as a replacement for `r`, then the table obtained by entering

```
VECTOR([k,s_bal(k,p,r1-1,a)], k, 0, 10)
```

and pressing $\boxed{=}$ presents the following pattern for `s_bal(k,p,r,a)`:

$$\begin{aligned}
&a(r_1^{k-1} + r_1^{k-2} + \cdots + r_1 + 1) + pr_1^k \\
&= a\frac{r_1^k - 1}{r_1 - 1} + pr_1^k \\
&= a\frac{(1+r)^k - 1}{r} + p(1+r)^k
\end{aligned}$$

Here we used (1) with $x = r_1 = 1 + r$. Thus, the geometric series arises naturally in compound interest problems and provides us with a useful formula.

Loan repayment. Suppose we borrow p at an annual rate of R. We are to pay this loan back by paying a monthly amount of a dollars for n years. Now the monthly interest is $r = \frac{R}{12}$. Thus, at the end of the first month we owe the p dollars plus the interest it would have earned, rp, for a total of $(1 + r)p$. We also make a payment of a dollars so the net amount we owe is $(1 + r)p - A$. The same computation is used month after month except that the p is replaced with the loan balance for the previous month. Hence, if let `l_bal(k,p,r,a)` denote the loan balance after k months on a loan amount of p dollars at a monthly interest rate r and monthly payment a then

```
l_bal(k,p,r):=IF(k=0,p,(1+r)*l_bal(k-1,p,r)-a).
```

which is very similar to our definition for `s_bal`.

Now suppose we are interested in a loan of \$20,000 at a monthly interest rate of $r = 0.01$. The problem is to compute the monthly payment a which will result in paying off the loan in four years. We can display a four year history of the loan in a table when the payments are a = \$500 by first authoring the vector `[k, l_bal(k,20000,0.01,500)]` and then using the Calculus/Vector menu to produce the expression

```
VECTOR([k, l_bal(k,20000,0.01,500)], k, 0, 48, 1).
```

We see[1] that after 4 year (so $k = 48$ payments) we still have an outstanding balance of \$1633.21 (of course, we could also discover this by just simplifying `l_bal(48,20000,0.01,500)`). This means that \$500 per month is not enough to pay off the loan in 4 years. At this point we could try increasing the payment a and then computing `bal(48,20000,0.01,a)` until we get nearly zero. We might start by incrementing a by \$10 until we get the answer within \$10 and then increment by a dollar until we get the answer within a dollar. For repeated computations this would be a rather tedious approach.

By comparing with the formula derived for `s_bal` using the geometric

[1] Due to a bug it's necessary to author a comment or any other expression before you can scroll through this matrix.

series we get a similar formula for `l_bal(k,p,r,a)`, namely,

$$
\begin{aligned}
(1+r)^k p - a\sum_{i=0}^{k-1}(1+r)^i &= (1+r)^k p - a\frac{1-(1+r)^k}{1-(1+r)} \\
&= (1+r)^k p - a\frac{(1+r)^k - 1}{r} \qquad (4)
\end{aligned}
$$

Using this formula we can easily get the general formula for a by solving `l_bal(k,p,r,a) = 0` for a. Thus, the monthly payments a on a loan of p dollars at a monthly interest rate r (divide the annual rate by 12) for a period of n years (so $k = 12n$ payments) is:

$$
a = \frac{r(1+r)^k p}{(1+r)^k - 1} \quad \text{where} \quad k = 12n. \qquad (5)
$$

Thus, in our \$20,000 example you need $a = \$526.68$, i.e.,

`bal(48,20000,0.01,526.68)` $= 0.$

Repeating Decimals. What exactly is meant by the decimal representation of a number $x = 0.d_1d_2d_3\cdots$, where each of the digits d_k are integers $0 \le d_k \le 9$? One explanation is that there is no difficulty as long as it is a finite decimal, i.e., $0.d_1 = \frac{d_1}{10}$, $0.d_1d_2 = \frac{d_1}{10} + \frac{d_2}{100}$, etc. For the infinite case, we can think of our decimal as the limit of an increasing sequence which is bounded from above:

$$
0.d_1 \le 0.d_1d_2 \le 0.d_1d_2d_3 \le \cdots \le 1\,.
$$

and hence this sequence has a limit, as mentioned above.

Another approach is to view the decimal as an *infinite series* as follows:

$$
x = 0.d_1d_2\cdots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots = \sum_{k=1}^{\infty}\frac{d_k}{10^k} \qquad (6)
$$

Now clearly, the partial sums form an increasing sequence since the terms are nonnegative numbers. However, maybe it is not completely obvious that they are bounded by 1! Here's a proof:

$$\begin{aligned}
&\frac{d_1}{10}+\frac{d_2}{10^2}+\cdots+\frac{d_n}{10^n}\leq\frac{9}{10}+\frac{9}{10^2}+\cdots+\frac{9}{10^n}\\
=&\frac{9}{10}+\frac{9}{10}\frac{1}{10}+\frac{9}{10}\frac{1}{10^2}\cdots+\frac{9}{10}\frac{1}{10^{n-1}}\\
\leq&\frac{9}{10}+\frac{9}{10}\frac{1}{10}+\frac{9}{10}\frac{1}{10^2}+\cdots=\frac{\frac{9}{10}}{1-\frac{1}{10}}=1.
\end{aligned}$$

Note that the key step above was recognizing that the geometric series $\sum_{k=0}^{\infty} a(1/10)^k$, where $a = 9/10$, sums to 1 by (2) on page 130.

Of course, we also notice that repeating decimals like $0.999\cdots = 1$ and $0.333\cdots = 1/3$ are all geometric series when represented as above. Try to figure out the a, x in (2) in each case. This turns out to be true of any repeating decimal and hence by the formula (2) these decimal numbers must be fractions a/b where a, b are integers. In fact, the converse is also true, namely, a decimal is a fraction if and only if it is eventually repeating.

Example. Consider the eventually repeating decimal $x = 0.5010101\cdots$. We express this as

$$\begin{aligned}
x &= \frac{5}{10}+\frac{1}{10^3}+\frac{1}{10^5}+\cdots=\frac{1}{2}+\sum_{k=0}^{\infty}10^{-3}\left(10^{-2}\right)^k\\
&=\frac{1}{2}+\frac{10^{-3}}{1-10^{-2}}=\frac{1}{2}+\frac{1}{990}=\frac{248}{495}\,.
\end{aligned}$$

We might notice that it is not possible to enter x in DERIVE as a decimal but we can define it by means of the infinite series above. Then, simplifying we get the above result.

9.4 Approximating Infinite Series

We can determine the sum of a geometric series exactly but for most convergent infinite series this is impossible. If the series converges to $s = \sum_{k=0}^{\infty} a_k$, then by definition s can be approximated arbitrarily closely by the partial sums $\sum_{k=0}^{n} a_k$ for large enough n. In this section we investigate two methods for approximating infinite series, with a given precision.

The Ratio Test. In a convergent geometric series, $a_k = ax^k$, and hence $a_{k+1} = a_k x$, i.e., the ratio of consecutive terms is x, where $|x| < 1$. In this section we consider positive series ($a_k > 0$) where the the ratio of consecutive is *approximately equal* to some x with $0 < x < 1$. It will turn out that all such series converge and that we can estimate their size by comparisons with appropriate geometric series. This technique is called the *ratio test*.

Theorem 1. *Let a_i be positive.*

(a) *If* $\lim_{i\to\infty} a_{i+1}/a_i = \lambda < 1$, *then the series* $\sum a_i$ *converges.*

(b) *Suppose that* $0 < x < 1$ *and that for some* n, $a_{i+1}/a_i \le x$ *for all* $i > n$. *Then*

$$0 \le \sum_{i=1}^{\infty} a_i - \sum_{i=1}^{n} a_i \le \frac{a_{n+1}}{1-x} \tag{7}$$

Proof. If $\lim_{i\to\infty} a_{i+1}/a_i = \lambda < 1$, then for any x satisfying $\lambda < x < 1$, we know that the ratios will be less than x for all large i. Thus, given x there is an n for which $a_{i+1} \le xa_i$ whenever $i > n$. So $a_{n+2} \le xa_{n+1}$ and $a_{n+3} \le xa_{n+2} \le x^2 a_{n+1}$. In general, $a_{n+1+k} \le x^k a_{n+1}$ and hence for all $m > n$

$$\begin{aligned} 0 \le \sum_{i=1}^{m} a_i - \sum_{i=1}^{n} a_i &= a_{n+1} + a_{n+2} + \cdots + a_m \\ &\le a_{n+1}(1 + x + x^2 + \cdots) = \frac{a_{n+1}}{1-x}. \end{aligned}$$

Thus, the partial sums $\{s_m\}$ are bounded and the series is convergent. Moreover, the inequality (7) follows by taking limits as $m \to \infty$. □

Example. Suppose we want to use Theorem 1 to prove that the series

$$\sum_{k=0}^{\infty} \frac{2^k}{k!} = \frac{1}{1} + \frac{2}{1} + \frac{4}{2} + \ldots$$

converges and estimate its value with an error of at most 10^{-6}. The first step is to show that $\lim_{k\to\infty} a_{k+1}/a_i < 1$. We think of the terms as a function of k by authoring `term(k):=2^k/k!`. Now simplifying the ratio `term(k+1)/term(k)` we see that

$$\frac{a_{k+1}}{a_k} = \frac{2}{k+1} \le \frac{1}{2}$$

for all $k > 2$. Thus, $\lim_{k\to\infty} a_{k+1}/a_k = \lim_{k\to\infty} 2/(k+1) = 0 < 1$ so the series converges and furthermore we can take $x = 1/2$ and $n = 2$ in Theorem 1(b).

Now we must determine n so that

$$\frac{a_{n+1}}{1-x} = 2a_{n+1} < 10^{-6}\,.$$

We do this by authoring

`VECTOR([n,2*term(n+1)],n,2,20)`,

approximating the result and then searching the entries (by scrolling) until we find one smaller than 10^{-6}. It turns out $n = 13$ works. The last step is to compute the partial sum $s_{13} = \sum_{k=0}^{13} 2^k/k!$ giving 7.38906.

We might observe how fortunate we were that k turned out to be so small. Recall some of our computations using the trapezoid method or Simpson's rule where similar accuracy required thousands or even millions of computations (using the left endpoint method, for example). It is one of the fundamental properties of geometric series that they converge very rapidly. Think about it, 6–decimal place accuracy with just 15 computations!

As it turns out this series is rather special since

$$\sum_{k=0}^{\infty} \frac{2^k}{k!} = e^2 = 7.38905\cdots$$

This important fact will be explained in the next chapter. For now, try authoring the above infinite series and have DERIVE simplify the result. What if the 2 is replaced with 3 or x?

Example*. Now consider the harder problem of approximating

$$\sum_{k=1}^{\infty} \frac{k!}{k^k}$$

with error again of at most 10^{-6}. Proceeding as before we author the formula `term(k):=k!/k^k`. Now by first simplifying and then taking limits we see that

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}=\lim_{k\to\infty}\frac{k^k}{(k+1)^k}=\frac{1}{e}<\frac{1}{2}$$

since $e>2$. Thus, the limit is less than 1 so the series converges. Furthermore, we can take $x=1/2$ in Theorem 1(b). But now we need to find an integer n so that $a_{k+1}/a_k=k^k/(1+k)^k<1/2$ for all $k>n$.

This step is harder than before. If we graph $f(x)=(x/(x+1))^x$ it *appears* to be decreasing for all $x\geq 0$. See Figure 9.1.

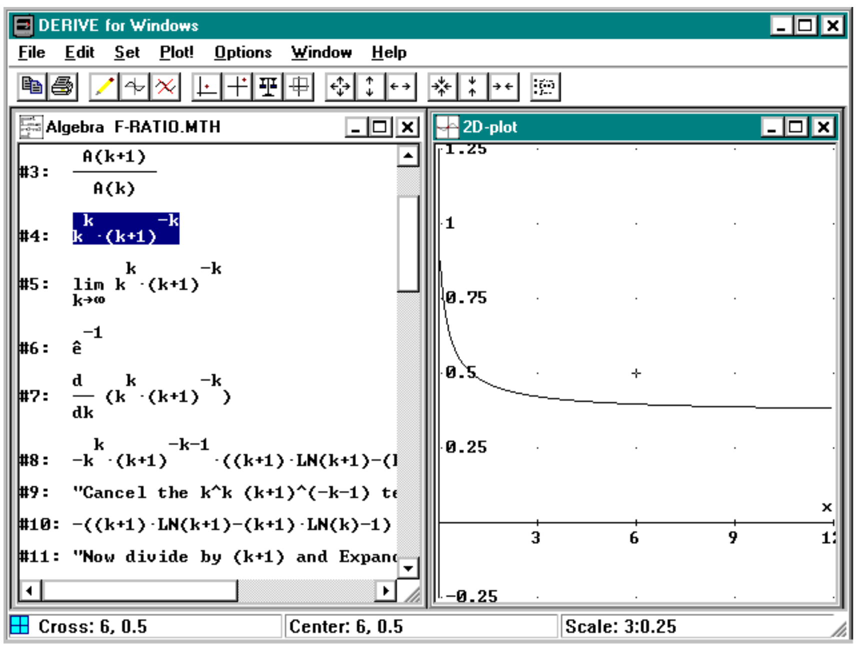


Figure 9.1: Ratio test example

In order to prove that $f(x)$ is a decreasing function we differentiate and show that $f'(x)<0$. Using DERIVE we get

$$\begin{aligned}f'(x)&=-\frac{x^x}{(x+1)^{x+1}}\left[(x+1)\ln(x+1)-(x+1)\ln x-1\right]\\&=-\frac{x^x}{(x+1)^x}\left[\ln(x+1)-\ln x-\frac{1}{x+1}\right]\end{aligned}$$

after some rearrangement (see the file F-RATIO.MTH for the step-by-step procedure). Since $x^x/(x+1)^x$ is positive for $x > 0$, we need to show that $\ln(x+1) - \ln x - 1/(x+1)$ is positive. At this point DERIVE can't help so we need an idea from calculus. One quick way to solve this problem is to use the Mean Value Theorem for the function $g(x) = \ln x$. The Mean Value Theorem says: $g(b) - g(a) = g'(c)(b-a)$ for some $a < c < b$. For $a = x$ and $b = x + 1$ this gives $\ln(x+1) - \ln x = 1/c$ for some $x < c < x + 1$. Thus

$$\ln(x+1) - \ln x - \frac{1}{x+1} = \frac{1}{c} - \frac{1}{x+1} > 0$$

where we obtain the desired inequality since $c < x + 1$. Thus, $f'(x) < 0$ for all $x > 0$ and so f is decreasing.

Now that we have established that the ratios decrease we need to know when they are less than 1/2. Since

$$\frac{a_{k+1}}{a_k} = \frac{k^k}{(k+1)^k} = f(k) \le f(1) = \frac{1}{2}.$$

for all $k \ge 1$, it follows that we may apply the theorem for any n. Finally, by (7), we must determine n so that

$$\frac{a_{n+1}}{1 - \frac{1}{2}} = 2a_{n+1} < 10^{-6}.$$

As before we author

```
VECTOR([n,2*term(n+1)],n,2,20),
```

approximate the result and then search the entries until we find one smaller than 10^{-6}. It turns out in this case that $k = 16$. Computing the partial sum s_{15} gives 1.87985.

Now suppose that your series $\sum a_k$ satisfies $\lim a_{k+1}/a_k = \lambda$ but that $\lambda \ge 1$. The case $\lambda > 1$ is pretty much like the case $\lambda < 1$ except that now the series diverges. The idea is to pick $1 < x < \lambda$ and observe that

$$\infty = a_n + a_n x + a_n x^2 \cdots \le a_n + a_{n+1} + a_{n+2} + \ldots$$

for some large n since now $a_{k+1}/a_k \ge x$ for all $k \ge n$. The case $\lambda = 1$ is much harder since, as we see in the next section, there are examples in which the series converges and examples where it diverges.

The Integral Test Suppose that $f(x)$ is a decreasing positive valued function, for $x \geq 1$. Let $a_n = f(n)$. We want to approximate $\sum_{i=1}^{\infty} a_i$ and determine whether the series is convergent or divergent.

In Section 6.4 we saw that, for a decreasing function like $f(x)$, the left endpoint method of estimating a definite integral of $f(x)$ always overestimates the integral while the right endpoint method underestimates it. This is quite obvious by looking at Figure 9.2 on the facing page where we use $f(x) = 1/x$ as our function and apply the box drawing function from Chapter 6; see the file F-SERINT.MTH for a demonstration. Now, we observe that since the interval size is one, the area of the box with height $f(n)$ is just a_n. From this we get that adding the area of boxes corresponds to partials sums of the series $\sum a_k$. Thus, for any $1 \leq n \leq m$

$$\sum_{i=n+1}^{m+1} a_i \leq \int_n^m f(x)\,dx \leq \sum_{i=n}^{m} a_i \tag{8}$$

The sum on the left is the right endpoint estimate and the sum on the right is the left endpoint estimate, when we use $\Delta x = 1$ as the subinterval size. From this inequality, we obtain the following theorem:

Theorem 2. *Suppose that $f(x)$ is a continuous, nonnegative, decreasing function for $x \geq 1$. Put $a_n = f(n)$.*

(a) *The sum $\sum_{i=1}^{\infty} a_i$ converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ does.*

(b) *Moreover, the inequality*

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} a_i + \int_{n+1}^{\infty} f(x)\,dx \leq \sum_{i=1}^{\infty} a_i \leq \sum_{i=1}^{n} a_i + \int_{n}^{\infty} f(x)\,dx \tag{9}$$

holds for all $n = 1, 2, \ldots$.

