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3-3 lattice inclusions imply congruence modularity

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The complexity of a lattice polynomial is defined inductively, with variables having complexity 0. If $\rho=\rho_1\vee\cdots\vee\rho_k$ or $\rho=\rho_1\wedge\cdots\wedge\rho_k$ is the canonical expression of the polynomial ρ, then the complexity $c(\rho)=1+\max\{c(\rho_i):1\le i\le k\}$. An $n-k$ *lattice inclusion* is an inclusion of the form $\rho\le\sigma$ with $c(\rho)\le n$ and $c(\sigma)\le k$. In this note we use the main result of Day [1] to show that if all the congruence lattices of algebras in a variety satisfy a fixed, nontrivial 3-3 lattice inclusion, then they are all modular.

Let q be an element of a distributive lattice D. Then $D[q]$ will denote the lattice obtained by "doubling" the element q, i.e., $D[q]=(D-\{q\})\cup\{(q,0),(q,1)\}$ ordered by $a<b$ if $a,b\in D-\{q\}$ and $a<b$ in D, $a<(q,0)$ if $a<q$ in D, $(q,1)<b$ if $q<b$ in D, and $(q,0)<(q,1)$. Day showed that if a lattice identity ε fails in $D[q]$ for some distributive lattice D and some $q\in D$, then any congruence variety satisfying ε is modular. We prove our result by constructing for each nontrivial 3-3 lattice inclusion $\lambda\le\rho$ a distributive lattice D and an element $q\in D$ such that $D[q]$ fails to satisfy $\lambda\le\rho$.

LEMMA. *In $FL(X)$, let π_{ij} $(i\in I,\ j\in J_i)$ be a meet of variables, and let σ_{kl} $(k\in K,\ l\in L_k)$ be a join of variables. Then*

$$(*)\quad \bigwedge_I\bigvee_{J_i}\pi_{ij}\le\bigvee_K\bigwedge_{L_k}\sigma_{kl}$$

fails in $FL(X)$ if and only if

$$(\forall i)(\exists j)(\forall k)(\exists l)\ \mathrm{var}\,(\pi_{ij})\cap\mathrm{var}\,(\sigma_{kl})=\varnothing$$

and

$$(\forall k)(\exists l)(\forall i)(\exists j)\ \mathrm{var}\,(\pi_{ij})\cap\mathrm{var}\,(\sigma_{kl})=\varnothing.$$

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Proof. By repeated applications of Whitman's conditions, we obtain that the failure of (*) is equivalent to

$$(\forall i)(\exists j)(\forall k)(\exists l) \operatorname{var}(\pi_{ij}) \cap \operatorname{var}(\sigma_{kl}) = \varnothing$$

and

$$(\forall k)(\exists l)(\forall i)(\exists j) \operatorname{var}(\pi_{ij}) \cap \operatorname{var}(\sigma_{kl}) = \varnothing$$

and

$$(\forall i)(\exists j)(\forall x \in \operatorname{var}(\pi_{ij}))(\forall k)(\exists l) x \notin \operatorname{var}(\sigma_{kl})$$

and

$$(\forall k)(\exists l)(\forall x \in \operatorname{var}(\sigma_{kl}))(\forall i)(\exists j) x \notin \operatorname{var}(\pi_{ij}).$$

Since the third and fourth conditions are consequences of the first two, the lemma follows.

THEOREM. *Let* ε *be a nontrivial* 3-3 *lattice inclusion. Then there exists a finite distributive lattice* D *and an element* $q \in D$ *such that the lattice* $D[q]$ *fails to satisfy* ε.

Proof. It is easy to see that we may assume that ε is of the form (*). Let X be the set of variables involved in ε. Then we may assume that ε holds in the free distributive lattice $FD(X)$, for otherwise it fails in **2**, which is of the form $D[q]$. Let λ be the evaluation of the left-hand side of ε in $FD(X)$, and let ρ be the evaluation of the right-hand side. Let θ be the smallest congruence on $FD(X)$ which identifies λ and ρ. We let $D = FD(X)/\theta$ and we let q be the congruence class containing λ, i.e., $q = \lambda/\theta$ $(= \rho/\theta)$.

