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Congruence Lattices of Finitely Generated Modular Lattices

Ralph Freese*

R. P. Dilworth [1] has shown that every finite distributive lattice is isomorphic to the congruence lattice of some finite lattice. E. T. Schmidt has shown that every finite distributive lattice is isomorphic to the congruence lattice of a modular lattice L. L cannot, in general, be taken to be finite, because the congruence lattice of a finite modular lattice is a Boolean algebra. However, we are able to prove the following theorem.

THEOREM. Every finite distributive lattice D is isomorphic to the congruence lattice of a finitely generated modular lattice.

First we require some lemmas. The notation $a \succ b$ means that a covers b in a lattice, i.e., a/b is a prime quotient. The congruence lattice of a lattice L is denoted $\Theta(L)$. In the case of $L = M_3$, the following lemma was obtained by E. T. Schmidt [5]; see also [4]. Moreover, the first part of this lemma was proved independently by E. T. Schmidt [7].

LEMMA 1. Let $a \succ b$ in a modular lattice L and let D be a distributive lattice with 0 and 1. Then L is isomorphic to a sublattice of a modular lattice L^* (in the variety generated by L) such that the quotient sublattice a/b in L^* is isomorphic to D. Moreover, $\Theta(a,b)/0$ in $\Theta(L^*)$ is isomorphic to $\Theta(D)$. If L is simple, then $\Theta(L^*) = \Theta(D)$.

Let D be a $(0,1)$-sublattice of 2^S, for some set S. For any function f defined on S, we let π_f be the partition on S associated with f; that is, $\pi_f = \{(s,t) \in S^2 : f(s) = f(t)\}$. Let $U = \{(s,t) \in S^2 : f(s) \leq f(t) \text{ for all } f \in D\}$. Let $D' = \{g \in 2^S : g(s) \leq g(t)$ for all $(s,t) \in U$ and $\pi_g \geq \pi_{d_1} \cdots \pi_{d_n}$ for

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some $d_1,\ldots,d_n \in D$, $n \in \omega\}$.

LEMMA 2. $D' = D$.

Proof. Clearly $D \subseteq D'$ and D' is a sublattice of 2^S. Suppose $g \in D' - D$ and let $\pi_g \geq \pi_{d_1} \cdots \pi_{d_n}$. Fixing $d_1,\ldots,d_n$ for the remainder of the proof, let $s_1,\ldots,s_k$ be a set of distinct representatives of the blocks of $\pi_{d_1} \wedge \ldots \wedge \pi_{d_n}$. Let $\phi : 2^S \to 2^k$ be the projection onto the $s_1,\ldots,s_k$ coordinates. Let $E' = \{f \in D' : \pi_f \geq \pi_{d_1} \cdots \pi_{d_n}\}$ and let $E = E' \cap D$. Notice that $0,1 \in E$ and (ϕ) is one to one on E'. Fix a coordinate among $s_1,\ldots,s_k$; say s_1. Let $a = (a_1,\ldots,a_k)$ be the meet of the elements of $d \in \phi(E)$ such that $d(s_1) = 1$. Let $b = (b_1,\ldots,b_k)$ be the join of the elements $d \in \phi(E)$ such that $d(s_1) = 0$. Hence $a_1 = 1$ and $b_1 = 0$. If for some i, $2 \leq i \leq k$, $a_i = 1$ then $b_i = 1$ also. For if $b_i = 0$, then, since s_1 and s_i are from different blocks, there is a d_j among $d_1,\ldots,d_n$ such that either $d_j(s_1) = 0$ and $d_j(s_i) = 1$ or $d_j(s_1) = 1$ and $d_j(s_i) = 0$. The former case gives a contradiction to the definition of b, the latter case, to a. Consequently, the elements a and $a \wedge b$ agree on the coordinates $2,\ldots,k$ but are unequal on the first coordinate. This shows that the embedding of $\phi(E)$ in 2^k is an irredundant subdirect representation. It is not hard to see that this shows that the length of $\phi(E)$ equals the length of 2^k, which is, of course, k.