(c) *The value of the series can be estimated using the following:*

$$\boxed{0 \leq \sum_{i=1}^{\infty} a_i - \left(\sum_{i=1}^{n} a_i + \int_{n+1}^{\infty} f(x)\,dx\right) \leq a_n} \tag{10}$$

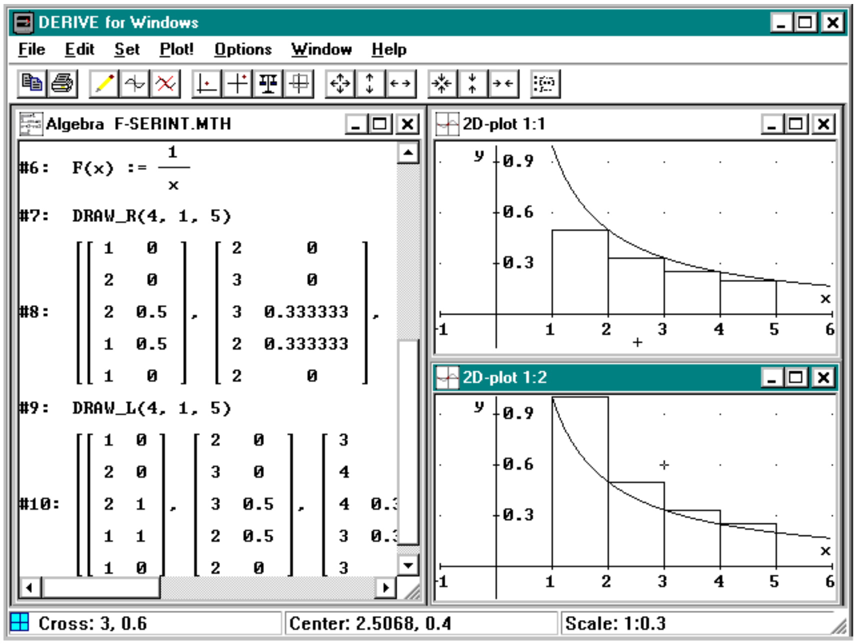


Figure 9.2: The geometric estimate used in the integral test

Proof. Suppose that the improper integral $\int_1^\infty f(x)\,dx$ is convergent so that the total area under the curve is finite. From (8) it follows that

$$\sum_{i=n+1}^{m+1} a_i \le \int_n^m f(x)\,dx \le \int_n^\infty f(x)\,dx < \infty$$

and hence the partial sums $\{s_m\}$ are *bounded* (the first n terms are irrelevant). Thus, the series converges and the second inequality in (9) follows from

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{n} a_i + \sum_{i=n+1}^{\infty} a_i \le \sum_{i=1}^{n} a_i + \int_n^\infty f(x)\,dx\,.$$

A similar argument shows that the integral is convergent if the series is and that the first inequality in (9) holds.

The first inequality in (10) is an immediate consequence of (9) and similarly it follows that the middle expression in (10) is bounded by $\int_n^{n+1} f(x)\,dx$. But since $f(x)$ is decreasing this integral is less than or equal to $f(n) = a_n$ and the theorem is proved. $\square$

We note that (9) actually gives us two methods for approximating the sum of a convergent series $s = \sum_{k=1}^{\infty} a_k$. The first technique looks more like the one used in the ratio test:

$$0 \leq s - s_n = \sum_{i=1}^{\infty} a_i - \sum_{i=1}^{n} a_i \leq \int_n^{\infty} f(x)\,dx$$

and the second one is the more refined estimate (10) which uses the quantity

$$\sum_{i=1}^{n} a_i + \int_{n+1}^{\infty} f(x)\,dx$$

to approximate s instead of the partial sum s_n. As we shall see in the examples, this more refined method has a dramatic computational advantage.

A curious formula. As our first application of the integral test let us prove that the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

is convergent. First note that $\lambda = 1$ in the ratio test so we cannot use that approach. Next, we take $f(x) = 1/x^2$ and observe (say using DERIVE) that

$$\int_1^n \frac{dx}{x^2} = -\frac{1}{x}\bigg|_1^n = 1 - \frac{1}{n} \to 1 \quad \text{as} \quad n \to \infty$$

and hence the improper integral $\int_1^{\infty} dx/x^2$ is convergent. Now, by the integral test the series $\sum 1/i^2 < \infty$, that is the series is convergent. Actually, a similar argument shows that $\sum 1/i^p < \infty$ whenever $p > 1$.

Now it is a remarkable fact that

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}\,. \tag{11}$$

You have to wonder how the π–term can possibly be involved in this computation. The proof of this fact is beyond the scope of this text but DERIVE can help us *believe* this result. One way to do this is to have DERIVE simplify the series and get $\pi^2/6$ as the answer. It works, try it. A more independent approach would be to compute the partial sums s_n for several n and compare with a decimal approximation to $\pi^2/6$. In Figure 9.3 on the next page we have DERIVE make these comparisons with $n = 100$, 1000, and 10,000. We also observe that DERIVE knows about (11) and simplifies the series accordingly.

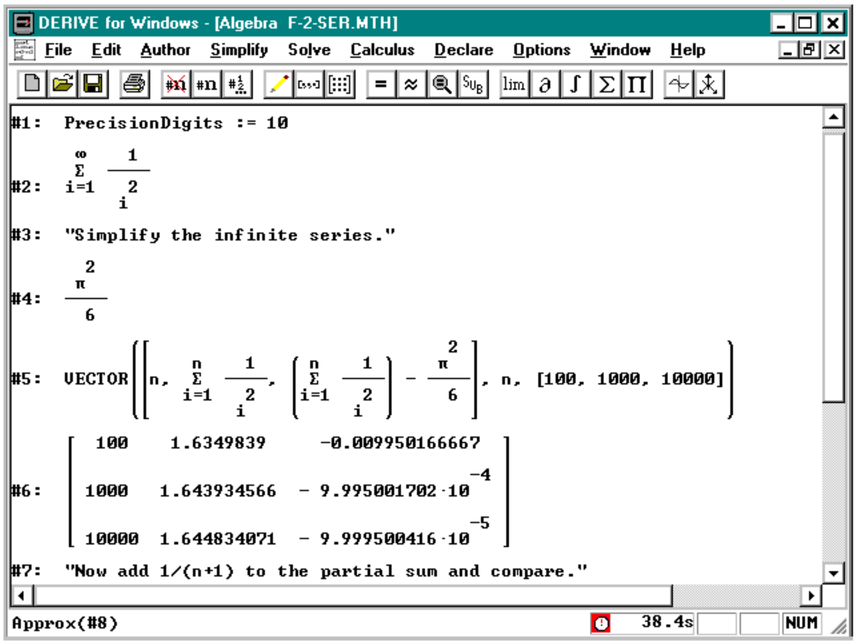


Figure 9.3: Summing the series $\sum 1/i^2$

Problem: Compute this series to m decimal places. We solve this problem by using (10) which in this case says:

$$0 \leq \sum_{i=1}^{\infty} \frac{1}{i^2} - \left(\sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{n+1} \right) \leq \frac{1}{n^2}. \tag{12}$$

Thus, to solve our problem we need only find n so that right-hand side of (12) is less than 10^{-m}, and then use

$$\sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{n+1} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \frac{1}{n+1}$$

for our estimate. For example, with $m = 6$ we need to take $n^2 > 10^m$ or $n > 1000$.

Something rather amazing occurred in this problem. In Figure 9.3 we used the partial sums $\{s_n\}$ to approximate the sum s. This is the natural thing to do since $s = \lim s_n$. But the accuracy in Figure 9.3 is only 3 or 4 decimal places with $n = 1000$. This error is to expected since (8) yields

$$0 \leq s - s_n = \sum_{i=n+1}^{\infty} \frac{1}{i^2} \leq \int_n^{\infty} \frac{dx}{x^2} = \frac{1}{n}$$

and the right-hand side is just less than 10^{-3}. On the other hand, adding the term $1/(n+1)$ (which is the integral in (10)) increases the accuracy to $1/n^2 = 10^{-6}$. This accuracy is *1000 times better* than the other estimate. Put another way, suppose for example that both computations take about 3 seconds with $n = 1000$ on your PC, the amount of computation time needed to produce 6 decimal place accuracy using the less efficient method is almost an hour! See the file F-2-SER.MTH which contains a comparison of these methods. This problem illustrates the potential value of a innovative approach to a computation compared to the conventional solution.

The Harmonic Series Let us apply the integral test to the *harmonic series*, namely,

$$\sum_{i=1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

We take $f(x) = 1/x$ in the theorem and observe that

$$\int_1^x \frac{dt}{t} = \ln x \to \infty \qquad \text{as} \qquad x \to \infty$$

and hence the integral is divergent. Thus, the series is divergent. Another way to express this is that $\sum_{i=1}^{\infty} \frac{1}{i} = +\infty$ or in other words, the partials sums are eventually larger than any given number.

Consider this: How many terms of the harmonic series are necessary before the partial sums exceed 100? Is the answer 1000? 1,000,000? 10^{10}? Amazingly, none of these answers are even close to the actual result. Suppose that $100 < \ln n$, then by (8)

$$100 < \ln n = \int_1^n \frac{dt}{t} \leq \sum_1^n \frac{1}{i}$$

so that any $n > e^{100} \approx 2.6 \times 10^{43}$ is certainly large enough. On the other hand, using (8) again we have

$$100 \leq \sum_{1}^{n} \frac{1}{i} = 1 + \sum_{2}^{n} \frac{1}{i} \leq 1 + \int_{1}^{n-1} \frac{dt}{t} < 1 + \ln n$$

so that when combined with the above we see that the best n satisfies: $e^{99} < n < e^{100}$.

9.5 Laboratory Exercises

For these problems it is a good idea to have more digits of precision: choose Declare/Algebraic State/Simplification and the Digits box to 10 or 12.

1. Formula (3) on page 132 shows the amount of money in an account after n years if the interest rate is r, the original amount is A, and the interest is compounded k times a year. In Problem 1 on page 112 you showed that if interest is compounded *continuously*, the amount of money would be Ae^{rn}.

- **a.** Show that the limit as $k \to \infty$ of compounding k times a year is the same as compounding continuously.
- **b.** If you put \$1000.00 into an account earning 4.5% interest, how much money will be in the account after one year if the interest is compounded yearly? quarterly? daily? continuously?
- **c.** Do the previous part only assume that the bank is paying 9%.

2. Suppose you get a 30 year mortgage loan for $\$200,000$ which is to be repaid in $30 \cdot 12$ equal monthly payments, based on an annual interest rate of 7.5%.

- **a.** Find your monthly payment.
- **b.** How much do you still owe after your first payment? How much of your first year's payment went to interest and how much went to paying off the principal?

c. Formula (4) gives the amount you still owe after k months. Replace the k with $12k$ in Formula (4) so that k now represents years, approximate the resulting expression and then plot. (You'll need to adjust the range in such a way that the visible x-axis contains the range 0 to 30 and the y–axis contains the range 0 to $200,000$.) Notice that at the beginning the amount you owe changes slowly but that near the end of the 30 years it changes quickly.

3. In some problems involving monthly payments or interest the monthly interest rate is computed by dividing the annual rate by 12. But sometimes the monthly rate m is not specified and instead the *effective annual rate* r is given. This means that compounding the monthly rate m 12 times gives the annual rate r, i.e. $(1+m)^{12} = 1+r$. Consider the previous problem but now suppose that the *effective* annual rate is 7.5%.

a. Calculate the monthly rate for this problem.

b. Find the monthly mortgage payments using this new rate.

4. The bank says that it will give you a car loan of $6,000 provided you make monthly payments of $135 for 5 years. What interest rate is the bank charging? (Hint: You may need to be a little careful how you compute this.)

5. Consider the fraction $1/7$.

a. Using DERIVE show that $1/7$ *appears* to have a repeating decimal expansion. What is it?

b. Express this repeating decimal from part **a** as an infinite series, see the example on page 135.

c. Have DERIVE simplify this series.

d. Identify the a and x terms from (2) and verify using that formula that your infinite series simplifies to $1/7$.

6. Have DERIVE evaluate the sum $\sum_{n=1}^{\infty} 1/n^2$. (Make sure you use Exact mode.) Evaluate the left and right sides of formula (9) in Theorem 2 on page 140 for $n = 1000$. You should approximate the sum rather than Simplify, otherwise the computation time is fairly long. Use these to estimate π giving upper and lower bounds.

7. For each of the following series $\sum_{k=0}^{\infty} a_k$ find $\lambda = \lim_{i\to\infty} a_{k+1}/a_k$ and show that $\lambda < 1/2$. Now use Theorem 1 with $x = 1/2$ to find n large enough so that $\sum_{k=0}^{n} a_k$ approximates the series with error at most 10^{-6}.

a. $\displaystyle\sum_{k=0}^{\infty} \frac{k^3}{4^k}$ **b.** $\displaystyle\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}$

8. Use Theorem 2(c) to evaluate each of the following series with an error of at most 10^{-6}. (The finite sum of Theorem 2(c) should be Approximated but the improper integral should be evaluated exactly.)

a. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ **b.** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$

c. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1+k^2}$

9. Some of the following series converge and some diverge. Decide which do which and state the required Theorem needed to prove your conclusion.

a. $\displaystyle\sum_{k=0}^{\infty} 2^k$ **b.** $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ln k}$

c. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{e^k}$ **d.** $\displaystyle\sum_{k=1}^{\infty} \frac{2\cdot 4\cdots 2k}{(2k)!}$

10. Consider the series $\sum_{k=0}^{\infty} 1/k!$.

a. Show that the series converges by the ratio test.

b. Have DERIVE simplify this series.

c. Use these results to approximate the Euler constant e with an accuracy of 10^{-6}

***11.** The following formula

$$1 + 2\sum_{k=1}^{\infty} e^{-k^2/t} = \sqrt{t\pi}\left(1 + 2\sum_{k=1}^{\infty} e^{-k^2\pi^2 t}\right) \tag{13}$$

is known to hold for all $t > 0$. The formula is derived from an important technique in the theory of Fourier transforms called *Poisson summation*. We will not attempt to prove this formula but instead try to use it as a method of approximating π more efficiently than in an earlier problem. It has a number of other useful applications too. We will fix the value of $t = 2$ for the rest of this problem.

a. Using the ratio test, show that both infinite series in (13) are convergent.

b. Use Theorem 1 with $x = e^{-6\pi^2}$ and $n = 1$ to show that $\sum_{k=1}^{\infty} e^{-2k^2\pi^2}$ is less than 10^{-8}. Thus, with an accuracy of $2\sqrt{2\pi}10^{-8}$ or roughly 7 decimal places we can take the right hand side of equation (13) to be $\sqrt{2\pi}$.

c. Using Theorem 1 again, show that

$$\begin{aligned} 0 &< \left(1 + 2\sum_{k=1}^{\infty} e^{-k^2/2}\right) - \left(1 + 2\sum_{k=1}^{6} e^{-k^2/2}\right) \\ &= 2\sum_{k=7}^{\infty} e^{-k^2/2} < 10^{-10} \end{aligned}$$

and hence

$$\pi \approx \frac{1}{2}\left(1 + 2\sum_{k=1}^{6} e^{-k^2/2}\right)^2.$$

d. Approximate the above expression using Simplify/Approximate with the number of precision digits set to 10. Compare the above approximation of π with DERIVE's. What is the decimal place accuracy?

e. To achieve more decimal places you should increase the value of t. Show that with $t = 10$, the analogous estimate in part **b** is

$$\sum_{k=1}^{\infty} e^{-10k^2\pi^2} < 10^{-42}$$

(This problem is essentially due to George Csordas.)

Chapter 10

Taylor Polynomials

10.1 Polynomial Approximations

Suppose we want to approximate a function $f(x)$ by a polynomial

$$f(x) \approx P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \sum_{k=0}^{n} a_k x^k.$$

One natural way to to do this is to require that $f(0) = P_n(0)$, $f'(0) = P_n'(0)$, $f''(0) = P_n''(0)$, etc., i.e., $f^{(k)}(0) = P_n^{(k)}(0)$ for $k = 0, \ldots, n$. This gives $n+1$ equations for the $n+1$ unknowns $a_0, \ldots, a_n$. If we differentiate, say $P_3(x)$, several times these equations become quite clear:

$$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{1}$$
$$P_3'(x) = 1 \cdot a_1 + 2 \cdot a_2 x + 3 \cdot a_3 x^2 \tag{2}$$
$$P_3''(x) = 2 \cdot 1 \cdot a_2 + 3 \cdot 2 \cdot a_3 x \tag{3}$$
$$P_3'''(x) = 3 \cdot 2 \cdot 1 \cdot a_3 \tag{4}$$

Setting $x = 0$ in the first of these equations gives $a_0 = f(0)$. Setting $x = 0$ in the second of these equations gives $1 \cdot a_1 = f'(0)$. Taking more derivatives and setting $x = 0$, we get $2 \cdot 1 \cdot a_2 = f''(0)$, $3 \cdot 2 \cdot 1 \cdot a_3 = f'''(0)$. By thinking about factorials, you can see the pattern evolving: the general term (solving for a_k) is

$$a_k = \frac{f^{(k)}(0)}{k!} \qquad \text{for} \qquad 0 \le k \le n \tag{5}$$

$P_n(x)$ is what is known as the n^{th} *Taylor polynomial* for $f(x)$:

$$\boxed{P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\, x^k} \tag{6}$$

The coefficient of x^k in $P_n(x)$ is just $f^{(k)}(0)/k!$ (which is the same for all n as long as $n \geq k$). This quantity is called the k^{th}*–Taylor coefficient for* $f(x)$.

As our first application, notice that it follows from (5) that the graph of $y = P_1(x)$ is just the *tangent line* to the curve $y = f(x)$ at the point $(0, f(0))$. We studied this method of approximation extensively in Section 2.2. Thus, since the tangent line yields the best degree-one approximation to the function, near the point $x = 0$, it is reasonable that guess that $P_n(x)$ is the best n^{th}–degree approximation, near $x = 0$.

Notice that if $m < n$, then the terms of degree m or less in the polynomial $P_n(x)$ equal $P_m(x)$, i.e., we obtain $P_n(x)$ from $P_m(x)$ by adding higher order terms. Now, we define the *Taylor series* for $f(x)$, about the point $x = 0$, as the corresponding *infinite series*:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, x^k = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\, x^k = \lim_{n\to\infty} P_n(x) \tag{7}$$

provided this series converges. Naturally, when this series converges we hope that it converges to $f(x)$ and hence the Taylor polynomials would converge to the function. The conditions under which this occurs will be explored throughout this chapter. The use of graphics in specific examples will make the success of this important approximation technique especially clear.

To have DERIVE compute a Taylor polynomial for a function first select the Calculus/Taylor menu, then enter the function in the form, enter some integer n, say $n = 5$, for the Degree and leave the Point[1] value at its default value of 0. This results in the expression `TAYLOR(f(x),x,0,5)`. An alternate approach after becoming familiar with its syntax is to simply author this expression. See Figure 10.1 on the next page for some of the basic examples and a comparison of the graph of $f(x) = 1/(1 - x)$ and it's 5^{th} degree Taylor polynomial approximation. An interesting exercise is to load the file F-TAY0.MTH which contains the expressions from Figure 10.1 and compare graphically the various functions with their Taylor polynomials of different degrees.

