Notes on tensor products

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ABSTRACT. Graduate algebra

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CHAPTER 1

Tensor products

1. Basic definition

Last semester, we quickly defined and constructed the tensor product of modules two modules M and N over a commutative ring R, defining it as the universal target $M \otimes_R N$ of the universal R-bilinear form. Here, we'll begin by considering the more general setup where R is allowed to be non-commutative. We first introduce the generalization of a bilinear map to this setting: a 'balanced' map.

Let R be a (possibly non-commutative, but always unital) ring. In this section, M will be a *right* R-module and N will be a *left* R-module, unless otherwise indicated.

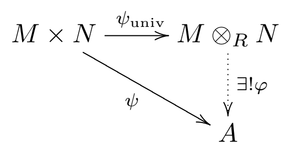
Definition 1.1. Let A be an abelian group (written additively). A function $\psi : M \times N \to A$ is called *R-balanced* (or *middle R-linear*) if for all $m_1, m_2, m \in M, n_1, n_2, n \in N$, and $r \in R$,

(i) $\psi(m_1 + m_2, n) = \psi(m_1, n) + \psi(m_2, n)$,
(ii) $\psi(m, n_1 + n_2) = \psi(m, n_1) + \psi(m, n_2)$,
(iii) $\psi(mr, n) = \psi(m, rn)$.

Let $R\text{-Balan}_{M,N}(A)$ denote the set of R-balanced A-valued functions $\psi : M \times N \to A$.

We may now define the tensor product $M \otimes_R N$ by a universal property as we did in the case of commutative R. For simplicity, we will eschew the language of natural transformations.

Definition 1.2. The *tensor product of M and N over R* is the abelian group $M \otimes_R N$ (if it exists) equipped with an R-balanced map $\psi_{\text{univ}} : M \times N \to M \otimes_R N$ defined by the following universal property. For every abelian group A and every $\psi \in R\text{-Balan}_{M,N}(A)$, there is a unique group homomorphism $\varphi : M \otimes_R N \to A$ such that



commutes.

Theorem 1.3. *The tensor product $M \otimes_R N$ exists.*

Proof. The proof is basically the same as the commutative case, but let's go through it again. First, we'll define the abelian group $M \otimes_R N$ and ψ_{univ}, then we'll show it satisfies the universal property. Let $F = \text{Free}(M \times N)$ be the free abelian group (i.e. the free $\mathbf{Z}$-module) on $M \times N$, i.e. the abelian group whose elements are formal finite linear combinations

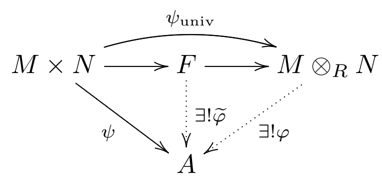
$$\sum_{(m,n) \in M \times N} a_{(m,n)} \cdot (m, n)$$

with $a_{(m,n)} \in \mathbf{Z}$. There is a natural group homomorphism $M \times N \to F$ sending (m, n) to $1 \cdot (m, n)$. Let $J \subset F$ be the subgroup generated by all elements of one of the following forms, as m_1, m_2, m varies over all elements of M, n_1, n_2, n varies over all elements of N, r varies over all elements of R:

(i) $(m_1 + m_2, n) - (m_1, n) - (m_2, n)$,
(ii) $(m, n_1 + n_2) - (m, n_1) - (m, n_2)$,
(iii) $(mr, n) - (m, rn)$.

Let $M \otimes_R N := F/J$. Let $\psi_{\text{univ}} : M \times N \to M \otimes_R N$ be the composition of the quotient map $F \to M \otimes_R N$ with the natural injection $M \times N \to F$. By the definition of J, ψ_{univ} is R-balanced.