Notice that our element g is in $E' - E$. Since ϕ is one to one on E', $\phi(g) \notin \phi(E)$. There is a least element of $\phi(E)$ which is above $\phi(g)$. Let g^+ be the unique inverse image of this element in E. Define g^- dually. We may assume that g is chosen in $E' - E$ such that the dimension of $\phi(g^+)/\phi(g')$ is minimal. Since this dimension must exceed one, there is an $f \in E$ such that $g^- < f < g^+$. It follows from the minimality of the dimension of $\phi(g^+)/\phi(g^-)$ that $f \vee g \in E$. Consequently $f \vee g = g^+$. Similarly, $f \wedge g = g^-$. If h is another element of E such that $g^- < h < g^+$, then, as above, $g \vee h = g^+$ and $g \wedge h = g^-$. Distributivity implies that $f = h$. Hence the dimension of $\phi(g^+)/\phi(g^-)$ is two, and $\phi(f)$ and $\phi(g)$ differ in exactly two coordinates, say 1 and 2. Thus we may assume that $f(s_1) = 0 = g(s_2)$ and $f(s_2) = 1 = g(s_1)$. Since $g \in D'$, $(s_1, s_2) \notin U$. Hence there is an element $d \in \{d_1, \ldots, d_n\}$ such that $d(s_1) = 1$ and $d(s_2) = 0$. From this it follows that $g = (d \wedge g^+) \vee g^- \in E$, contradicting $g \in D' - D$.

To prove Lemma 1, let S and U be as above and define $L^* = \{f \in L^S : f(s) \leq f(t)$ for all $(s,t) \in U$ and $\pi_f \geq \pi_{d_1} \cdots \pi_{d_n}$ for some $d_1, \ldots, d_n \in D$ and some $n \in \omega\}$. Clearly L^* is a sublattice of L^S, and L is embedded into L^* by $x \to f_x$ where $f_x(s) = x$ for all $s \in S$. It follows from lemma 2 that a/b in L^* is isomorphic to D.

Let $\theta \in \Theta(D)$. If $x \succeq y$ in a lattice, we define $x - y = 0$ if $x = y$ and $x - y = 1$ if $x \succ y$. If $f, g \in L^S$ are such that $f(s) \succeq g(s)$ for all $s \in S$, then $f - g$ is the element of 2^S given by $(f - g)(s) = f(s) - g(s)$. We let θ^* be the set of all ordered pairs $(f,g) \in (L^*)^2$ such that there exists a finite sequence $f_0 = f \wedge g$, $f_1, \ldots, f_n = f \vee g$ with $f_i(s) \preceq f_{i+1}(s)$, all $s \in S$, $i = 0,1,\ldots,n-1$ and $f_{i+1} - f_i \leq g_i - h_i$ for some $(g_i, h_i) \in \theta$ $i = 0,1,\ldots,n-1$. θ^* is a congruence on L^*. Let $a \succ b$ in L. Then, if we identify D with the sublattice a/b, $\theta^* \cap (a/b)^2 = \theta$. To see this, suppose that $h \geq k$ are elements of a/b in L^* such that $h - k \leq f - g$ for some $(f,g) \in \theta$. If, for some $s \in S$, $f(s) = k(s) = b$ then the condition $h - k \leq f - g$ implies $h(s) = b$ also. It follows that $f \vee k = f \vee h$. Dually, $g \wedge k = g \wedge h$. From this it follows that h/k is projective to $(h \wedge f) \vee g/(k \wedge f) \vee g$ which is contained in f/g. Hence $(h,k) \in \theta$.

In order to establish the isomorphism between $\Theta(D)$ and the sublattice $\theta(a,b)/0$ in $\Theta(L^*)$, we define $\alpha: \Theta(D) \to \theta(a,b)/0$ by $\alpha(\theta) = \theta^* \wedge \theta(a,b)$ and

$\beta: \Theta(a,b)/0 \to \Theta(D)$ by $\beta(\psi) = \psi \cap (a,b)^2$. Clearly both α and β are monotone. Since $\theta^* \cap (a,b)^2 = \theta$, $\beta \circ \alpha$ is the identity map on $\Theta(D)$. Now let $\psi \leq \theta(a,b)$ in $\Theta(L^*)$ and let $(f',g') \in \psi$. There exists a finite chain between $f' \wedge g'$ and $f' \vee g'$ such that the quotient formed by any two consecutive elements of the chain is projective to a subquotient of a/b. Let f/g be such a quotient. Thus f/g is projective to $h/k \subseteq a/b$. Consequently, $(h,k) \in \psi \cap (a,b)^2$, and, since f/g is projective to h/k, $f - g = h - k$. Hence, $(f,g) \in [\psi \cap (a,b)^2]^*$ It follows that $(f',g') \in [\psi \cap (a,b)^2]^* \wedge \theta(a,b)$. Thus $\alpha \circ \beta$ is the identity on $\theta(a,b)/0$. This completes the proof of Lemma 1.