[1]For now we just take the Point value to be 0. Later, in Section 10.6 we discuss how

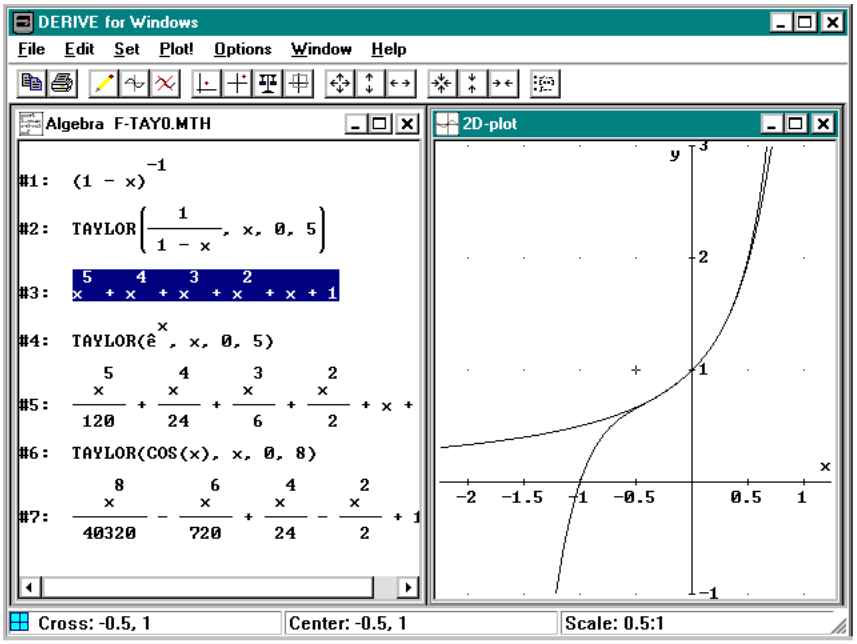


Figure 10.1: Basic examples of Taylor polynomials

10.2 Examples

As we see from Figure 10.1, the formula for the geometric series in Chapter 9 looks to be very closely related to the Taylor polynomials for the function $f(x) = 1/(1-x)$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \lim_{n\to\infty} \sum_{k=0}^{n} x^k. \tag{8}$$

This suggests that the partial sums above, $\sum_{k=0}^{n} x^k$, are the n^{th} Taylor polynomials. To verify this we must use (5) to compute the Taylor coefficients. We will need to show $f^{(k)}(0) = k!$. Using DERIVE we can make a table of derivatives by authoring

```
VECTOR([k, DIF((1-x)^-1,x,k)], k, 0, 4)
```

to use this variable.

and then simplifying. The answers seem to follow the pattern $k!/(1-x)^{k+1}$ which can be verified by having DERIVE check that:

```
DIF((1-x)^(-k-1) k!,x) = (1-x)^(-k-2) (k+1)!
```

(Try this for yourself).

Note that in Figure 10.1 the 5^{th} degree Taylor polynomial approximation gives a very good approximation on the interval $[-.5, .5]$. As we mentioned earlier, you should Load the file F-TAY0.MTH and experiment with higher degree approximations to see how the interval size improves, but we must bear in mind that the infinite series is *only* valid for $-1 < x < 1$ (even though the function appears well behaved near $x = -1$).

Three other important examples are the series for e^x, $\sin x$, and $\cos x$. If we look carefully at Figure 10.1 we might guess the pattern for the exponential function because the denominators 1, 1, 2, 6, 24, 120 are just $k!$ as k varies from $0 \le k \le 5$. On the other hand, equation (5) gives the required formula easily since all derivatives $f^{(k)}(x) = e^x$ and so are 1 at $x = 0$. Thus, in this case $f^{(k)}(0)/k! = 1/k!$ so the n^{th} Taylor polynomial is simply $\sum_{k=0}^{n} x^k/k! = 1 + x + x^2/2! + \cdots + x^n/n!$. Now, if we could take the limit as in the case of the geometric series, then

$$(9) \qquad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!}.$$

We encountered this series earlier on page 136 with $x = 2$ and also in Problem 10 on page 147 with $x = 1$. In fact, the series above does converge, for all values of x, to the exponential function. Moreover, it is this series that forms the basis for numerical calculations of the exponential function on computers and calculators. Section 10.5 will give a complete explanation of this matter.

We can proceed in a like manner to compute the Taylor polynomials for the sine and cosine functions. The only problem is that the pattern for the successive derivatives is a little trickier. Let us discuss the function $f(x) = \sin x$ since the analysis of the cosine function is similar. If we make a vector of $f^{(k)}(x)$ with $0 \le k \le 4$ we get $[\sin x, \cos x, -\sin x, -\cos x, \sin x]$ and it is clear that the pattern will repeat in groups of 4 with $f^{(4k)}(x) = \sin x$. Substituting $x = 0$ gives the pattern $[0, 1, 0, -1]$ and hence every *even* power of x, i.e., $x^0, x^2, x^4, \ldots$, will have a zero coefficient; whereas, the odd power x^{2k+1} will have the coefficient $(-1)^k/(2k+1)!$ by (5). See Figure 10.2 on the next page for several Taylor polynomials of the sine function. The only

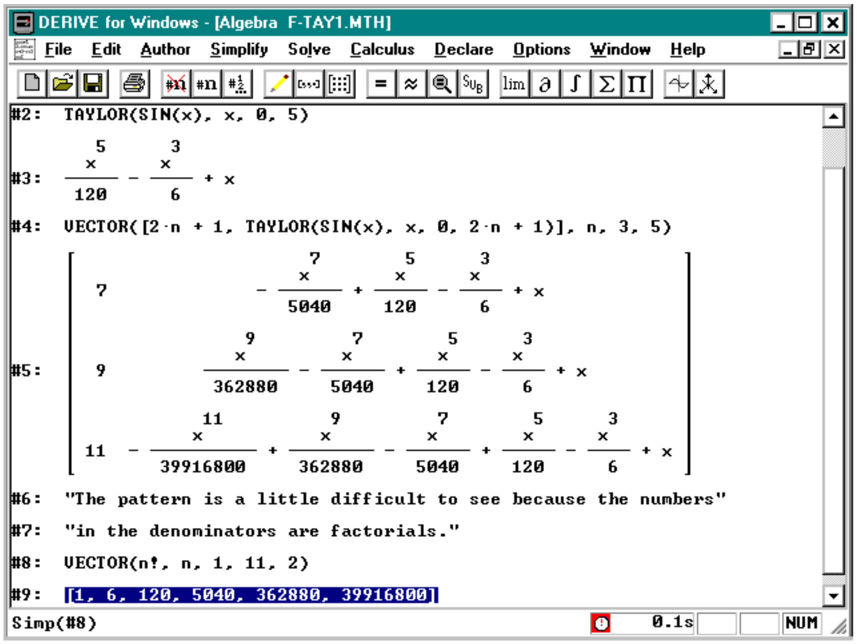


Figure 10.2: Taylor polynomials for $\sin x$

unfortunate part about making these computations in DERIVE is that the factorials are expanded to their integer values which makes it difficult to recognize the patterns. On the other hand, it's easy to see how fast the factorials in the denominator grow which means that the added terms are quite small in magnitude. At any rate, the Taylor polynomials form the partial sums of an infinite series representation of $\sin x$ which is convergent for all $-\infty < x < \infty$. This series and the one for $\cos x$ are given below:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{10}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{11}$$

Next we want to graph several of these Taylor polynomials and compare them with the graph of $\sin x$. This is done in Figure 10.3. Another instructive exercise is to plot 3 or more Taylor polynomials all at once by making a vector of the functions and then plotting the vector. As each successively higher

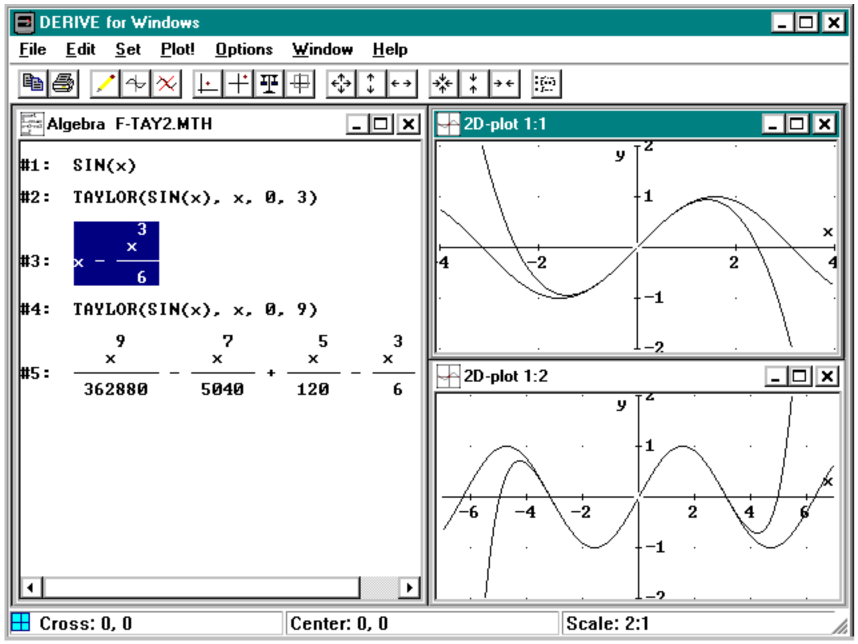


Figure 10.3: Approximating $\sin x$ with its Taylor polynomials

degree polynomial is plotted, the range of close approximation gets larger and larger. Curiously, one sees from Figure 10.3 that the approximation is good up to a point and then is very bad thereafter. The basic idea in approximating is simply to take more terms; i.e., use a higher degree Taylor polynomial, to obtain more accuracy. An example of a more precise question we shall be interested in is: What degree n is needed for approximating the sine function on the interval $[0, \pi/2]$ to within 6 decimal places?

10.3 Taylor's Theorem with Remainder

We are interested in how accurately a Taylor polynomial approximates $f(x)$ and for what values of x does the Taylor series converge to $f(x)$. The basic result is the following theorem:

Theorem 1. *Suppose that $f(x)$ is $(n+1)$–times continuously differentiable on the interval $[0, b]$. Let the n^{th} degree Taylor polynomial be denoted by*

$P_n(x)$. *Then, for any* $0 \le x \le b$ *we have*

$$\boxed{|f(x) - P_n(x)| \le \max_{0\le t\le x} |f^{(n+1)}(t)| \cdot \frac{|x|^{n+1}}{(n+1)!} = M \cdot \frac{|x|^{n+1}}{(n+1)!}} \tag{12}$$

where we abbreviate the maximum by M.

Furthermore, the theorem also holds when the defining interval is $[-b, 0]$ *for some positive* b. *The only change is that now* $-b \le x \le 0$ *and the maximum in* (12) *is taken over the interval* $x \le t \le 0$.

Observe that the error estimate in (12) is similar to those we obtained for the approximate integral formulas (Trapezoid method, Simpson's rule) in that they depend on the maximum of a high order derivative, look back at the formulas on page 91. Also, notice that when $n = 0$ then (12) follows immediately from the Mean Value Theorem and in fact, you can think of (12) as a *higher order* Mean Value Theorem.

The proof is based on a simple application of the integration by parts formula; namely, for any continuously differentiable function $g(t)$ which satisfies $g(0) = 0$, then

$$\int_0^x g(t)\frac{(x-t)^m}{m!}\,dt = \int_0^x g'(t)\frac{(x-t)^{m+1}}{(m+1)!}\,dt \qquad m = 0, 1, \ldots \tag{13}$$

Just put $u = g(t)$ and $v = (x-t)^{m+1}/(m+1)!$ and apply the integration by parts formula. Notice that the integrated terms, i.e., the $uv|_0^x$ vanish because $g(0) = 0$ at the left endpoint and $(x-t)^{m+1}$ is zero when $t = x$.

Proof. Put $g(t) = f(t) - P_n(t)$ let M be the maximum of $|f^{(n+1)}|$ on the interval $[0, x]$. By the definition of the Taylor polynomial, observe that

$$g^{(m)}(0) = 0 \quad \text{for} \quad m = 0, 1, \ldots, n \quad \text{and} \quad g^{(n+1)}(t) = f^{(n+1)}(t)$$

where the second fact follows since the $(n+1)^{\text{st}}$–derivative of any degree n polynomial is zero (look back at (1) on page 149). Now we get to apply (13) to $g', g'', \ldots, g^{(n)}$ with the result that

$$\begin{aligned}\int_0^x g'(t)\,dt = \int_0^x g''(t)(x-t)\,dt = \int_0^x g'''(t)\frac{(x-t)^2}{2}\,dt \\ = \cdots = \int_0^x g^{(n+1)}(t)\frac{(x-t)^n}{n}\,dt\end{aligned}$$

and hence that

$$
\begin{aligned}
f(x) - P_n(x) = g(x) - g(0) &= \int_0^x g'(t)\,dt \\
&= \int_0^x g^{(n+1)}(t)\frac{(x-t)^n}{n}\,dt = \int_0^x f^{(n+1)}(t)\frac{(x-t)^n}{n}\,dt\,.
\end{aligned}
$$

We take absolute values of the left and right-hand sides of the above to get

$$
\begin{aligned}
|f(x) - P_n(x)| &\le \int_0^x |f^{(n+1)}(t)|\frac{(x-t)^n}{n}\,dt \\
&\le \int_0^x M\frac{(x-t)^n}{n}\,dt = M\frac{x^{n+1}}{(n+1)!}
\end{aligned}
$$

which proves the theorem. □

10.4 Computing the Sine Function

First observe that we don't need to compute, for example, $\sin 100$ directly since the sine function is 2π periodic. We just set $x = 100 - 2k\pi$ where the integer k is chosen so $0 \le x < 2\pi$. In Derive we simplify the function `MOD(100, 2π)` to get $k = 15$ and $x = 5.75221$ approximately. Now it's an interesting exercise to use the properties of the sine function to reduce the computation to the interval $[0, \pi]$. For example, if $\pi < x < 2\pi$ then $\sin x = -\sin(2\pi - x)$ where now $0 \le 2\pi - x < \pi$. Similarly, you can use the identity $\sin(\pi - x) = \sin x$ to reduce the problem to the smaller interval $[0, \pi/2]$. It's even possible to reduce the interval to $[0, \pi/4]$.

We can use formula (12) to estimate the error in using the Taylor polynomial to estimate $\sin x$. The computation of $M = \max|f^{(n+1)}|$ might look a little formidable at first but we observe that *any* derivative is equal to either $\pm\sin t$ or $\pm\cos t$ and in either case $M \le 1$. Thus, we can take $M = 1$ and achieve 6 decimals of accuracy by determining the smallest integer n satisfying

$$
\frac{|x|^{n+1}}{(n+1)!} \le 10^{-6} \tag{14}
$$

For approximations on the interval $[0, \pi/2]$ we could just take the worst case by setting $x = \pi/2$ in the above.

We now have reduced the problem to solving (14) for the smallest possible integer n. Unfortunately, the factorial expression means that we can't use simple algebra to solve this inequality. A simple numerical approach would be to make a table with n in the first column and the above expression in the second column. Examining the data will result in an answer provided n is reasonably small. We did this earlier on page 136 when we studied the ratio test. If this fails, as with the $1/k^2$ series, you might try testing various powers of 10. Both of these techniques are easy to do using the `vector` function. (In the next section we present another way of finding n.) In Figure 10.4, see the file F-TAY3.MTH, we analyze $\sin 100$ by reducing the computation to a smaller value of x ($x = 0.530973$), determining which n yields an error of less than 10^{-6} ($n = 7$) and then computing using $P_7(0.530973)$. Observe that for the sine function $P_{2n+1}(x) = P_{2n+2}(x)$ and so for the error computation (14) we use the higher power $2n + 3$ instead $2n + 2$ and hence

$$(15) \qquad |\sin x - \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}| \le \frac{|x|^{2n+3}}{(2n+3)!} \quad \text{for} \quad n = 0, 1, \ldots$$

Lastly, let us observe that the right hand side of (15) tends to zero for any x. After all, for x fixed, the ratio of terms above is

$$\frac{|x|^{2n+5}}{(2n+5)!} \cdot \frac{(2n+3)!}{|x|^{2n+3}} = \frac{|x|^2}{(2n+5)(2n+4)} \le \frac{1}{2}$$

for all large n. Thus, $|x|^{2n+3}/(2n+3)! \le c_x/2^n$ (or for that matter $|x|^n/n! \le c_x/2^n$) for some constant c_x and the sequence tends to zero because $1/2^n \to 0$. By applying Theorem 1 we see that the Taylor series converges for all x and we indeed have the representation

$$(16) \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

which is valid for all $-\infty < x < \infty$. In a similar manner we establish (11) on page 153.

10.5 Computing the Exponential Function.

Now let's repeat the above procedure for e^x. We use the partial sums of (9) for approximating and (12) for determining the number of terms to use.

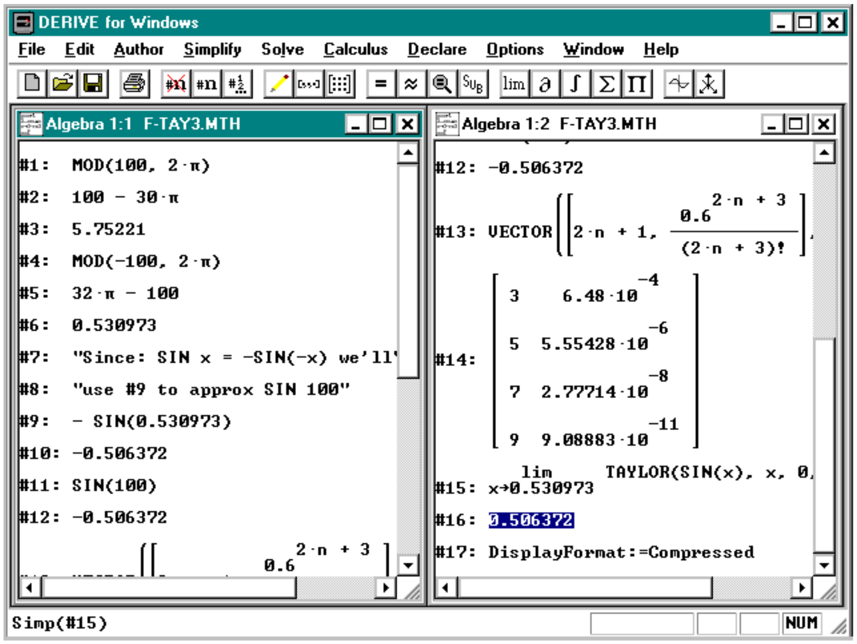


Figure 10.4: Approximating $\sin 100$ within 6 decimals

Let's assume $x > 0$. Since $f^{(n+1)}(x) = e^x$ is an increasing function, we can take $M = e^x$ or more conveniently we will replace e with the larger value 3. Thus,

$$(17) \qquad |e^x - (1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!})| < \frac{3^x x^{n+1}}{(n+1)!}$$

and so we need only find n so that the right-hand side is sufficiently small.