Let's now show that this constructed $\psi_{\text{univ}} : M \times N \to M \otimes_R N$ satisfies the universal property. Let A be an abelian group and let $\psi : M \times N \to A$ be an R-balanced map. By the universal property of free $\mathbf{Z}$-modules, there is a unique homomorphism $\widetilde{\varphi} : F \to A$ such that $\widetilde{\varphi}((m,n)) = \psi(m,n)$ for all $(m,n) \in M \times N$. Since ψ is R-balanced, $\widetilde{\varphi}(J) = 0$, i.e. $J \subseteq \ker(\widetilde{\varphi})$. By the universal property of quotients, there is a unique homomorphism $\varphi : M \otimes_R N \to A$ such that $\varphi((m,n) \bmod J) = \widetilde{\varphi}((m,n))$. We thus have the commutative diagram



yielding the desired universal property. □

The image of (m,n) in $M \otimes_R N$ is denoted $m \otimes n$ as is called a *pure tensor* or a *simple tensor*. Note that it denotes an equivalence class and hence may be equal to some other expression $m' \otimes n'$. A general element of $M \otimes_R N$ is a linear combination (with $\mathbf{Z}$ coefficients) of pure tensors.

As you may have noticed, unlike the case where R is commutative, the tensor product may not be an R-module. In order to get that extra structure, we can proceed as follows.

Definition 1.4. Let R and S be two rings. An abelian group M is called an (R,S)-*bimodule* if it is a left R-module, a right S-module, and

$$r(ms) = (rm)s$$

for all $r \in R, s \in S, m \in M$.

A simple, and important, example is the case where R is commutative and M is a left (or right) R-module. In this case, you can define a right (or left) R-module structure on M by $m \cdot r := r \cdot m$. This makes M into a (R,R)-bimodule called the *standard bimodule structure on* M.

Proposition 1.5. *Let R and S be rings. Let M be an (R,S)-bimodule and let N be a left S-module. Then, $M \otimes_S N$ is a left R-module with scalar multiplication defined by*

$$r \cdot \left(\sum_{i=1}^{k} m_i \otimes n_i \right) := \sum_{i=1}^{k} (r \cdot m_i) \otimes n_i.$$

Proof. We'll take advantage of the universal property. For each $r \in R$, you can check that the map $\psi_r : (m,n) \mapsto (rm) \otimes n$ is S-balanced; indeed, checking condition (iii),

$$\begin{aligned} \psi_r(ms,n) &= r(ms) \otimes n \\ &= (rm)s \otimes n \\ &= (rm) \otimes (sn) \\ &= \psi_r(m,sn). \end{aligned}$$

By the universal property of tensor products, there is a unique homormophism

$$\varphi_r : M \otimes_S N \to M \otimes_S N$$

such that

$$\varphi_r(m \otimes n) = \psi_r(m,n) = (rm) \otimes n.$$

The existence of φ_r shows that the definition of the scalar multiplication in the statement of the proposition is well-defined (independent of the way of representing the input as a sum of pure tensors) and shows that it gives a left R-module structure. □

In particular, if R is a commutative ring, M and N are two left R-modules, and we view M with its standard bimodule structure, then $M \otimes_R N$ is a left R-module and this left R-module structure is the same we defined near the end of last semester (i.e. the one that satisfies the universal property of being the target of the universal R-bilinear form).

Lemma 1.6. *In any tensor product $M \otimes_R N$, the pure tensors $m \otimes 0$ and $0 \otimes n$ are 0 for all $m \in M$ and all $n \in N$.*

Proof. For $m \in M$,

$$\begin{aligned} m \otimes 0 &= m \otimes (0+0) \\ &= m \otimes 0 + m \otimes 0, \end{aligned}$$

so canceling one $m \otimes 0$ on each side gives the desired result. Similarly, for $0 \otimes n = 0$. $\square$

Lemma 1.7. *For $m, n \in \mathbf{Z}_{\geq 1}$,*

$$\mathbf{Z}/m\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/n\mathbf{Z} \cong \mathbf{Z}/\gcd(m,n)\mathbf{Z}.$$

Proof. First, note that $\mathbf{Z}/m\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/n\mathbf{Z}$ is a cyclic abelian group generated by $1 \otimes 1$; indeed,

$$a \otimes b = a \otimes (b \cdot 1) = (ab) \otimes 1 = ((ab) \cdot 1) \otimes 1 = ab(1 \otimes 1),$$

so every tensor is a multiple of $(1 \otimes 1)$.

