## Name:

## 241 Practice and Extra Credit

Instructions: Write very clearly and put a box around your answer. Not all questions will be graded. You can ask about problems in class and you can work with your friends. Open book.

1. (15 points) Compute the following limits or show that they do not exist:
(a)

$$
\lim _{t \rightarrow 4}\left(\frac{\sqrt{t}-3}{t-9}\right)
$$

Solution: The limit is $1 / 5$.
(b)

$$
\lim _{x \rightarrow 0} \frac{x}{\sin 3 x}
$$

Solution: The limit is $1 / 3$.
(c)

$$
\lim _{x \rightarrow \infty} \frac{x \sin \left(x^{2}\right)}{x^{2}+1}
$$

Solution: The limit is 0 .
2. (15 points) Find the derivative of the following functions. Do not simplify.
(a) $f(x)=x^{3} \sqrt{\sin x}$

$$
f^{\prime}(x)=3 x^{2} \sqrt{\sin x}+\frac{x^{3} \cos x}{2 \sqrt{\sin x}}
$$

(b) $g(x)=\tan \left(2+x^{2}\right)$

$$
g^{\prime}(x)=2 x \sec ^{2}\left(2+x^{2}\right)
$$

(c) $h(x)=\int_{2}^{x^{3}} \frac{\sin x}{x} d x$

$$
h^{\prime}(x)=3 x^{2} \frac{\sin \left(x^{3}\right)}{x^{3}}
$$

3. (10 points) Using the definition of derivative, NOT differentiation rules, find $f^{\prime}(2)$ if

$$
\begin{gathered}
f(x)=\frac{1}{3 x} \\
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}
\end{gathered}
$$

So substituting 2 for $x$

$$
\begin{aligned}
f^{\prime}(2)=\lim _{h \rightarrow 0} \frac{1 /(3(2+h))-1 / 6}{h} & =\lim _{h \rightarrow 0}(1 / h)\left(\frac{1}{3(2+h)}-\frac{1}{6}\right) \\
& =\lim _{h \rightarrow 0}(1 / h) \frac{6-(6+3 h)}{6(6+3 h)} \\
& =\lim _{h \rightarrow 0}(1 / h) \frac{-3 h}{6(6+3 h)}=-\frac{1}{12}
\end{aligned}
$$

4. Evaluate the following integrals. Show your work!
(a) (5 points) $\int \sin (x) \cos ^{3}(x) d x$

Substitute $u=\cos x, d u=-\sin x d x$
(b) (5 points) $\int_{0}^{1}\left(t^{2}+1\right)(4 t-1) d t$

Multiply out the polymonial and then integrate.
(c) (5 points) $\int x^{2} \sqrt{x+1} d x$

Substitute $u=x+1, d u=d x$. Note $x=u-1$ so the integral becomes

$$
\int(u-1)^{2} \sqrt{u} d u=\int u^{5 / 2}-2 u^{3 / 2}+u^{1 / 2} d u
$$

5. (10 points) Find the area of the region bounded by the lines $y=x, y=8 x$ and the curve $y=1 / x^{2}$.
6. (10 points) Find an equation for the tangent line to the curve defined by

$$
x^{3}+y^{3}=9 x y
$$

at the point $(4,2)$. Show your work!
7. (10 points) Consider a point $P=(x, y)$ that moves along the graph of the function $y=8 / x$ with a horizontal velocity of 3 units per second. (This means that $d x / d t=3$.) At what rate does the distance between $P$ and the origin $(0,0)$ change as the point passes through $(4,2)$ ?