We would like to define a function in DERIVE to determine the number of terms n necessary to achieve 6 decimals places, rather than looking at tables. First of all, recall from the previous section that

$$\lim_{n\to\infty} \frac{3^x x^{n+1}}{(n+1)!} = 0$$

for all values of x. Hence we are guaranteed that there is a first n for which the above quantity is less than 10^{-6}. Moreover, this proves that the Taylor series converges for all x, by Theorem 1, to e^x. Thus, as stated earlier

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!}$$

for all $-\infty < x < \infty$.

Now consider the functions

1: `N1(x,k) := IF(`$\dfrac{3^x x^k}{k!} < 10^{-6}$`, k-1, N1(x,k+1))`

2: `N(x) := N1(x,1)`

and consider what happens when you Simplify $N(5)$. The N-function computes the $N1$-function with a starting value of $k = 1$. The error expression is compared to 10^{-6} and if successful then $k - 1$ is the value of $N(5)$; otherwise, k is increased by one and the process continues. Eventually, we get to a large enough k so that the comparison with 10^{-6} is successful and that value of $k - 1$ is returned as the value of the function. The function $N1$ is called a *recursive function* because its definition refers to itself. Care has to exercised with such functions to make sure that they eventually return a value and don't continue computing forever (press the Esc if this happens). See Figure 10.5 and load the file F-TAY4.MTH where these functions are used to define a new version of the exponential function (for $x \geq 0$) which is accurate to 6 decimal places. A comparison of this function with the built in version obtained by approximating shows that the build in function is faster but the accuracy is the same for the first 6 decimals using $P_{25}(x)$.

10.6 Taylor Expansions About $x = c$

Up to this point we have been approximating functions near $x = 0$. Suppose instead we want to approximate $f(x)$ near $x = c$. A simple approach is to define $g(x) = f(x + c)$ and approximate $g(x)$ near $x = 0$ as before. Observe, that for $x \approx c$ we then have

$$f(x) = g(x - c) \approx P_n(x - c) = \sum_{k=0}^{n} a_k (x - c)^k$$

where P_n is the Taylor polynomial for $g(x)$ and hence

$$a_k = \frac{g^{(k)}(0)}{k!} = \frac{f^{(k)}(c)}{k!}\,.$$

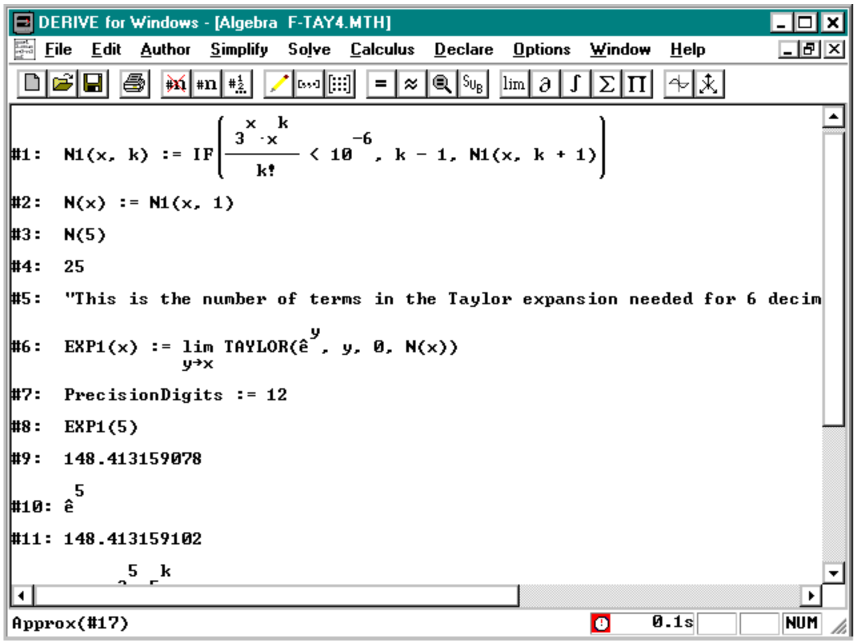


Figure 10.5: Approximating e^5 within 6 decimals

By the above observation it makes sense to define

$$P_n(x,c) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

to be the n^{th}–Taylor polynomial of $f(x)$, *expanded about the point* $x = c$. In DERIVE we just enter `TAYLOR(f(x),x,c,n)` or put the Point variable equal to c if we use the menu method.

Similarly, the Taylor series expansion about the point c is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

provided this series converges to $f(x)$. To discuss convergence of the above we use Theorem 1 applied to the function $g(x) = f(x+c)$. We do the same thing when we are computing with DERIVE. The advantage of this method for DERIVE is that if the fifth Taylor polynomial of $f(x)$ around c is say

$\sum_{k=0}^{5} a_k(x-c)^k$, DERIVE will expand the powers of $x - c$ so you get an expression like $b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$, and you won't be able to see what the a_k's are. Thus,

> If you want a Taylor polynomial of $f(x)$ expanded about the point c, it is best to find the Taylor polynomial of $f(x+c)$ expanded about 0.

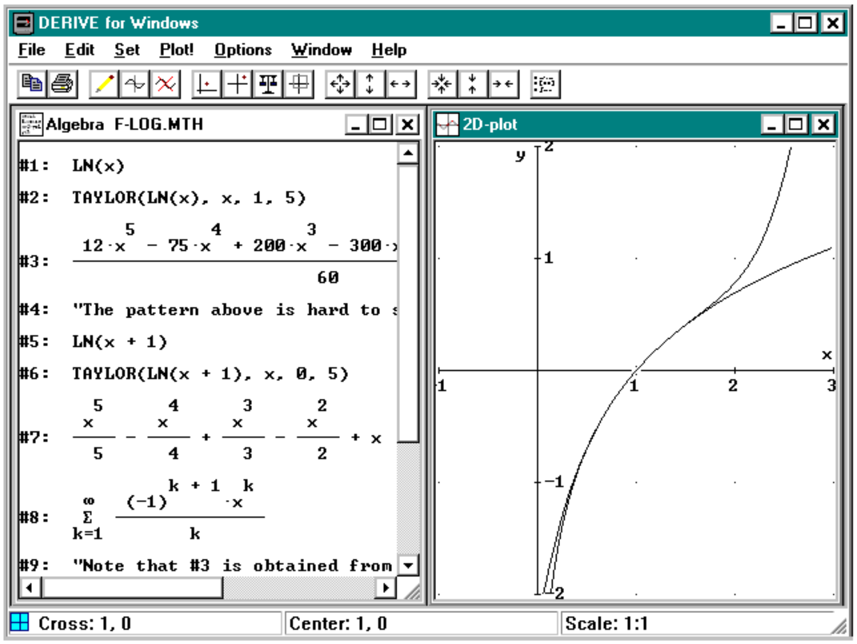


Figure 10.6: Taylor expansion of the logarithm function

A nice illustration of this technique is to examine Figure 10.6 where $f(x) = \ln x$ is plotted along with $P_5(x, 1)$. Since $\ln 0$ is not even defined it would be foolish to think about its Taylor expansion about $x = 0$, however, expanding about $x = 1$ is a reasonable alternative. Notice that `TAYLOR(ln x,x,1,5)` produces a messy result in which the 6^{th} term is hard to guess but that there is a clear pattern in `TAYLOR(ln (x+1),x,0,5)`. In fact, it can be shown that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k, \qquad -1 < x \le 1 \tag{18}$$

although the proof that the series converges to $\ln x$ on the interval $0 < x \le 1/2$ is straightforward, see Exercise 4, is quite a bit harder than our earlier examples to get the full interval $0 < x \le 1$. You can load the file F-LOG.MTH and try approximating $\ln x$ with higher degree Taylor polynomials to see if you can confirm the above representation. The full convergence problem for the logarithm function will be studied in the next chapter.

10.7 Interval of Convergence

The Taylor series for $f(x) = \sin x$, $\cos x$, or e^x converges to $f(x)$ for all values of x. This means that, by taking the degree large enough, the Taylor polynomials of these functions will approximate $f(x)$ accurately on arbitrarily large intervals. However, the geometric series (8) only converged for $|x| < 1$ and so the Taylor polynomial will approximate $1/(1-x)$ only on this interval. Of course, $1/(1-x)$ is not continuous at $x = 1$ and hence it is not surprising that the Taylor polynomials will not converge at $x = 1$. Surprisingly, this divergence at $x = 1$ turns out to influence the convergence of the series for negative values of x! It is an important basic theorem about the convergence of Taylor series that if the series converges at a point $x_1 \neq 0$, then it also converges at *all* $|x| < |x_1|$. Thus, any Taylor series which diverges at $x = 1$ cannot converge at any $x < -1$. Why? If it did converge say at $x_1 = -2$, then it would also converge at $x = 1$. But it diverges for $x = 1$ so it cannot possibly converge at $x_1 = -2$ (or any $|x| > 1$). This fact also leads to the observation that the set of points x where the Taylor series converges must be an interval which is centered about the origin. Actually, there are four possibilities for the interval of convergence: $(-r, +r)$, $(-r, +r]$, $[-r, +r)$ or $[-r, +r]$ for some $0 \le r \le +\infty$.[2] This number r is called the *radius of convergence.*

Now consider the function $1/(1 + x^2)$. We can obtain the Taylor series for this function by substituting $-x^2$ for x in (8):

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{k=0}^{\infty} (-1)^k x^{2k} \tag{19}$$

As before we can use DERIVE to plot several of the Taylor polynomials for this function; see Figure 10.7.

[2]We need to allow the notation $r = +\infty$ so that the set of all real numbers can be represented as the interval $(-r, r)$.

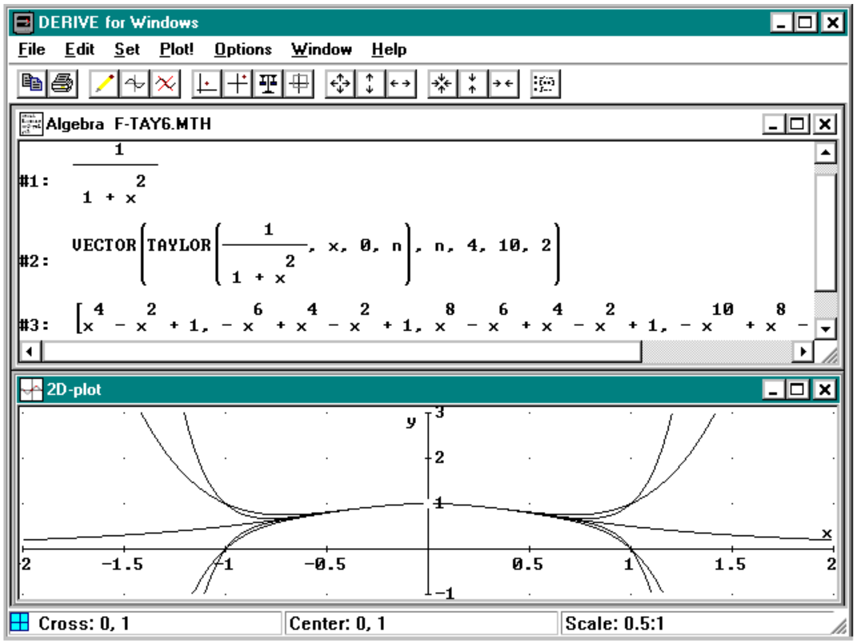


Figure 10.7: Graphically finding the radius of convergence

Notice that although the higher degree polynomials do a better job of approximating the function for $|x| < 1$, none of them work outside of this region. This strongly suggests that the radius of convergence of $1/(1+x^2)$ is $r = 1$. This can be proved by observing the following: if the series $s = \sum_{k=1}^{\infty} a_k$ converges then the terms $a_k \to 0$. This is because $a_n = s_n - s_{n-1} \to s - s = 0$. Now, in our case $|a_k| = |x|^{2k} \to \infty$ whenever $|x| > 1$. So even though the function $1/(1+x^2)$ is defined and differentiable to all orders on the whole real line, the radius of convergence of its power series is $r = 1$. It is therefore impossible to deduce the radius of convergence for a function by looking at its graph.

In the case of our example $1/(1+x^2)$, an interesting explanation as to why $r = 1$ can be based on the fact that $x^2 + 1$ has a *complex* root at the point $x = i$, which a distance one from the origin. We will not pursue this approach here but let us say that this application of complex numbers turned out to be one the the great triumphs for this man-made invention.

10.8 Laboratory Exercises

1. Start by declaring $f(x)$ to be a function, i.e., Author `F(x):=`. Use the Calculus/Taylor series menu to produce the expression `TAYLOR( F(x), x, 0, 5)` and then edit this expression by replacing the 5 with n (note that the Taylor menu requires integer values for the degree). With this last expression highlighted, use the Calculus/Vector menu twice, with the Variable set to n, Start value 4, End value 10 and Step size 1 to produce the two expressions:

```
VECTOR(TAYLOR( F(x), x, 0, n), n, 4, 10, 1)
VECTOR([n, TAYLOR( F(x), x, 0, n)], n, 4, 10, 1)
```

For each of the functions below do the following:

(i) Define $f(x)$ to be the given function.

(ii) Simplify the first `vector` function above to make a 7–vector which has the degree n Taylor polynomial, expanded about $x = 0$, for $n = 4, \ldots, 10$ as its entries.

(iii) Graph this vector to plot each of these Taylor polynomial in succession. Then, plot the function, say in the color red, and compare the graphs using an appropriate scale.

(iv) Simplify the second `vector` function to make a 7×2–table that has the degrees n in the first column and $P_n(x)$ in the second column.

(v) Use your table to guess what the infinite Taylor series expansion is.

(vi) Prove that in each case, the Taylor series expansion converges to the function and determine the interval of x's for which it is valid. Use the series techniques of the previous chapter to do this. (Hint: Try using (2) on page 130.)

a. $f(x) = \dfrac{x^8}{8} + \cdots + \dfrac{x^2}{2} + x$

b. $f(x) = \dfrac{1}{3 - x}$

c. $f(x) = \dfrac{1}{x^2+2}$ (Hint: For the pattern recognition you will need to change the output mode to Rational. Use the Declare/Algebra state menu to access the Output menu.)

2. Let $f(x) = \ln\left(\frac{1+x}{1-x}\right)$ for $-1 < x < 1$.

a. Do parts (i)–(iv) of Problem 1 using $f(x)$. Show that your analysis suggests that

$$\ln\left(\frac{1+x}{1-x}\right) = 2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} \quad \text{for} \quad -1 < x < 1 \tag{20}$$

b. Plot $g(x) = \frac{1+x}{1-x}$ and show that $y = g(x)$ is a strictly increasing function on $-1 < x < 1$ with range $0 < y < \infty$.

c. Solve $\frac{1+x}{1-x} = 3$ for x. Let x_3 be your answer.

d. Assuming that (20) above is indeed valid, we get an infinite series for $\ln 3$ by substituting $x = x_3$ into (20). Use the ratio test on page 136 to prove that the series converges.

e. Compare the numerical values of $P_n(x_3)$ for various n with the approximate value of $\ln 3$.

3. If we take $x = 1/3$ in (20) above we get

$$\int_1^2 \frac{dt}{t} = \ln 2 = \sum_{k=0}^{\infty} \frac{2}{(2k+1)3^{2k+1}}.$$

In Chapter 6 we studied numerical integration techniques for approximating the above integral with the most efficient method being Simpson's rule. One the other hand, using the ratio test on page 136 we approximated the infinite series similar to the above.

a. Using the error in Simpson's rule, formula (5) on page 91, determine approximately how many subdivisions (and hence how many computations) are needed to obtain 8 decimal place accuracy.

b. For completeness, also do part (a) using the left endpoint method and the trapezoid method.

c. Show that the ratio of terms in the above series is less than $1/9$.

d. Using formula (7) of Theorem 1 on page 136 with $x = 1/9$, determine how many terms are needed to approximate $\ln 2$ to 8 decimal places.

e. Now compare all four approximation techniques. Which method is the most efficient?

4. Let $f(x) = -\ln(1-x)$. We want to determine the Taylor series for $f(x)$ and prove that it converges to $f(x)$ using Theorem 1 on page 154.

a. Compute the first several derivative of $f(x)$ and guess at a general formula for $f^{(n)}(x)$ for all $n = 0, 1, 2, \ldots$.

b. Use part **a** to establish the Taylor series of $f(x)$ and hence if the Taylor series converges to $f(x)$ we would have:

$$\ln\left(\frac{1}{1-x}\right) = \sum_{k=1}^{\infty} \frac{x^k}{k} = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \tag{21}$$

c. Use Theorem 1 to show that the (1) holds for any $-1 \le x \le 1/2$. (Hint: Carefully compute the right-hand side of (12) on page 155. Then, show that the error estimate tends to zero as $n \to \infty$ only for $-1 \le x \le 1/2$.)

d. Show that taking $x = -1$ in (21) leads to another series representation of $\ln 2$. Analyze how quickly the partial sums of this series converge to $\ln 2$ by making tables of numerical computations. How efficient is this approach compared with the previous problem?

e. It turns out that (21) actually holds for all $-1 \le x < 1$ and the radius of convergence is $r = 1$. By computing $P_n(x)$ for several n and comparing their graphs with $f(x)$, show that the Taylor polynomials *seem* to converge to $f(x)$ on the full interval $-1 \le x < 1$

5. For each of the functions below do the following:

(i) Do parts (i)–(iv) of Problem 1 using these functions.