Remarks. 1. The lattice L^* of Lemma 1 has the property that all prime quotients of L become D in L^*. In some situations we want a/b to become D while certain other prime quotients remain prime. Let a/b and c/d be prime quotients of L which are not projective to each other. Then $(c,d) \in \psi = \psi(a,b)$ where $\psi(a,b)$ is the unique largest congruence separating a from b. Since L is modular, $\psi(a,b) \wedge \theta(a,b) = 0$. Hence L is a subdirect product of L/ψ and L/θ, where $\theta = \theta(a,b)$. We apply Lemma 1 to L/ψ, obtaining a lattice $(L/\psi)^*$ in which a/b is isomorphic to D. Now L/ψ is embedded in $(L/\psi)^*$ by the diagonal embedding, which we denote $x/\psi \to (x/\psi)^*$. L can be embedded in $(L/\psi)^* \times L/\theta$ by $x \to ((x/\psi)^*, x/\theta)$. Under this embedding, a/b gets mapped to $((a/\psi)^*, a/\theta)/((b/\psi)^*, b/\theta)$ which is isomorphic to D since $(a/\psi)^*/(b/\psi)^* \cong D$ and $a/\theta = b/\theta$. On the other hand, c/d gets mapped to $((c/\psi)^*, c/\theta)/((d/\psi)^*, d/\theta)$ which is isomorphic to 2 as $c/\psi = d/\psi$ and $c/\theta \succ d/\theta$. We obtain the desired lattice by taking the sublattice of this direct product generated by L and a/b.

2. The congruence lattice of the lattice constructed in the first remark is the lattice obtained from $\Theta(L)$ by replacing the prime quotients of the form $\theta(a,b) \vee \sigma/\sigma$, where $\sigma \leq \psi(a,b)$, by $\Theta(D)$.

Proof of the Theorem. Let K_0 be the two element lattice. Let K_1 be the lattice diagramed in Figure 1 (see [2]). Notice that K_1 is four-generated and any two prime quotients in K_1 are projective. If θ is the congruence relation generated by collapsing these prime quotients, then K_1/θ is the six element lattice of length two. Hence the congruence lattice of K_1 is the three element chain.

We apply the construction of Lemma 1 to K_1 using for D a chain of length n. Let K_n denote the resulting lattice. In K_n, the quotient d/e is isomorphic to a chain of length n, $d = d_0 \succ d_1 \succ \ldots \succ d_{n-1} \succ d_n = e$. It follows from Lemma 1 or from direct inspection, that the congruence lattice of K_n is a 2^n element Boolean algebra with a new greatest element adjoined.

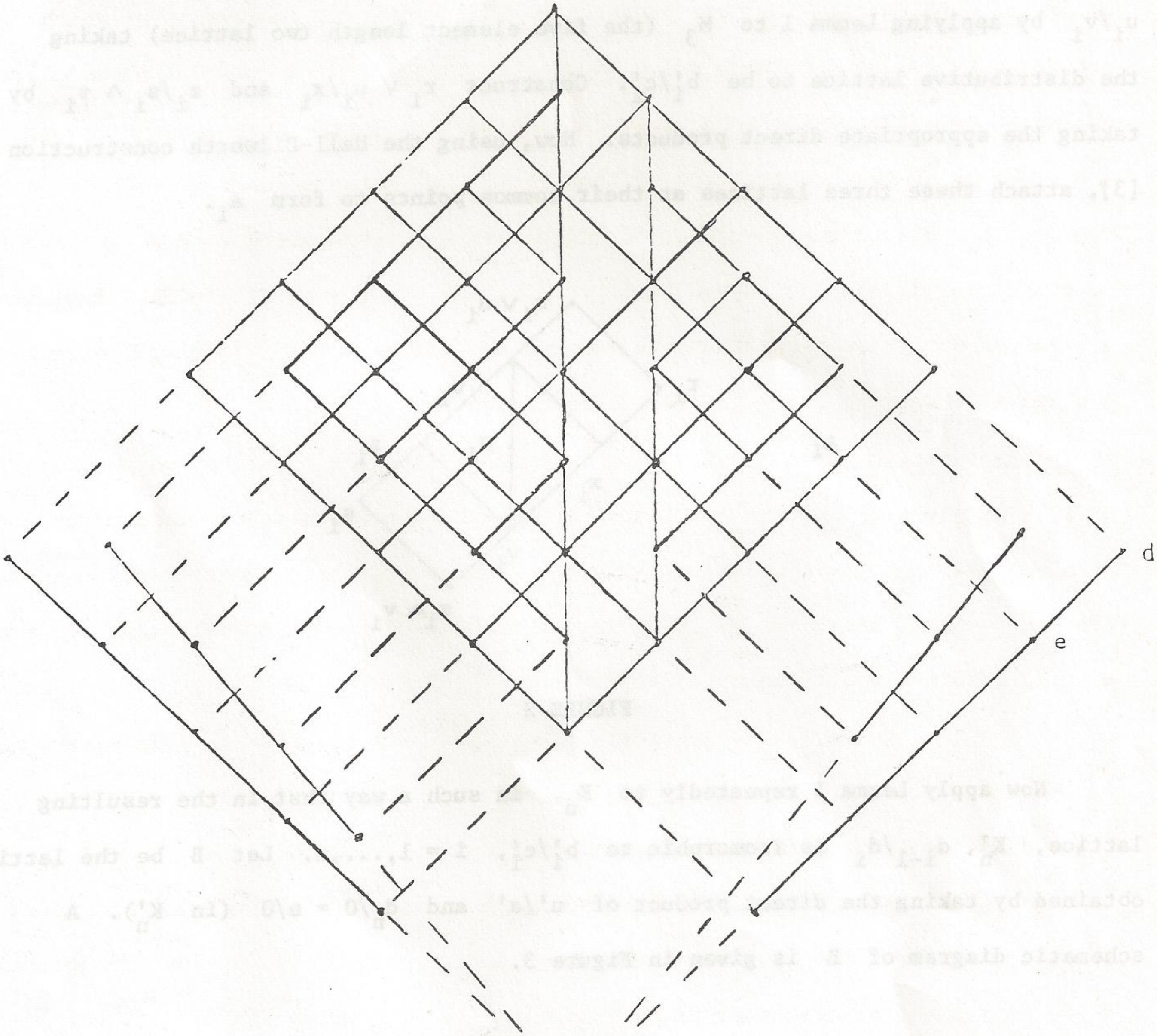
In order to prove the theorem, we prove a stronger result by induction: if D is a finite distributive lattice, then D is isomorphic to the congruence lattice of a finitely generated modular lattice L. Moreover, there exists $a \in L$ such that u/a is a chain, where u is the greatest element of L, and every congruence on L is determined by its restriction to u/a.