Let $g := \gcd(m,n)$. By Bézout's identity, there are integers $u, v \in \mathbf{Z}$ such that $g = mu + nv$. Thus,

$$\begin{aligned} g(1 \otimes 1) &= (mu + nv)(1 \otimes 1) \\ &= mu(1 \otimes 1) + nv(1 \otimes 1) \\ &= (mu \cdot 1) \otimes 1 + 1 \otimes (nv \cdot 1) \\ &= 0 \otimes 1 + 1 \otimes 0 \\ &= 0, \end{aligned}$$

and so $\mathbf{Z}/m\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/n\mathbf{Z}$ is cyclic of order dividing g. To show its order is at least g, we'll take advantage of the universal property of tensor products. Consider the map

$$\begin{array}{cccc} \psi & : & \mathbf{Z}/m\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z} & \to & \mathbf{Z}/g\mathbf{Z} \\ & & (a,b) & \mapsto & ab \pmod{g} \end{array},$$

which is well-defined because g divides both m and n. It is straightforward to check that ψ is $\mathbf{Z}$-bilinear and so, by the universal property, there is a $\mathbf{Z}$-module homomorphism $\varphi : \mathbf{Z}/m\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/n\mathbf{Z} \to \mathbf{Z}/g\mathbf{Z}$. Since φ 'agrees with' ψ, $\varphi(1 \otimes 1) = 1$. But $1 \in \mathbf{Z}/g\mathbf{Z}$ has order g, so $1 \otimes 1$ must have order at least g. $\square$

Lemma 1.8. *Suppose R is commutative and M and N are two left R-modules with their standard bimodule structures. If M is a free R-module with basis $B = \{m_i : i \in I\}$ and N is a free R-module with basis $C = \{n_j : j \in J\}$, then $M \otimes_R N$ is a free R-module with basis $B \otimes C := \{m_i \otimes n_j : i \in I, j \in J\}$, so that $\mathrm{rk}_R(M \otimes_R N) = \mathrm{rk}_R(M) \cdot \mathrm{rk}_R(N)$.*

Proof. $B \otimes C$ spans: we know that $M \otimes_R N$ is generated by pure tensors $m \otimes n$. Furthermore, a given m is a finite linear combination

$$m = \sum_{i \in I} a_i m_i$$

and similarly

$$n = \sum_{j \in J} b_j n_j.$$

Thus,

$$m \otimes n = \sum_{i \in I, j \in J} a_i b_j (m_i \otimes n_j),$$

and so $B \otimes C$ is a generating set.

$B \otimes C$ is linearly independent: suppose

$$\sum_{i \in I, j \in J} \alpha_{i,j}(m_i \otimes n_j) = 0$$

(where only finitely many $\alpha_{i,j}$ are non-zero). We want to show that all $\alpha_{i,j}$ are zero. To do this, we will 'extract' this coefficient. This is done by constructing, for each pair (i, j), a linear functional $f_{i,j} : M \otimes_R N \to R$ that takes the value 1 on $m_i \otimes n_j$ and 0 on $m_{i'} \otimes n_{j'}$ for $i' \neq i$ and $j' \neq j$. If we have such linear functionals, then, applying each one independently to the above relation gives

$$0 = f_{i,j}(0) = f_{i,j}\left(\sum_{i \in I, j \in J} \alpha_{i,j}(m_i \otimes n_j)\right) = \alpha_{i,j}.$$

So, how do you construct a linear function $M \otimes_R N \to R$? By the universal property of tensor products, you just have to define a bilinear form $\psi_{i,j} : M \times N \to R$. Given what we want from this bilinear form, we define it as follows. For $i_0 \in I$, $j_0 \in J$, and

$$m = \sum_{i \in I} a_i m_i \quad \text{and} \quad n = \sum_{j \in J} b_j n_j,$$

define

$$\psi_{i_0,j_0}(m, n) = a_{i_0} b_{j_0}.$$

This gives an R-valued bilinear form and a linear functional on $M \otimes_R N$ with the desired property. □

Bibliography