(ii) By comparing the graph of the function with several Taylor polynomials make a guess at the interval of x's for which the Taylor expansion is valid, see Section 10.7.

(iii) Give further support for your answer in (ii) by picking some nonzero x_1 (where the Taylor representation is valid) and numerically comparing the function's value at x_1 with that of several of its Taylor polynomials. Recall that $\tan^{-1} x$ is entered as `atan x` in DERIVE.

a. $\dfrac{1}{(x+1)^2}$ **b.** $\tan^{-1} x$

c. e^{-x^2}

6. Load the file F-TAY3.MTH and following the methods of Section 10.4 compute $\sin 7$. Which degree Taylor polynomial should you use to get an error of less than 10^{-6}?

7. In this problem we approximate e^5 using the methods in Section 10.5.

a. Express e^5 using the Taylor series representation of the exponential function.

b. Compare the numerical value of e^5 using approximate with the value of the first several Taylor polynomials. How many terms appear to needed for 6 decimal place accuracy?

c. We now want to use Theorem 1 on page 154 to obtain a precise estimate of e^5 within 10^{-6}. Compute an upper bound on the error estimate on the right-hand side of (12) for several n. Do this by first giving an upper estimate for M and then making a list of several error estimates until the value becomes less than 10^{-6}.

d. What is your estimate for e^5 and how many terms do you need?

8. The functions $f(x) = \sin x$ has only odd powers in its Taylor series expansion. This property can be explained by the fact that $f(x)$ satisfies the equation $f(-x) = -f(x)$ as do all odd powers of x. It is because of this that we call any such $f(x)$ an *odd function*. Similarly, a function is an even function if $f(-x) = f(x)$ holds for all x, as do all even powers of x.

a. Prove that $f^{(2k)}(0) = 0$ for $k = 0, 1, \ldots$ and any odd function $f(x)$.

b. Prove that $f^{(2k+1)}(0) = 0$ for $k = 0, 1, \ldots$ and any even function $f(x)$.

***9.** We discussed complex number in Section 5.4. Find the 6^{th} Taylor polynomial for the function e^{ix}. (Recall that i is entered in DfW using the symbol bar and with Alt-i in DERIVE FOR DOS) Now find the 6^{th} Taylor polynomial for $\cos x$ and for $\sin x$. Multiply the one for $\sin x$ by i and add it to the one for $\cos x$. Compare the result with the polynomial for e^{ix}. What relation between e^{ix}, $\cos x$, and $\sin x$ does this suggest?

***10.** In this problem we use the Taylor polynomials for the arc tangent function $\tan^{-1} x$ to estimate π. Recall that $\tan^{-1} x$ is entered as `atan x`.

a. Use DERIVE to verify the formula

$$\frac{\pi}{4} = 4\tan^{-1}(1/5) - \tan^{-1}(1/239)$$

b. Compute the eighth degree Taylor polynomial $P_8(x)$ for $\tan^{-1} x$.

c. Use $P_8(x)$ to approximate on the right side of the above formula and use your answer to estimate π.

d. Let M be the maximum value of the 9^{th} derivative of $tan^{-1}(x)$ on the interval $[0, 1/5]$. Use the error estimate (12) on page 155 to give an upper bound for the error in your estimate of π in terms of M. For example, give an answer like $M \cdot 10^{-2}$.

e. Use graphical techniques to give an upper bound on M.

f. Combine the last two parts to give an estimate on the number of decimal places your estimate to π valid for. How does this compare with DERIVE approximation to π?

g. Use DERIVE to show that the *absolute maximum* value for $|f^{(9)}(x)|$, where $f(x) = \tan^{-1}(x)$, is achieved at $x = 0$.

Chapter 11

Approximating Integrals with Taylor Polynomials

11.1 Introduction

In Chapter 6 we developed several techniques for approximating a definite integral $I = \int_0^b f(x)\,dx$ by applying the trapezoid method or Simpson's rule. In the last chapter we saw that many of the important functions in Calculus can be represented by a Taylor series and hence can be approximated by their Taylor polynomials. This suggest *another* approach to approximating I; namely, approximate the integrand $f(x)$ by its Taylor polynomials and then use

$$\int_0^b f(x)\,dx \approx \int_0^b P_n(x)\,dx = \int_0^b \sum_{k=0}^{n} a_k x^k\,dx = \sum_{k=0}^{n} a_k \frac{b^{k+1}}{k+1}$$

where $a_k = f^{(k)}(0)/k!$ to obtain the desired estimate.

The advantage to this approach was strongly suggested by Problem 3 on page 165. In that problem it was shown that approximating $\ln 3$ using Taylor series techniques gave 8–decimal place accuracy with approximately 8 computations. Whereas the standard approach using Simpson's rule require approximately 100 computations.

11.2 The Basic Error Estimate

Recall from (12) of Chapter 10 on page 155 that if $\sum_{k=0}^{n} a_k x^k$ is the n^{th} Taylor polynomial for $f(x)$, then an upper bound for the error made in estimating $f(x)$ with this Taylor polynomial is given by

$$\left| f(x) - \sum_{k=0}^{n} a_k x^k \right| \leq M \frac{|x|^{n+1}}{(n+1)!} \tag{1}$$

where M is the maximum of $|f^{(n+1)}(t)|$ on the interval connecting 0 to x. This can be written as

$$-M \frac{|x|^{n+1}}{(n+1)!} \leq f(x) - \sum_{k=0}^{n} a_k x^k \leq M \frac{|x|^{n+1}}{(n+1)!} \tag{2}$$

If we integrate this from 0 to b, we get

$$-M \frac{|b|^{n+2}}{(n+2)!} \leq \int_0^b f(x)\,dx - \sum_{k=0}^{n} \frac{a_k}{k+1} b^{k+1} \leq M \frac{|b|^{n+2}}{(n+2)!} \tag{3}$$

Writing this with absolute values:

$$\left| \int_0^b f(x)\,dx - \sum_{k=0}^{n} \frac{a_k}{k+1} b^{k+1} \right| \leq M \frac{|b|^{n+2}}{(n+2)!} \tag{4}$$

This technique works for integrals going from 0 to b. If you want to approximate $\int_a^b f(x)\,dx$, you can make the substitution $u = x - a$ so the integral becomes $\int_0^{b-a} f(u)\,du$.

11.3 The Logarithm Series

Consider the logarithm function $f(x) = \ln \dfrac{1}{1-x}$ where we shift the variable so that $x = 0$ yields $\ln 1 = 0$. First of all, we have the integral representation:

$$\ln \frac{1}{1-x} = \int_0^x \frac{dt}{1-t} \qquad \text{for} \qquad -\infty < x < 1\,.$$

which can be easily checked using DERIVE.

Now, the idea is to approximate the integrand by its Taylor series but in this case we recognize the connection with the geometric series; namely,

$$\sum_{k=0}^{\infty} t^k = \frac{1}{1-t} \qquad \text{for} \qquad -1 < x < 1\,.$$

We'll actually use the following more refined estimate from Section 9.2:

$$\frac{1}{1-t} - \sum_{k=0}^{n} t^k = \sum_{k=n+1}^{\infty} t^k = t^{n+1} + t^{n+2} + \cdots = \frac{t^{n+1}}{1-t} \tag{5}$$

whenever $|t| < 1$. Now we integrate this equation from 0 to x, where we assume that $-1 \le x < 1$, to get

$$\ln\frac{1}{1-x} - \sum_{k=0}^{n} \frac{x^{k+1}}{k+1} = \int_0^x \left(\frac{dt}{1-t} - \int_0^x \sum_{k=0}^{n} t^k \right) dt \tag{6}$$

$$= \int_0^x \frac{t^{n+1}}{1-t}\, dt\,. \tag{7}$$

Taking absolute values of the above we need to evaluate the integral in (7). Since this looks complicated, we instead try to obtain an upper bound. For positive x, we uses the inequality $0 < \frac{1}{1-t} \le \frac{1}{1-x}$ to obtain that

$$\left| \int_0^x \frac{t^{k+1}}{1-t}\, dt \right| \le \int_0^x \frac{t^{n+1}}{1-x}\, dt = \frac{x^{n+2}}{(1-x)(n+2)}\,. \tag{8}$$

On the other hand, for negative x, we instead use $0 < \frac{1}{1-t} \le 1$ to get a similar bound:

$$\left| \int_x^0 \frac{t^{k+1}}{1-t}\, dt \right| \le \int_x^0 |t|^{n+1}\, dt = \frac{|x|^{n+2}}{(n+2)}\,. \tag{9}$$

Hence, we have the desired approximation result because

$$\lim_{n\to\infty} \frac{x^{n+2}}{(1-x)(n+2)} = 0 \quad \text{whenever} \qquad 0 \le x < 1$$

and

$$\lim_{n\to\infty} \frac{|x|^{n+2}}{(n+2)} = 0 \quad \text{whenever} \quad -1 \le x \le 0$$

One small point, the polynomial approximations in (6) look a little different from the standard Taylor polynomials because the powers are expressed with the index $k+1$. This is just an artificial difference since

$$P_{n+1}(x) = \sum_{j=1}^{n+1} \frac{x^j}{j} = x + \frac{x^2}{2} + \cdots + \frac{x^{n+1}}{n+1} = \sum_{k=0}^{n} \frac{x^{k+1}}{k+1}$$

and hence we ultimately obtain that

$$\left| \ln \frac{1}{1-x} - P_n(x) \right| \leq \begin{cases} \dfrac{x^{n+1}}{(1-x)(n+1)}, & 0 \leq x < 1; \\ \dfrac{|x|^{n+1}}{(n+1)}, & -1 \leq x \leq 0 \end{cases} \tag{10}$$

which tends to zero as $n \to \infty$. This leads to the Taylor series representation

$$\ln \frac{1}{1-x} = \sum_{j=1}^{\infty} \frac{x^j}{j} \quad \text{for} \quad -1 \leq x < 1\,. \tag{11}$$

11.4 An Integral Approximation

Suppose we wanted to estimate the definite integral

$$\int_0^1 \frac{\sin x}{x}\, dx.$$

At first glance there appears to be a problem at $x = 0$ because we are dividing by zero. However, L'Hospital rule shows that $\lim_{x\to 0} \sin x/x = 1$. An interesting alternative way of proving this fact is use the Taylor series representation for the $\sin x$, i.e.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

for all $-\infty < x < +\infty$. Now for $x \neq 0$ we can divide both sides by x to get

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots .$$

Formally then, the right-hand side above approaches 1 as $x \to 0$ because all the x^n–terms tend to zero. Of course, there is always the problem of making estimates for *infinite* series, as opposed to finite sums, which can be difficult.

One way around this difficulty is use the approach we adopted for approximating $\sin x$ in Section 10.4. There we used Taylor's Theorem with remainder to show that

$$\left|\sin x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)\right| \leq \frac{|x^{2n+3}|}{(2n+3)!} \tag{12}$$

for all $n = 0, 1, \dots$. For example, even taking $n = 0$ in the above yields a nice result; namely, $|\sin x - x| \leq |x|^3/6$. Hence, $|\dfrac{\sin x}{x} - 1| \leq x^2/6 \to 0$ as $x \to 0$ and thus $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

Similarly, if we take a larger value of n, say $n = 3$, we get

$$\left|\sin x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right)\right| \leq \frac{|x^9|}{9!} \tag{13}$$

and so dividing by x and integrating from 0 to 1 yields

$$\begin{aligned}
&\left|\int_0^1 \frac{\sin x}{x}\,dx - \sum_{n=0}^{3} \frac{1}{(2n+1)(2n+1)!}\right| \\
=&\left|\int_0^1 \frac{\sin x}{x}\,dx - \left(1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \frac{1}{7\cdot 7!}\right)\right| \\
=&\left|\int_0^1 \left(\frac{\sin x}{x} - \sum_{n=0}^{3} \frac{(-1)^n x^{2n}}{(2n+1)!}\right)\,dx\right| \leq \left|\int_0^1 \frac{|x|^8}{9!}\,dx\right| = \frac{1}{9\cdot 9!}.
\end{aligned}$$

Finally, since $1/(9\cdot 9!) \approx 3.0619 \times 10^{-7}$ we get 6 decimal place accuracy by approximating the integral using 4 terms from the series.

In Figure 11.1 on the following page we have DERIVE approximate our integral using 20 digit precision. This computation, which uses Simpson's rule, is actually quite slow, Load the file F-SININT.MTH and try this yourself. On the other hand, we enter the partial sums of the series solution and make a table comparing the first several sums with the answer from DERIVE. Notice that the theoretical error that we calculated above is practically the same as the actual error when $n = 3$.

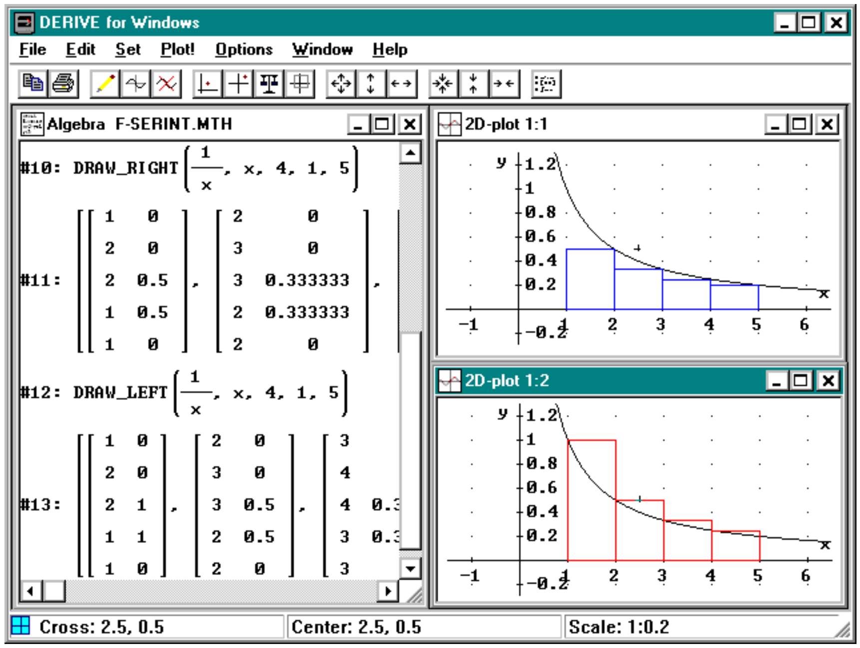


Figure 11.1: Using Taylor series to approximate integrals

11.5 Laboratory Exercises

1. Use (11) to prove that

$$\ln\left(\frac{1+x}{1-x}\right) = 2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1} \quad \text{for} \quad -1 < x < 1$$

Look back at Problems 2 and 3 on page 165 to verify that the series representation in those problems is valid.

2. Use the formula

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2}$$

and the techniques of this chapter to prove that the Taylor representation

$$\tan^{-1} x = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \quad \text{for} \quad -1 \le x \le 1$$

holds. Look back at Problem 8 on page 168 and also Problem 5**b**. What are the implications of this problem to the earlier ones?

3. In this problem you are to estimate

$$\int_0^2 e^{\sqrt{x}}\,dx$$

using the method outlined in the text.

a. Define and simplify `P(x):=` the 8$^{\text{th}}$–degree Taylor polynomial for e^x.

b. Author $P(\sqrt{x})$ and integrate the result from 0 to 2. Simplify this integral and then express the answer as a decimal.

c. Compute the maximum value of the ninth derivative of e^x on the interval 0 to $\sqrt{2}$. Denote this maximum by M. (Note: This is the M value associated with the Taylor polynomial $p(x)$ in (12) on page 155 corresponding to the interval $0 \le x \le \sqrt{2}$. The reason we use $\sqrt{2}$ and not 2 is that if $|e^x - p(x)| \le c$ for $0 \le x \le \sqrt{2}$, then $|e^{\sqrt{x}} - p(\sqrt{x})| \le c$ for $0 \le \sqrt{x} \le \sqrt{2}$, *i.e.*, for $0 \le x \le 2$.)

d. In a manner similar to what was done in Section 11.4, find the error in the approximation you obtained in part **b.**

e. Have DERIVE evaluate $\int_0^2 e^{\sqrt{x}}\,dx$ and then approximate it and compare the answer with what you obtained in part **b.**

4. Do parts **a.** to **e.** but this time for

$$\int_0^1 e^{-x^2}\,dx.$$

Instead of starting with the Taylor polynomial for e^x, start with the Taylor polynomial of e^{-x}.

Chapter 12

Harmonic Motion and Differential Equations

12.1 Introduction

In this chapter we consider differential equations of the form

(1) $$y'' + by' + cy = 0$$

These are *second order* differential equations because of the y'' Let's first look at the special case $b = 0$ and $c = 1$:

(2) $$y'' = -y$$

If we try $y = e^{rt}$ then $y'' = r^2e^{rt}$ so for y to satisfy (1) we need $r^2 = -1$. While there is no real number r satisfying $r^2 = -1$, the complex number i does. And so does $-i$. (Recall in DERIVE you input i by using the symbol bar or typing `#i`. This is displayed with $\hat{i}$. Try inputing i^2. It should simplify to -1.) Thus both $y = e^{it}$ and e^{-it} are solutions.

What is the function e^{it}? If we author this expression and then ask for the 6th Taylor approximation we get

$$-\frac{t^6}{720} + \frac{t^4}{24} - \frac{t^2}{2} + 1 + \hat{i}\left(\frac{t^5}{120} - \frac{t^3}{6} + t\right)$$

If we find the 6th Taylor polynomial for $\cos(x)$ and $\sin(x)$ we get $-t^6/720 + t^4/24 - t^2/2 + 1$ and $t^5/120 - t^3/6 + t$. This suggests that

(3) $$e^{it} = \cos(t) + i\sin(t)$$

We can check this by having DERIVE simplify $e^{it} - (\cos(t) + i\sin(t))$. Equation (3) breaks the function e^{it} into it real and imaginary parts. Each one is a solution of (2) as you can easily check. We will use this trick to solve (1), first finding the solution as a complex function and then taking the real and imaginary parts to get the 'real' solutions.