Since $y=8 / x$ on the curve, the point $P$ has coordinates $(x, 8 / x)$. So the distance $D$ from $P$ to the origin is

$$
D=\sqrt{x^{2}+(8 / x)^{2}}
$$

So the derivative with respect to $t$ of this distance is

$$
\frac{d D}{d t}=\frac{2 x-2 \cdot 64 / x^{3}}{2 \sqrt{x^{2}+(8 / x)^{2}}} \frac{d x}{d y}=\frac{x-64 / x^{3}}{\sqrt{x^{2}+(8 / x)^{2}}} \frac{d x}{d y}
$$

Now plug in $d x / d t=3, x=4$ and $y=2$
8. (10 points) A rectangular box with volume 6 cubic feet is to be built with a square base and no top. The material used for the bottom panel costs $\$ 3.00$ per square foot and the material for the side panels costs $\$ 2.00$ per square foot. Find the minimum cost of such a box. Justify your answer using the methods of calculus.

Let the bottom have dimensions $x$ by $x$ and let $h$ be the height. The $V=6=x^{2} h$ so $h=6 / x^{2}$ and the cost $C(x)$ is

$$
C(x)=3 x^{2}+4 \cdot 2 x h=3 x^{2}+8 \cdot 6 / x
$$

Now find $C^{\prime}(x)$ and set it equal to 0 . There is only one root to this equation, $x=2$. Since $C^{\prime}(x)>0$ for $x>2$ and $C^{\prime}(x)<0$ for $x<2$, this is a global minimum. So the answer is $C(2)$.
9. (30 points) Let $f(x)=x(x-3)^{2}=x^{3}-6 x^{2}+9 x$.
(a) Compute the first and second derivatives:

$$
f^{\prime}(x)=\quad f^{\prime \prime}(x)=
$$

(b) Find the interval(s) where $f$ is increasing and those where $f$ is decreasing.
(c) Find the local and global maxima and minima on the interval $[-1,3.5]$. Give both the values and where they are attained.
(d) Find the intervals where $f$ is concave up and those where it is concave down.
(e) Find the inflection points of $f$.
(f) Graph $y=f(x)$.
10. (10 points) Let $R$ be the region bounded by the graph of $f(x)=1-(x-2)^{2}$ and the x -axis. Set up but do not evaluate integrals for the following:
(a) the volume of the region arrived at by rotating $R$ about the x -axis.

$$
\int_{1}^{3} \pi f(x)^{2} d x=\int_{1}^{3} \pi\left(1-(x-2)^{2}\right)^{2} d x
$$

(b) the volume of the region arrived at by rotating $R$ about the $y$-axis.

$$
\int_{1}^{3} \pi x f(x) d x=\int_{1}^{3} \pi x\left(1-(x-2)^{2}\right) d x
$$

11. (20 points)


The graph of a function $f$ consists of a semicircle and two line segments as shown above. Let $g$ be the function given by

$$
g(x)=\int_{0}^{x} f(t) d t
$$

(a) Find $g(3)$.
$g(3)=\pi-1 / 2$.
(b) Find all values of $x$ on the open interval $(-2,5)$ at which $g$ has a relative maximum. Justify your answer.

By the fundamental theorem $g^{\prime}(x)=f(x)$. So $g^{\prime}(x)=0$ where $f(x)=0$ which is at -2 , 2 and 4. Since we are considering the open interval $(-2,5)$ we discard -2 since it is not in this interval. Since $g^{\prime}(x)=f(x)$ is positive for $x<2$ and negative for $x>2$, there is a local maximum at 2. (At 4 there is a relative minimum.)
(c) Write an equation for the line tangent to the graph of $g$ at $x=3$.

The tangent line is $y-y_{0}=m\left(x-x_{0}\right)$ where $m$ is the slope. So $m=g^{\prime}(3)=f(3)=-1$. So the line is

$$
y-(\pi-1 / 2)=-1(x-3)
$$

(d) Find the $x$-coordinate of each point of inflection of the graph of $g$ on the open interval $(-2,5)$. Justify your answer.

Inflection points occur only when $g^{\prime \prime}(x)=f^{\prime}(x)=0$ or is undefined. $f^{\prime}(x)=0$ when $x=0$ and $f^{\prime}(x)$ is undefined when $x=2$ and $x=3$. There is an inflection point at $x=0$ and $x=3$ but not at $x=2$ because $g^{\prime \prime}(x)=f^{\prime}(x)$ does not change signs there.