In order to show that ε fails in $D[q]$, we interpret the variables as follows: for $x \in X$, let $\bar{x} = x/\theta$ if $x/\theta \neq q$, and $\bar{x} = (q, 1)$ if $x/\theta = q$. For $\sigma \in FL(X)$, let $\bar{\sigma}$ denote the image of σ in $D[q]$ under this interpretation, and let $\tilde{\sigma}$ denote the evaluation of σ in $FD(X)$. We shall show that for each $i \in I$, $\overline{\bigvee_{J_i} \pi_{ij}} > (q, 1)$ and dually for each $k \in K$, $\overline{\bigwedge_{L_k} \sigma_{kl}} < (q, 0)$. The desired conclusion will then follow.

First observe that in $FD(X)$, $\rho = \bigvee_K \bigwedge_{L_k} \tilde{\sigma}_k = \bigwedge_{L^K} \bigvee_K \tilde{\sigma}_{k,f(k)}$ where L^K denotes the set of all functions $f: K \to \bigcup L_k$ such that $f(k) \in L_k$. By the lemma, for each $i \in I$ there exist $j \in J_i$ and $f_i \in L^K$ such that $\operatorname{var}(\pi_{ij}) \cap [\bigcup_K \operatorname{var}(\sigma_k, f_i(k))] = \varnothing$. It follows that in $FD(X)$, $\tilde{\pi}_{ij} \nleq \bigvee_K \tilde{\sigma}_{k,f_i(k)}$, and consequently $\bigvee_{J_i} \tilde{\pi}_{ij} \nleq \rho$, for each $i \in I$.

Recall that in a distributive lattice, if $a \leq b$ and $a \leq c \nleq b$, then $(a, c) \notin \operatorname{con}(a, b)$. Now in $FD(X)$ we have $\lambda \leq \rho$ and for each $i \in I$, $\lambda \leq \bigvee_{J_i} \bar{\pi}_{ij} \nleq \rho$, so $(\lambda, \bigvee_{J_i} \bar{\pi}_{ij}) \notin \theta$. Now an easy induction on the complexity of a polynomial ω yields that if $\bar{\omega}/\theta \neq q$, then $\bar{\omega} = \omega/\theta$. Thus for each $i \in I$ we have $\overline{\bigvee_{J_i} \pi_{ij}} > (q, 1)$. Dually we obtain $\overline{\bigwedge_{L_k} \sigma_{kl}} < (q, 0)$ for each $k \in K$. This completes the proof.

COROLLARY. *Let ε be a lattice inclusion. The following are equivalent:*

(i) *ε fails in $D[q]$ for some distributive lattice D and some $q \in D$.*

(ii) *For some $m, n \geq 2$ ε implies the inclusion*

$$\bigwedge_{1 \leq i \leq n} \left(x_i \vee \bigvee_{1 \leq j \leq m} y_j\right) \leq \bigvee_{1 \leq j \leq m} \left(\hat{y}_j \wedge \bigwedge_{1 \leq i \leq n} \hat{x}_i\right)$$

where

$$\hat{x}_i = \bigvee_{k \neq i} x_k \vee \bigvee_{1 \leq j \leq m} y_j \quad \textit{and} \quad \hat{y}_j = \bigvee_{1 \leq i \leq n} x_i \vee \bigvee_{k \neq j} y_k.$$

(iii) *ε implies some nontrivial* 3-3 *lattice inclusion.*

Proof. In [1] Day shows that (i) is equivalent to (ii), and that (ii) implies (iii) is obvious. By the Theorem, (iii) implies (i).

Not every nontrivial lattice inclusion implies a nontrivial 3-3 inclusion. Consider the conjugate inclusion for the splitting lattice N_6 of [2]:

$$(\nu) \quad y \wedge ((x \wedge (w \vee (x \wedge z))) \vee (z \wedge (w \vee (x \wedge z))))$$
$$\leq x \vee ((x \vee y \vee (w \wedge (x \vee z))) \wedge (z \vee (w \wedge (x \vee z)))).$$

If ν implied some nontrivial 3-3 inclusion, then it would imply some inclusion β of the form of condition (ii) of the Corollary. By [1] β is the conjugate inclusion for a splitting lattice of the form $D[q]$ where D is a finite Boolean algebra and q is a doubly reducible element of D. Since $D[q]$ fails β, it would fail ν, and N_6 would be in the variety generated by $D[q]$. By Jónsson's Lemma we could imbed N_6 in $D[q]$, which however is easily seen to be impossible. Thus ν does not imply any nontrivial 3-3 lattice inclusion. Similarly, the conjugate inclusions for the lattices Q_0, Q_1, and Q_4 of [2] and their duals do not imply any nontrivial 3-3 inclusions.

REFERENCES

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