So now let's try $y = e^{rt}$ in (1). We get

$$r^2e^{rt} + bre^{rt} + ce^{rt} = (r^2 + br + c)e^{rt} = 0$$

so we want to solve

$$r^2 + br + c = 0 \tag{4}$$

This equation is called the *characteristic equation*. Its roots are

$$r_1 = \frac{-b + \sqrt{b^2 - 4c}}{2} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4c}}{2} \tag{5}$$

and so both $y = e^{r_1 t}$ and $y = e^{r_2 t}$ are solutions to (1). If $b^2 - 4ac \geq 0$ then these are real solutions. If $b^2 - 4ac < 0$ then both r_1 and r_2 are complex numbers. We can write $r_1 = \alpha + i\beta$ where $\alpha = -b/2$ and $\beta = \sqrt{4c - b^2}$ (since $b^2 - 4ac < 0$, $4ac - b^2 > 0$). Note $r_2 = \alpha - i\beta$ which is known as the *complex conjugate* of r_1.

To find the real solutions corresponding to $e^{(\alpha+i\beta)t}$ we calculate

$$e^{\alpha+i\beta} = e^{\alpha t}e^{i\beta t} = e^{\alpha t}\big(\cos(\beta t) + i\sin(\beta t)\big)$$

This suggests that $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$ are both solutions and we can easily check that they are. (Just substitute $e^{\alpha t}\cos(\beta t)$ for y in (1) and show the left side simplifies to 0.)

When $b^2 - 4ac = 0$, $r_1 = r_2$. In this case both $e^{r_1 t}$ and $te^{r_1 t}$ are solutions; see Exercise 1 on page 189. Finally notice that if both $y(t)$ and $z(t)$ are solutions to (1), then $C_1 y(t) + C_2 z(t)$ is a solution for any constants C_1 and C_2.

Summarizing, with r_1, r_2, α, and β as above, the solutions of (1) are

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t} \qquad \text{if } b^2 - 4c > 0 \tag{6}$$

$$y = C_1 e^{r_1 t} + C_2 t e^{r_1 t} \qquad \text{if } b^2 - 4c = 0 \tag{7}$$

$$y = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t) \qquad \text{if } b^2 - 4c < 0 \tag{8}$$

Notice that all of these solutions have two arbitrary constants C_1 and C_2. These will be determined by the initial value of y, denoted y_0, at t_0, and the initial velocity v_0 at this time. Of course by velocity we mean dy/dt. Assuming you have loaded the ADD-HEAD.MTH file, you can solve (1) subject to these initial conditions by authoring

```
DE2(b, c, t, t0, y0, v0)
```

12.2 Applications

Springs and Hooke's Law. Suppose we have a mass m attached to the end of a spring hanging from the ceiling. If we pull the mass down a little it will bounce (oscillate) up and down. We image it moving along the y–axis with $y = 0$ denoting the rest position. Newton's law says $F = ma$ where F is the force on the mass and $a = y''$ is the acceleration. A reasonably good approximation of the force is given by *Hooke's Law* which states

$$F = -ky$$

where k is a positive constant. Since $F = my''$ this gives the following differential equation.

$$y'' + \frac{k}{m}y = 0 \tag{9}$$

As an example suppose we pull the mass down a units and let go. Then $y_0 = -a$ and $v_0 = 0$ so we can find the motion by authoring `DE2(0, k/m, t, 0, -a, 0)`. DERIVE gives an answer in terms of two exponential functions because it does not know that k and m are positive but if you use the Declare/Variable Domain to tell DERIVE that k is positive and do the same for m, the answer simplifies to

$$-a\cos\left(\frac{\sqrt{k}}{\sqrt{m}}\,t\right)$$

Figure 12.1 on the next page shows the graph of this function when $k = 2$ and $m = 1$ and a varies between -2 and 2 in increments of 0.5. Notice all of the graphs cross the x–axis at the same place; that is, at the same time. So it doesn't matter how far the spring is pulled down it will take the same amount of time to return to its original position. If we let $\omega = \sqrt{\frac{k}{m}}$ then time to return to the original position is $2\pi/\omega$ independent of a. This is called the *period* of oscillation. ω is called the *angular frequency* while the reciprocal of the period, $\omega/2\pi$ is the *frequency*. In Exercise 2 on page 189 you investigate what happens if we start with $y_0 = 0$ but vary the velocity.

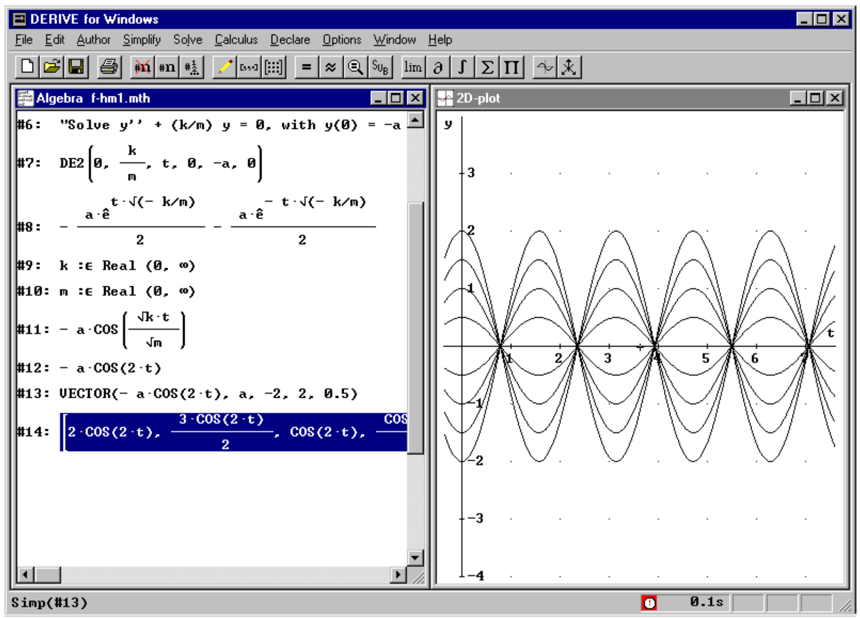


Figure 12.1: Spring motion starting at different positions

Damped oscillation. The frictional force due to air resistance is proportional to the velocity of the mass. If we take this into account our differential equation (9) becomes

$$y'' + \frac{a}{m}y' + \frac{k}{m}y = 0 \tag{10}$$

where a is a positive constant. The characteristic equation for this equation is

$$r^2 + \frac{a}{m}r + \frac{k}{m} = 0$$

It roots are

$$\frac{-a \pm \sqrt{a^2 - 4km}}{2m}$$

The solutions to (10) are given in (6)–(8). The sign of $\sqrt{a^2 - 4km}$ determines which equation applies. If $a^2 > 4km$ we say the spring is *over damped*. In this case the solutions of (10) have the form of (6) with r_1 and r_2 the solutions to the characteristic equation given above. Notice that since a, k, and m are all positive, both r_1 and r_2 are negative. So the general solution is the sum of two decaying exponentials.

If $a^2 = 4km$ the spring is *critically damped*. The solutions are given by (7). Again r_1 is negative.

If $a^2 \leq 4km$ the spring is *under damped*. The solutions are given by (8). As an illustration we take $a/m = 1$ and $k/m = 4$ in (9). To solve (9) with initial conditions $t_0 = 0$, $y_0 = 2$, and $v_0 = 0$ we author `DE2(1, 4, t, 0, 2, 0)`. Simplifying this gives

$$e^{-t/2}\left(2\cos(\frac{\sqrt{15}}{2}t) + \frac{2\sqrt{15}}{15}\sin(\frac{\sqrt{15}}{2}t)\right)$$

We use the Declare/Algebra State/Simplification menu to set Trigonometry to Collect and simplify again we get

$$\frac{8\sqrt{15}}{15}e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}t) + 2\arctan\frac{\sqrt{15}}{5}\right)$$

$$\approx 2.06559e^{-t/2}\sin(1.93649\,t + 1.31811)$$

Figure 12.2 on the next page graphs this function as well as $\pm\frac{8\sqrt{15}}{15}e^{-t/2}$.

Pendulums. Suppose we have a mass m at the end of a pendulum of length l. It swings along a circular arc. When the pendulum is at rest it hangs straight down and has velocity 0. Let $s(t)$ denote the arc length from this rest position as a function of time. Let $\theta(t)$ be the angle the pendulum makes from the vertical position. Then $s = l\theta$ and so the acceleration is $d^2s/dt^2 = ld^2\theta/dt^2$. The force on the mass due to gravity is downward and has magnitude mg, where g is the gravitational constant. This force can be broken into the part in the same direction as the pendulum rod and a part part tangent to the arc traced out by the mass; see Figure 12.3 on page 183.

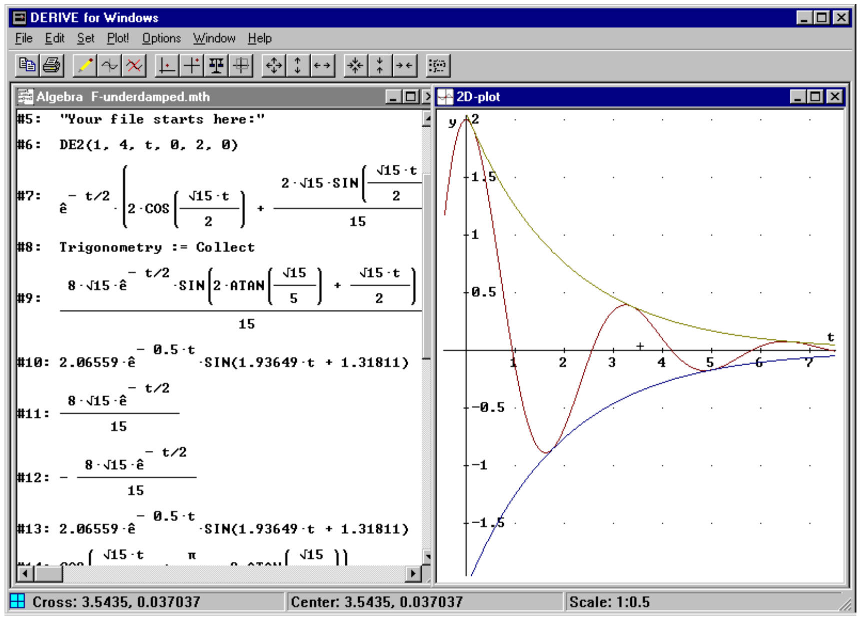


Figure 12.2: Under Damped Oscillations

The component of the force in the direction of the pendulum rod is cancelled out by the rod. The force along the arc is $-gm\sin\theta$. Newton's law, $F = ma$ says $md^2s/dt^2 = -gm\sin\theta$. In terms of θ we get the differential equation:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta \tag{11}$$

This is not a linear equation because of the $\sin\theta$. But the Taylor series is $\sin\theta = \theta - \theta^3/6 + \cdots$, so if θ is small we can approximate $\sin\theta$ with θ. Using this (11) becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \tag{12}$$

If we start by pulling the pendulum back by an angle θ_0 and letting go we can solve the equation by authoring `DE2(0, g/L, t, 0,` θ_0`, 0)`. This gives the solution

$$\theta_0 \cos\left(\sqrt{\frac{g}{l}}\,t\right)$$

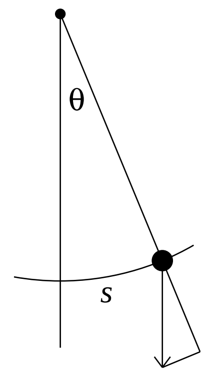


Figure 12.3: The pendulum

Notice that the period depends on l but not on θ_0. This is why old clocks often use pendulums. As the spring runs down the pendulum will continue to swing with the same frequency (until it stops completely of course). You can adjust the speed by making a small change in the length of the pendulum.

Later in this chapter we will show how to find approximate solutions to (11).

12.3 Systems of Differential Equations

Predator prey population growth. Suppose we have a population of rabbits. Let $R(t)$ be the population at time t and let $R_0 = R(0)$ be the initial population. In Chapter 7 we had two models for $R(t)$. The first was $R' = kR$ was the standard exponential growth model. The second was the Verhulst modification of this: $R' = kR/(1 - R/R_1)$, where R_1 is a constant representing the ideal population. But suppose we also have a population, $F(t)$, of foxes which prey on the rabbits. This gives us a *system* of differential equations for $R(t)$ and $F(t)$. It is reasonable to assume that number of rabbits eaten by foxes is proportional to $R \cdot F$. Then the population of rabbits and foxes can be modeled by the equations

$$\begin{aligned} R' &= kR(1 - R/R_1) - cRF \\ F' &= dRF - eF \end{aligned} \tag{13}$$

where k, c, d, and e are positive constants. If $R = 0$ the second equation becomes $F' = -eF$. This means that if there are no rabbits the fox population will dwindle because there is nothing to eat. If $F = 0$ then the rabbit population will follow the Verhulst model.

The Runge-Kutta method of approximation. A general system of two first order differential equations has the form

$$\begin{aligned} y' &= r(t, y, z) \\ z' &= s(t, y, z) \end{aligned} \tag{14}$$

These can be solved exactly if $r(t, y, z)$ has the form $ay + bz$ and $s(t, y, z)$ has a similar form. (When r and s have this form, the system of equations (14) is called *linear*.) Since the examples we are interested are not linear, we concentrate on finding approximate solution to (14).

In Chapter 7 we described Euler's method for finding an approximate solution of a single first order equation $y' = f(t, y)$ subject to the initial conditions $y(t_0) = y_0$. We start with the point (t_0, y_0). Since we know the slope of y at this point is $f(t_0, y_0)$ we draw a short line segment from (t_0, y_0) to $(t_1, y_1) = (t_0 + h, y_0 + hf(t_0, y_0))$, where h is a small increment. The $(n + 1)^{\text{st}}$ point is obtained the n^{th} by

$$(t_{n+1}, y_{n+1}) = (t_n + h, y_n + hf(t_n, y_n))$$

Figure 12.4 on the next page gives the direction field for the simple differential equation $y' = -4(t - 1)$. Of course we can find the solutions by integration. If $y(0) = 0$ this gives $y = 2t(2 - t)$, which we have also graphed. If we use Euler's method with $h = 1/2$ the first three points are $(0, 0)$, $(1/2, 2)$, and $(1, 3)$. As the graph indicates these points are not very close to the true solution.

If instead of using the slope at (t_n, y_n) we average this slope with the slope at the next point (t_{n+1}, y_{n+1}) we obtain a much more accurate approximation of the solution. This is known as the *second order Runge-Kutta method*. The precise formulae for t_{n+1} and y_{n+1} are

$$\begin{aligned} t_{n+1} &= t_n + h = t_0 + (n + 1)h \\ y_{n+1} &= y_n + \frac{h}{2}\big(f(t_n, y_n) + f(t_n + h, y_n + hf(t_n, y_n))\big) \end{aligned} \tag{15}$$

As we can see from Figure 12.4 this is much more accurate. If we also take into account the slope at the midpoint of the two points we obtain the *fourth order Runge-Kutta method*. This is usually just called the *Runge-Kutta method*.

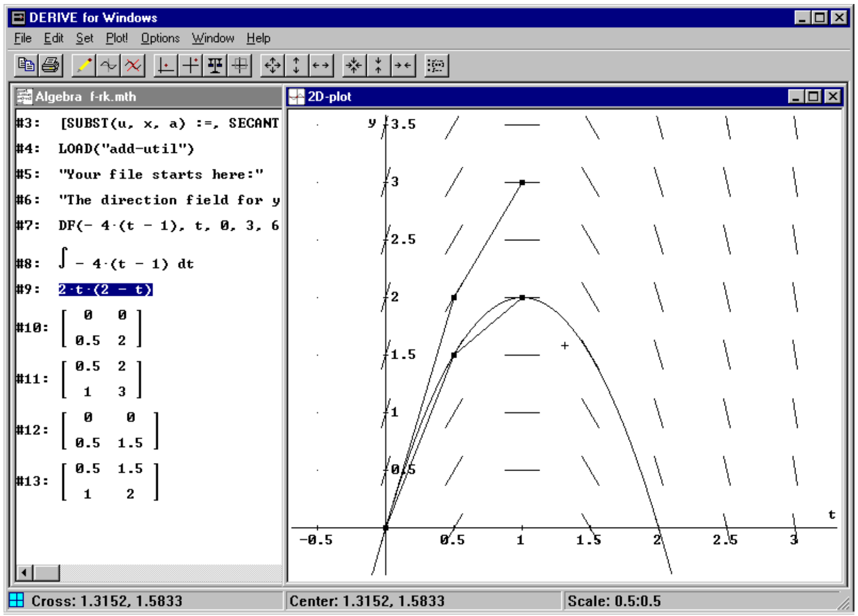


Figure 12.4: Euler and the 2$^{\text{nd}}$ Runge-Kutta methods

If we have a system of equations like (14) we calculate triples of points (t_n, y_n, z_n) instead of pairs, but the formula is essentially the same. The DERIVE utility function `RK`, which is included in ADD-HEAD.MTH, will calculate approximate solutions to system of differential equations using the Runge-Kutta method.[1] To approximately solve (14) with initial conditions $y(t_0) = y_0$ and $z(t_0) = z_0$, we author

```
RK([r(t,y,z), s(t,y,z)], [t, y, z], [t0, y0, z0], h, n)
```

where h is the step size and n is the total number of steps you want. When we approXimate this we get a matrix of triples. To graph $y(t)$ we use the function

[1] `RK` is the same as the one in DERIVE's utility file ODE-APPR so the description of it in DERIVE's Help applies.

`extract_2_columns(m,1,2)` (where `m` is the matrix we got). This gives a matrix of pairs which we can plot. To plot $z(t)$ we use `extract_2_columns(m,1,3)`.

Returning to the predator-prey problem let's look at the rabbits and foxes problem with specific data for the constants in (13):[2]

$$\begin{aligned}
k &= .1 \text{ rabbit per month per rabbit} \\
R_1 &= 10000 \text{ rabbits} \\
c &= .005 \text{ rabbit per month per rabbit-fox} \\
d &= .0004 \text{ fox per month per rabbit-fox} \\
e &= .04 \text{ fox per month per fox} \\
t_0 &= 0 \text{ months} \\
R_0 &= 2000 \text{ rabbits} \\
F_0 &= 10 \text{ foxes}
\end{aligned}$$

To use the Runge-Kutta method to find an approximation of the solution we author:

```
RK([.1r(1 - r/10000) - .005rf, .00004rf - .04f], [t,r,f],
                                    [0,2000,10], 0.5, 600)
```

We approximate this and then use `extract_2_columns` for columns 1 and 2 to see $R(t)$. The result is graphed in the upper right window of Figure 12.5 on the next page. Extracting column 1 and 3 gives the fox population graphed in the lower right. Notice both populations oscillate with the fox population following the rabbit population. After the rabbits increase the foxes will then increase but when the fox population gets large the rabbit population will decrease which in turn will cause the fox population to decrease and so on.

The window in the lower left of Figure 12.5 on the facing page shows the results of extracting columns 2 and 3. The point near the crosshair in that window is $(2000, 10)$, the initial rabbit and fox populations. At the beginning both the rabbit and fox populations increase. When they reach the point furthest to the right the rabbit population starts to decrease while the fox population continues to increase. As we continue along the curve it spirals inward indicating that the oscillation in the populations get smaller. In Exercise 4 on page 189 your find the point to which the spiral approaches.

[2] This example is taken from J. Callahan and K. Hoffman *Calculus in Context*, W. H. Freeman, 1995.

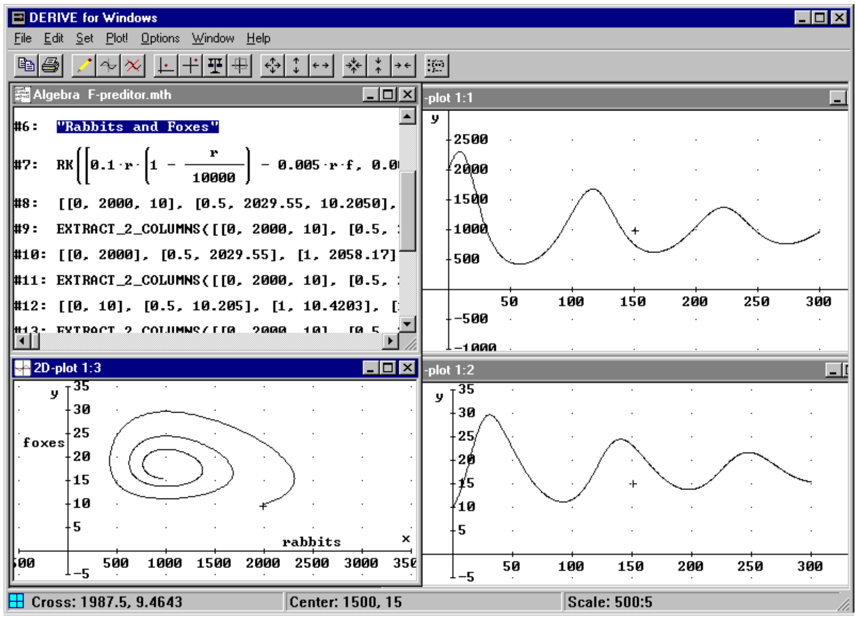


Figure 12.5: Rabbits and Foxes

The pendulum revisited. A second order differential equation such as (11) can be reduced to a system of two first order equations. To do this we introduce a new variable $w(t) = d\theta/dt$. Then (11) becomes

$$\begin{aligned} \theta' &= w \\ w' &= -\frac{g}{l}\sin\theta \end{aligned} \tag{16}$$

As an example suppose $g/l = 25$. Then to get an approximate solution of (11) we author and then approximate the following.

(17) `EXTRACT_2_COLUMNS(`
`RK([w,-25 SIN(`θ`)], [t,`θ`,w], [0,`θ_0`,0], 0.05, 60), 1, 2)`

Figure 12.6 on the following page shows the resulting graphs for $\theta_0 = \pi/8$, $\pi/4$, $\pi/2$, and $15\pi/16$. The graph of the solution of (11), namely $\theta_0 \cos(5t)$, is also shown on each graph. Looking at these graphs we can see several things. First for $\theta_0 = \pi/8 = 22.5°$ the curves are almost identical showing that using

(12) rather than (11) works well for small and even moderate angles. Even for $\theta_0 = \pi/4 = 45°$, shown in the lower left, is fairly close to the true graph.

Since we are considering a pendulum without friction (undamped) we expect that when we release it with an initial angle of θ_0 it will swing to the other side reaching the angle $\theta = -\theta_0$ and then return back to the original position with $\theta = \theta_0$. Then of course it will just repeat this. This means that the solution of (11) will be periodic. The linear approximation (12) has a shorter period than the true equation (11). This makes sense since the magnitude of the force pushing the pendulum back towards its rest position ($\theta = 0$) is proportional to $\sin\theta$ for the true equation and to θ in the linear approximation and $\sin\theta \leq \theta$ for $\theta > 0$.

The lower right frame of Figure 12.6 gives the graphs when $\theta_0 = 15\pi/16$. This corresponds to starting the pendulum almost at the top. Notice that not only is the true period much greater than the linear approximation but that the shape of curve is different.

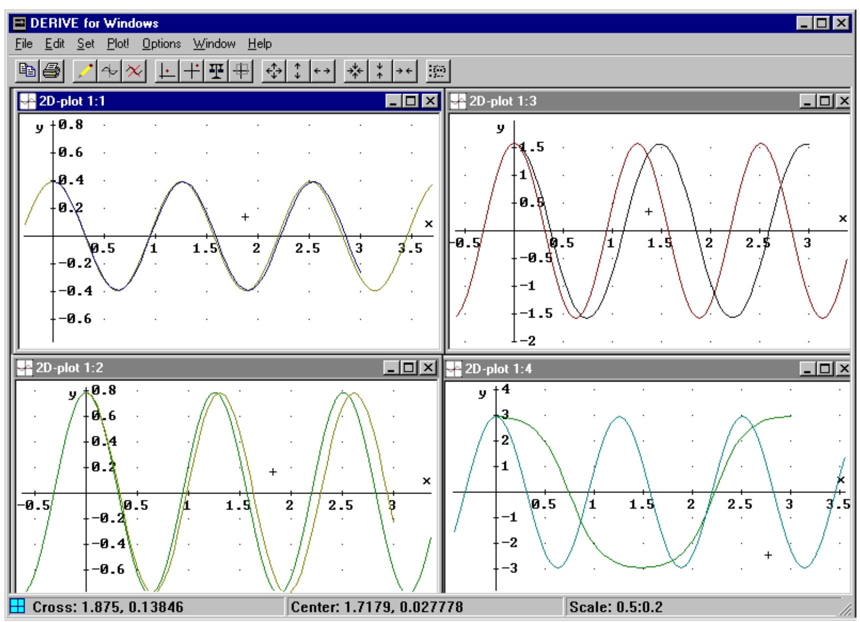


Figure 12.6: Pendulums

12.4 Laboratory Exercises

1. Show that if the characteristic equation (4) has a double root, that is, if $r_1 = r_2$, then both $y = e^{r_1 t}$ and $y = te^{r_1 t}$ satisfy (1) on page 177.

2. Hooke's Law is given by equation 9 on page 179.

a. Solve this with the initial conditions $y_0 = 0$ and $v_0 = v$. (You should use DERIVE's Declare/Variable Domain to tell DERIVE that m and k are positive.)

b. Graph your solutions with with $k = 2$, $m = 1$, and v varying between -2 and 2 in increments of 0.5.

3. A third order differential equation of the form

$$y''' + by'' + cy' + dy = 0$$

has the characteristic equation $r^3 + br^2 + cr + d = 0$. The roots of the characteristic equation determine the solutions of the differential equation in the same way as for second order differential equations. The solutions of second order differential equations involve two arbitrary constants but for third order there are three.

a. Find the solutions to $y''' - 2y'' - y' + 2y = 0$.

b. Find the solutions to $y''' - 2y'' + y' - 2y = 0$.

c. Find the solutions to $y''' - 4y'' + 5y' - 2y = 0$.

4. Suppose we want to find solutions to (13) such that both $R(t)$ and $F(t)$ are constant. One (trivial) solution is $R(t) = 0 = F(t)$ but we would like something more interesting than this. If the populations are constant then $R'(t) = 0$ and $F'(t) = 0$.

a. Solve (13) for R and F when $R'(t) = F'(t) = 0$. **Hint:** Use the second equation of (13) to find R, substitute this into the first equation and then solve for F.

b. The curve in the lower left window of Figure 12.5 on page 187 spirals inward. Use your answer from the previous part with the constants on page 186 to guess where it is heading.

5. Suppose in our rabbits and foxes example instead of (13) we use the simpler equations

$$R' = kR - cRF$$
$$F' = dRF - eF$$

without the Verhulst modification.

a. In Figure 12.5 on page 187 we used `RK` and `EXTRACT_2_COLUMNS` to make 3 graphs. Make the same 3 graphs but using these simpler equations. Describe the differences between your graphs and those of Figure 12.5.

b. Find the solutions which are constant as in the previous problem.

6. We saw in the lower right frame of Figure 12.6 on page 188 on page 188 that for $\theta_0 = 15\pi/16$ the solution to (11) and to the linear approximation (12) were quite different. In this exercise we will compare the solution to (11) with a cos function of the same amplitude and period.

a. Author the expression (17) with $\theta_0 = 15\pi/16$ and then approximate it. Then graph the result.

b. Using this graph estimate the period P of this function

c. Graph $\frac{15\pi}{16}\cos(\frac{2\pi}{P}t)$ using the P you found in the previous part.

d. Notice the graph you found in the first part is pretty flat at the top and bottom compared to the cos curve. What is the solution of (11) if $\theta_0 = \pi$ (and $w_0 = 0$, of course)? You should be able to just guess the solution.

7. In this problem we explore what happens to a pendulum with initial position $\theta_0 = 0$ but with a nonzero value for w_0. If the initial velocity is not too large the pendulum will swing up to an angle α and then swing back. The motion will be the same as if we started with $\theta_0 = \alpha$ and $w_0 = 0$ except that starting place for the graph will be different. That is, the curves will be the same except one will be shifted to the right. However if w_0 is large enough, the pendulum will swing all the way over the top.

a. Author

```
EXTRACT_2_COLUMNS(
   RK([w,-25 SIN(θ)], [t,θ,w], [0,0,w₀], 0.05, 60), 1, 2)
```

Substitute $w_0 = 9.5$ and approximate and then graph the result. Do the same for $w_0 = 10$ and $w_0 = 10.5$.

b. The behavior of the solutions of the previous part are quite different depending on whether w_0 is large enough to send the pendulum over the top (so it keeps spinning around and around) or it doesn't make it to the top and so falls back. In this problem we look for a value of w_0 so that it never falls back but also never goes over the top. We will try to choose w_0 so that it will have the exact amount of energy to just get to the top. Since we are assuming our pendulum is frictionless, there are two kinds of energy in our system, kinetic and potential energy. Kinetic energy is $\frac{1}{2}mv^2$ which in our case is $\frac{1}{2}m(ds/dt)^2 = \frac{1}{2}ml(d\theta/dt)^2$. Potential energy gained as the pendulum swings above its rest position is mgh where h is the height above the rest position. So the potential energy at the top is $2mgl$. Use the law of conservation of energy to show that if the kinetic energy at the bottom equals the potential energy at the top then

$$w_0 = \theta'(0) = 2\sqrt{\frac{g}{l}}$$

which is 10 when $g/l = 25$.

c. While the solution to (11) on page 182 cannot be expressed in terms of elementary functions, the solution when $\theta(0) = 0$ and $\theta'(0) = 2\sqrt{\frac{g}{l}}$ can. Show that

$$\theta(t) = 4\arctan(e^{\sqrt{\frac{g}{l}}t}) - \pi$$

is a solution of (11) and that $\theta'(0) = 2\sqrt{\frac{g}{l}}$. You will need set the trigonometry mode to Expand under the Declare/Algebra StateSimplification menu. Also remember that DERIVE uses ATAN for the arctan function.

d. When $g/l = 25$, $\theta(t) = 4\arctan(e^{5t}) - \pi$. Graph this function and compare it with the graph you made in the first part with

$w_0 = 10$. Also graph

```
EXTRACT_2_COLUMNS(
   RK([w,-25 SIN(θ)], [t,θ,w], [0,0,10], 0.01, 500), 1, 2)
```

e. Why do these graphs differ?

Appendix A

Utility Files

This book defines 19 DERIVE functions. All of these functions are defined in the utility file ADD-UTIL.MTH. This file and its companion, ADD-HEAD.MTH, can be downloaded from our web page

`http://www.math.hawaii.edu/206L/`

Listings of these files are given at the end of this appendix in the (hopefully unlikely) event you have trouble downloading them. Additional information on the use of the functions as well as examples are included on the web site above.

How to use these files. When a student first starts to work on a lab assignment he should:

1. Enter (Author) his name and the lab number as a comment. (Comments are entered using the double quotes, ".)
2. Do File/Load/Math `add-head`.
3. Begin working on his assignment.

The file ADD-HEAD.MTH has only four lines. Two of these are comments and one gives the variable syntax for the commands. The other line is `LOAD("add-util")`. This automatically silently loads ADD-UTIL.MTH.[1] For this load command directory to work correctly, ADD-HEAD.MTH and ADD-UTIL.MTH should both be in the current directory. It is best to put them in the default directory.

[1]This is only true for DfW version 4.08 or greater. If you have an earlier version you can download the latest version from the DERIVE web site: `http://www.derive.com`. You

Home use of these files. If you have DERIVE for your own computer you can install ADD-HEAD.MTH and ADD-UTIL.MTH. When DERIVE starts it has a default directory it looks for MTH files. If you haven't changed this it is `..\DfW\Math`. That directory contains the utility files that come with the system. You should add a new subdirectory, we use `..\DfW\M206L`, using the Win95 file manager and then make that your default directory using the File/Change Directory command in DfW. Next, you put both ADD-UTIL.MTH and ADD-HEAD.MTH in this default directory and the above directions should work fine. You can also add all the `F-*.MTH` files that are used in the book's figures to this directory. All of these files are available from our web site.

A.1 The Functions

Table A.1 on page 197 and A.2 on page 198 list the functions defined in ADD-UTIL.MTH. In each of the examples below it is assumed that the utility file ADD-UTIL.MTH has been loaded as described above. Here are some examples on their use.

Example 1. If you want to compute a tangent line for say $f(x) = x^3/3$ at the point $x = 1$ you would Author and Simplify `TANGENT(x^3/3, x, 1)`. The result will be $y = x - 2/3$. We describe the variables for this and the other functions typically as `TANGENT( u, x, a)` where the u refers to any expression in the variable x and a is a parameter in the function which in this case it is the point we are interested in.

Example 2. Suppose that you want to find the quadratic polynomial $ax^2 + bx + c$ that passes through the three points $(0, 0)$, $(1, 2)$, and $(2, 8)$. You Author `CURVEFIT( x, [[0,1], [1,2], [2,8]])`. After simplifying the result will be $2x^2$. Probably the best way to do this is to start by defining the 3×2 matrix of points using the matrix button and then plotting the 3 points on a graph. Next you Author the `CURVEFIT(x,` part and then right click and insert the matrix of data points. Simplify and plot to make sure the answer function does indeed pass through the 3 data points. The

can use earlier versions of DERIVE but after loading ADD-HEAD you need to highlight and then evaluate (by pressing the = button) the line $\boxed{\text{LOAD(“add-util”)}}$. This should evaluate to ‘true.’

CURVEFIT function will find the appropriate degree polynomial through the data regardless of the number of points.

Example 3. Suppose that now that you want to find the quadratic polynomial $ax^2 + bx + c$ that passes through the two points: (0,0) and (1,2). In addition, you want the derivative to be 1 when $x = 0$. You Author `CURVEFIT( x, [[0,1], [1,2]], [[0,1]])`. In other words, you enter one matrix for the points satisfied by the function and another matrix for the points satisfied by the derivative. The degree of the answer polynomial is always one less than the total number of equations for both the function and its derivative.

Example 4. Let's solve the equation $x^2 + x - 1 = 0$ using Newton's method of Chapter 5. We'll use $x0 = 5$ as our initial guess. We obtain our first approximation by Authoring `NEWT(x^2+x-1, x, 5)` and then Simplifying to get 2.63263. We repeat this process using this new value as our starting point. After 4–5 iterations we obtain an approximation we good to 6 decimal places.

Example 5. More generally, to approximate the solution to the equation $u = 0$, where u is an expression in x, using Newton's method with initial guess a you author and approximate `NEWT(u, x, a)`. Suppose instead that you want the first k approximates starting with $x = a$, then you approximate `NEWT(u, x, a, k)`. The 4^{th} argument is optional. You get a nice picture of the Newton method in action by approximating `DRAW_NEWT(u, x, a, k)` and then plotting the result. Notice that the starting point can be a complex number in which case the approximates are also complex. The function `DRAW_COMPLEX(v)` can be applied to the solution vector to get a matrix of $[x, y]$ points which can then be plotted in a 2D-plot window.

Example 6. Suppose that you want to approximate the integral which defines the natural logarithm of 2, i.e.,

$$\ln 2 = \int_1^2 \frac{dx}{x}$$

using say the trapezoid rule or Simpson's rule for numerical integration. We do this for $n = 100$ subdivisions by Authoring either the expression `TRAP(1/x, x, 100, 1, 2)` or the expression `SIMP(1/x, x, 100, 1, 2)`. Now since we are interested in a decimal approximation we use the $\boxed{\approx}$ button to simplify the expression. More generally, suppose you approximate the integral of the expression u, in the variable x over the interval $[a, b]$, using Simpson's rule with n subdivisions. You Author `SIMP(u,x,n,a,b)` and press ≈ .

Example 7. Suppose that you want to solve a Newton cooling type differential equation: $y' = -(y - 2)$ with initial conditions $y(0) = 4$. You start by manipulating the equation to the form $y' + py = q$ where $p = 1$ and $q = 2$. The function `DE(p,q,x,y,x0,y0)` solves this equation so we just substitute in the right values which in this case means that we Author `DE(1,2,x,y,0,4)` and press = .

Example 8. Suppose that you want to look at the direction field for the equation $y' = r$ where r is an expression in x and y. You use the function

```
DF(r,x,x0,xm,m,y,y0,yn,n))
```

where the grid of points is determined by $x0 < x < xm$ with m subdivisions and $y0 < y < yn$ with n subdivsions. Doing this for the previous example would mean Authoring say `DF(-(y-2),x,0,6,12,y,-2,4,12)` and then approximating the expression by pressing ≈ . You get a graph with a slope line at every half integer in an appropriate range of x and y's by plotting the result.

Table A.1: Functions Defined in ADD-UTIL.MTH

SUBST(u, x, a)	Substitutes $x = a$ in the expression u.
SECANT(u, x, a, h)	Secant line of $u(x)$ through $x = a$ and $x = a + h$.
TANGENT(u, x, a)	Tangent line of $u(x)$ at $x = a$.
CURVEFIT$(x, data)$ CURVEFIT$(x, data, ddata)$	Fits a polynomial in the variable x, though the points data := $[[x0, y0], [x1, y1], ...]$ provided ddata is either omitted or []. Otherwise, the graph of the derivative must pass through the ddata points.
NEWT$(u, x, x0)$ NEWT$(u, x, x0, k)$	Newton algorithm for root of $u(x) = 0$ with initial guess $x0$. If the optional k argument is used then a vector of k iterates is returned.
DRAW_NEWT$(u, x, x0, k)$	Draws a picture of Newton method applied to $u(x) = 0$ with initial guess $x0$ and k iterates. Simplify expression and plot the result.
DRAW_COMPLEX(v)	Converts the vector of complex numbers $[x0 + iy0, x1 + iy1, ...]$ into a matrix of points $[[x0, y0], [x1, y1], ...]$ which can then be plotted in a 2D-plot window.
BISECT(u, x, v) BISECT(u, x, v, k)	Bisection method for solving $u(x) = 0$ with interval $v = [a, b]$. The answer is either the left or right half of the interval depending on the root. If the optional k argument is used then a vector of k iterates is returned.
LEFT(u, x, n, a, b)	Numerical approximation to the integral of $u(x)$ over $[a, b]$ using the left-endpoint method with n rectangles.
MID(u, x, n, a, b)	Numerical approximation to the integral of $u(x)$ over $[a, b]$ using the midpoint method with n rectangles.

Table A.2: Functions Defined in ADD-UTIL.MTH (cont.)

RIGHT(u, x, n, a, b)	Numerical approximation to the integral of $u(x)$ over $[a, b]$ using the right-endpoint method with n rectangles.
TRAP(u, x, n, a, b)	Numerical approximation to the integral of $u(x)$ over $[a, b]$ using the trapezoid method with n trapezoids.
SIMP(u, x, n, a, b)	Numerical approximation to the integral of $u(x)$ over $[a, b]$ using Simpson's method with n subdivisions.
DRAW_LEFT(u, x, n, a, b)	Draws graphic demonstration of the left-endpoint method for numerically integrating $u(x)$ over the interval $[a, b]$ using n rectangles.
DRAW_RIGHT(u, x, n, a, b)	Same as above except for the right-endpoint method.
DRAW_TRAP(u, x, n, a, b)	Draws graphic demonstration of the trapezoid method for numerically integrating $u(x)$ over the interval $[a, b]$ using n trapezoids.
DE$(p, q, x, y, x0, y0)$	Solves the differential equation (DE) $y' + p(x)y = q(x)$ with y$(x0) = y0$.
DF$(r, x, x0, xm, m, y, y0, yn, n)$	The direction field (DF) for the differential equation: $y' = r(x, y)$ with a grid determined by $x0 < x < xm$ with m subdivisions and $y0 < y < ym$ with n subdivisions.
EULER$(r, x, y, x0, y0, xn, n)$	This gives an approximate solution to: $y' = r(x, y)$ with y$(x0) = y0$. The answer is a vector of points $[[x0, y0], [x1, y1], ...]$ from which one makes a piecewise linear approximating function, i.e., connect the points with straight line segments to get the approximating function's graph.

A.2 Listings of the Utility Files

In the event you are unable to download these files, you can type them in. Probably the easiest way to do this is to start DERIVE and author each line. Then save the first as `add-head` and the second as `add-util`.

A.2.1 The ADD-HEAD.MTH File

```
"The vector below declares all the utility functions in add-util.mth."

[SUBST(u,x,a):=,SECANT(u,x,a,h):=,TANGENT(u,x,a):=,CURVEFIT(x,data):=,SPLINE(~

x,data,m1):=,NEWT(u,x,x0,k):=,DRAW_NEWT(u,x,x0,k):=,DRAW_COMPLEX(v):=,BISECT(~

u,x,v0,k):=,LEFT(u,x,n,a,b):=,MID(u,x,n,a,b):=,RIGHT(u,x,n,a,b):=,TRAP(u,x,n,~

a,b):=,SIMP(u,x,n,a,b):=,DRAW_LEFT(u,x,n,a,b):=,DRAW_TRAP(u,x,n,a,b):=,DE(p,q~

,x,y,x0,y0):=,DF(r,x,x0,xm,m,y,y0,yn,n):=,EULER(r,x,y,x0,y0,xn,n):=]

LOAD("add-util")

"Your file starts here:"
```

A.2.2 The ADD-UTIL.MTH File

```
"File add-util.mth, (c) 1997 Ralph Freese and David Stegenga."

"See add-summary.mth for a summary of new functions defined below:"

"Substitute x=a into the expression u."

SUBST(u,x,a):=LIM(u,x,a)

"The secant line of u(x) through x = a and x = a + h."

SECANT(u,x,a,h):=y=(SUBST(u,x,a+h)-SUBST(u,x,a))/h*(x-a)+SUBST(u,x,a)

"The tangent line of u(x) at x = a."

TANGENT(u,x,a):=y=SUBST(u,x,a)+SUBST(DIF(u,x),x,a)*(x-a)

"Helper functions for CURVEFIT."

POLY(x,a,n):=SUM(a SUB (i+1)*x^i,i,0,n)

DPOLY(x,a,n):=SUM(i*a SUB (i+1)*x^(i-1),i,1,n)

EQNS(data,ddata,a,n):=APPEND(VECTOR(POLY(data SUB i SUB 1,a,n)=data
```

```
SUB i SUB~
2,i,1,DIMENSION(data),1),VECTOR(DPOLY(ddata SUB i SUB 1,a,n)=ddata
SUB i SUB~
 2,i,1,DIMENSION(ddata),1))

UNK(n):=RHS(VECTOR(SOLVE(upsilon=upsilon,upsilon),i,1,n,1)‘ SUB 1)

ANS(x,data,ddata,a,n):=IF(DIMENSION(ans_:=SOLVE(EQNS(data,ddata,a,n),[a]))=0,~

[],POLY(x,(RHS(ans_)) SUB 1,n))

"Finds the polynomial of the right degree through the nx2-data matrix."

CURVEFIT1(x,data):=ANS(x,data,[],UNK(DIMENSION(data)),DIMENSION(data)-1)

CURVEFIT2(x,data,ddata):=ANS(x,data,ddata,UNK(DIMENSION(data)+DIMENSION(ddata~

)),DIMENSION(data)+DIMENSION(ddata)-1)

CURVEFIT(x,data,ddata):=IF(DIMENSION(ddata)>0,CURVEFIT2(x,data,ddata),CURVEFI~

T1(x,data),CURVEFIT1(x,data))

"Quadratic spline function interpolating data points with initial
slope m1."

SPLINE_AUX(x,data,m):=SUM(CURVEFIT(x,[data SUB k,data SUB
(k+1)],[[data SUB k~
SUB 1,m SUB k SUB 2]])*CHI(data SUB k SUB 1,x,data SUB (k+1) SUB
1),k,1,DIME~
NSION(data)-1)

SLOPE(data,m1):=ITERATES([v SUB 1+1,2*(data SUB (v SUB 1+1) SUB
2-data SUB (v~
SUB 1) SUB 2)/(data SUB (v SUB 1+1) SUB 1-data SUB (v SUB 1) SUB 1)-v
SUB 2]~
,v,[1,m1],DIMENSION(data)-1)

"Note that SLOPE returns the matrix [[1,m1], [2,m2], ...]."

SPLINE(x,data,m1):=SPLINE_AUX(x,data,SLOPE(data,m1))

"Newton algorithm"

NEWT_ITERATES(u,x,x0,k):=ITERATES(x-u/DIF(u,x),x,x0,k)

NEWT(u,x,x0,k):=IF(k>0,NEWT_ITERATES(u,x,x0,k),?,SUBST(x-u/DIF(u,x),x,x0))


"This produces a vector which when plotted demonstrates Newton’s method."

DRAW_NEWT(u,x,x0,k):=VECTOR([[v,0],[v,SUBST(u,x,v)],[NEWT(u,x,v),0]],v,ITERAT~

ES(NEWT(u,x,w),w,x0,k))

DRAW_COMPLEX(v):=VECTOR([RE(z),IM(z)],z,v)
```

```
"Bisection method helper"

BIS_AUX(u,x,a,b):=IF(SUBST(u,x,a)*SUBST(u,x,(a+b)/2)<0,[a,(a+b)/2],[(a+b)/2,b~

])

"Bisection method"

BISECT(u,x,v0,k):=IF(k>0,ITERATES(BIS_AUX(u,x,v SUB 1,v SUB
2),v,v0,k),?,BIS_~
AUX(u,x,v0 SUB 1,v0 SUB 2))

"Formula for the left-endpoint method for integrating u(x) over [a,b]
with n ~
subdivisions."

LEFT(u,x,n,a,b):=(b-a)/n*SUM(SUBST(u,x,a+k*(b-a)/n),k,0,n-1)

"Formula for the midpoint method for integrating u(x) over [a,b] with
n subdi~
visions."

MID(u,x,n,a,b):=(b-a)/n*SUM(SUBST(u,x,a+(k+0.5)*(b-a)/n),k,0,n-1)

"Formula for the right-endpoint method for integrating u(x) over
[a,b] with n~
 subdivisions."

RIGHT(u,x,n,a,b):=(b-a)/n*SUM(SUBST(u,x,a+k*(b-a)/n),k,1,n)

"Formula for the trapezoid method for integrating u(x) over [a,b]
with n subd~
ivisions."

TRAP(u,x,n,a,b):=(b-a)/(2*n)*(SUBST(u,x,a)+SUBST(u,x,b)+2*SUM(SUBST(u,x,a+k*(~

b-a)/n),k,1,n-1))

"Formula for Simpson's rule for integrating u(x) over [a,b] with n
subdivisio~
ns."

SIMP(u,x,n,a,b):=(b-a)/(6*n)*(SUBST(u,x,a)+SUBST(u,x,b)+2*SUM(SUBST(u,x,a+k*(~

b-a)/n),k,1,n-1)+4*SUM(SUBST(u,x,a+(k+1/2)*(b-a)/n),k,0,n-1))

"The box and trapezoid drawing functions used in the graphical
demonstrations~
 of numerical integration techniques."

D_BOX(x1,y1,x2,y2):=[[x1,y1],[x2,y1],[x2,y2],[x1,y2],[x1,y1]]

D_TRAP(x1,y1,x2,y2,x3,y3,x4,y4):=[[x1,y1],[x2,y2],[x3,y3],[x4,y4],[x1,y1]]


DRAW_LEFT(u,x,n,a,b):=VECTOR(D_BOX(t,0,t+(b-a)/n,SUBST(u,x,t)),t,a,b-(b-a)/n,~
```

```
(b-a)/n)

DRAW_RIGHT(u,x,n,a,b):=VECTOR(D_BOX(t,0,t+(b-a)/n,SUBST(u,x,t+(b-a)/n)),t,a,b~

-(b-a)/n,(b-a)/n)

DRAW_TRAP(u,x,n,a,b):=VECTOR(D_TRAP(t,0,t+(b-a)/n,0,t+(b-a)/n,SUBST(u,x,t+(b-~

a)/n),t,SUBST(u,x,t)),t,a,b-(b-a)/n,(b-a)/n)

"The solution to the differential equation (DE) y'+p(x)y=q(x) with
y(x0)=y0."

DE(p,q,x,y,x0,y0):=y=(y0+INT(q*#e^INT(p,x,x0,x),x,x0,x))/#e^INT(p,x,x0,x)

"Direction field helper function."

SEG(rc,x,y,x,y):=IF(ABS(rc)>1 AND
y<ABS(rc)*x,[[x-y/rc,y-y],[x+y/rc,y+~
y]],[[x-x,y-rc*x],[x+x,y+rc*x]])

"The direction field (DF) for the differential equation: y'=r(x,y)
with a de~
termined by x0<x<xm with m subdivisions and y0<y<ym with n subdivisions."

DF(r,x,x0,xm,m,y,y0,yn,n):=VECTOR(VECTOR(SEG(LIM(r,[x,y],[x0+j*(xm-x0)/m,y0+k~

*(yn-y0)/n]),x0+j*(xm-x0)/m,y0+k*(yn-y0)/n,(xm-x0)/(4*m),(yn-y0)/(4*n)),j,0,m~

),k,0,n)

"The EULER function gives an approximate solution to: y'=r(x,y) with
y(x0)=y0~
."

EULER(r,x,y,x0,y0,xn,n):=ITERATES(v+(xn-x0)/n*[1,LIM(r,[x,y],v)],v,[x0,y0],n)
```

Appendix B

Instructors' Manual

This appendix, which obviously isn't complete, will contain general information about using this book and for each chapter

- a short description on what we hope to accomplish,
- things to remind the students about,
- possible class demos, and
- advanced topics.

Chapter 5

This chapter primarily deals with solving equations using Newton's method. Even though DERIVE has built in method for numerically solving equations, there are many reasons for choosing this topic. The method itself is a nice application of both differential calculus and the geometry behind it. It introduces the students to the idea of approximation. More interestingly it forms a subtle introduction to dynamic systems and includes such topics as fixed points, attractors, super attractors, cycles, chaos and fractals.

You should take some time to prove and explain Theorem 1 on page 71 to your students. It is easy to prove and shows the importance the derivative, $\mathrm{NG}'(x)$, of the Newton iterate. The discussion after it explains why we get convergence as long as $\mathrm{NG}'(r) < 1$ and why is is so fast if $\mathrm{NG}'(r) = 0$.

Advanced topics. There are several advanced topics in the text and the exercises that you might want to cover. For an open ended project for students or for a more advanced demonstration you might consider the function $x^3 - 5x$ from Exercise 4 and 5 on page 83. This function has three roots 0 and $\pm\sqrt{5}$. By Theorem 1 on page 71 there are intervals around each of these such that

$$\mathrm{NG}(t) = \mathrm{NEWT}(x^3 - 5x, x, t) = \frac{2t^3}{3t^2 - 5}$$

applied to any elements of these intervals moves closer to the root. The project would be to have the student investigate what happens to the seqence of Newton iterates,

$$t,\ \mathrm{NG}(t),\ \mathrm{NG}(\mathrm{NG}(t)),\ \mathrm{NG}(\mathrm{NG}(\mathrm{NG}(t))),\ \ldots, \tag{1}$$

for an arbitrary real number t.

The exercise shows that there is also a cycle of length two, namely 1 and -1. And, of course, $\mathrm{NG}(t)$ is undefined when $t = \pm\sqrt{5/3}$. This means it will become undefined if we start with t such that $\mathrm{NG}^k(t) = \sqrt{5/3}$, for some k. Examining the graph of $x^3 - 5x$ we can see that there is exactly one t_1 with $\mathrm{NG}(t_1) = \sqrt{5/3}$ and that there is a unique t_2 with $\mathrm{NG}(t_2) = t_1$, etc. DERIVE can find t_1 by solving $\mathrm{NG}(t) = \sqrt{5/3}$ numerically. This will return the vector `[t = -1.04111]`. We can make this into a DERIVE function which just returns the number as follows:

```
INV(t) := RHS(SOLVE(2x^3/(3x^2-5) = t, x, -3, 3) SUB 1)
```

and then

```
ITERATES(INV(t), t, (5/3)^(1/2), 7)
```

will show $t_1, t_2, \ldots, t_7$. These alternate in sign and their absolute values converge to 1. Interestingly if $|t_{k+1}| < t < |t_k|$ then the sequence (1) of Newton iterates will converge to $\sqrt{5}$ or $-\sqrt{5}$, depending on the parity of k. Thus we get smaller and smaller intervals near 1 (and -1) whose elements alternately converge to $\sqrt{5}$ and $-\sqrt{5}$.

Chapter 7

This chapter covers differential equation in more detail than is usually done in the first year of calculus and, if the more advanced parts are covered, would be suitable for second year students. Nevertheless it still concentrates primarily on tradition population growth and related problems.

The DERIVE function `DE(p,q,x,y,x0,y0)` solves the general first order linear differential equation

$$y' + p(x)y = q(x), \qquad y(x_0) = y_0 \tag{3}$$

In order to use this the student needs to rewrite his differential equation into this form so he can identify $p(x)$ and $q(x)$.

For population problems, where the general solutions has the form $y(t) = y_0 e^{k(t-t_0)}$, the student is usually give some information which allows him to find y_0, k, and t_0 and then ask for the population at some other time. It may be a good idea to do one such problem in class. Also note in half-life and doubling time problems it may not be necessary to solve for y_0.

Advanced topics. We consider the Verhulst population model:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{P_1}\right) \tag{9}$$

This makes a very nice demonstration of the methods of this chapter. We first draw the direction field for this equation; see Figure 7.3 on page 108. This screen clearly shows that under this population model the population tends to P_1 whether it starts below or above P_1. We then note that even though (9) is not of the form of (3), the substitution $Q = 1/P$ transforms the equation into a linear first order equation. This is then solved and plotted.

We also introduce Euler's method of finding an approximate solution to an equation of the form $y' = f(x, y)$.

Index

asymptote, 31
attractor, 71
    super, 71

basin of attraction, 72
bisection method, 77–79

chaos, 73
character input mode, 53
complex numbers, 71
complex plane, 72
complex roots, 9
critical points, 30

defining variables, 17
differential equations, 56, 97–108
    linear first order, 98
direction field, 102
dynamic system, 71

Euler's constant, 5
Euler's method, 105
exponential growth, 46

fixed point, 81
    attractive, 72

Gamma function, 42

half-life, 101

integral test, 126, 134–139
interest, 127
    continuous, 139
    effective annual rate, 140

loan repayment, 127–128

Newton's Law of Cooling, 99–101
Newton's method, 65–75

Poisson summation, 142
population growth, 51, 97–99, 103–105

radioactive decay, 101
ratio test, 126, 130–134
recursion, 19
right mouse button, 5

series
    convergent, 125
    divergent, 125
    geometric, 125, 126
    harmonic, 139
Simpson's rule
    derivation, 92–93
    error estimate, 87
slope field, 102
solving equations, 8–9
    numerically, 45
spline functions, 58
subscripts, 23
super attractor, 71

tangent line, 37

trapezoid rule
    error estimate, 87

vectors, 22

web page, xii
word input mode, 18