Let P be the partially ordered set of nonzero join irreducible elements of D. We induct on the size of P. If P has only one element, then D is the two element lattice and we may take L to be K_0.

Now suppose $|P| \geq 2$. Choose a maximal element $p \in P$ and let p cover $p_1, \ldots, p_n$ in P. Let D' be the lattice of order ideals of $P - p$. By induction, there exists a finitely generated modular lattice L' with $\Theta(L') \cong D'$, and L' has an element a' satisfying the above conditions. Thus for each p_i there exists b_i' and c_i', $u' \geq b_i' > c_i' \geq a'$, such that $\theta(b_i', c_i')$ corresponds to p_i under the isomorphism $\Theta(L') \cong D'$. Moreover, we can choose b_i' and c_i' such that $b_i'/c_i' \cap b_j'/c_j'$ has at most one element for $i \neq j$.

We wish to construct a modular lattice A_i such that A_i contains a sublattice isomorphic to the lattice diagramed in Figure 2, and such that, in A_i, $v_i/s_i \wedge v_i$

FIGURE 1



is isomorphic to c_i'/a', x_i/v_i is isomorphic to b_i'/c_i', and $r_i \vee u_i/u_i$ is isomorphic to u'/b_i'. This is done in three steps. First construct the sublattice u_i/v_i by applying Lemma 1 to M_3 (the five element length two lattice) taking the distributive lattice to be b_i'/c_i'. Construct $r_i \vee u_i/x_i$ and $z_i/s_i \wedge v_i$ by taking the appropriate direct products. Now, using the Hall-Dilworth construction [3], attach these three lattices at their common points to form A_i.

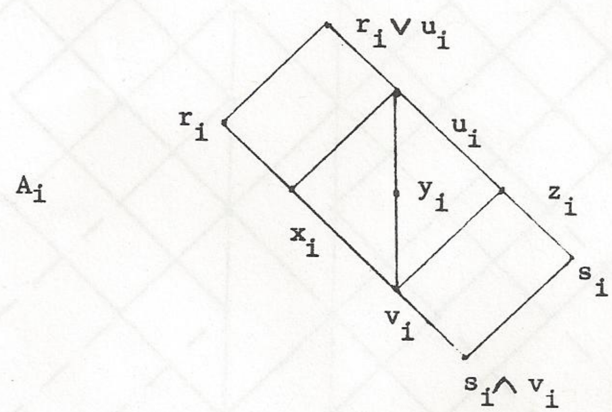


FIGURE 2

Now apply Lemma 1 repeatedly to K_n, in such a way that in the resulting lattice, K_n', d_{i-1}/d_i is isomorphic to b_i'/c_i', $i = 1,\ldots,n$. Let B be the lattice obtained by taking the direct product of u'/a' and $d_n/0 = e/0$ (in K_n'). A schematic diagram of B is given in Figure 3.

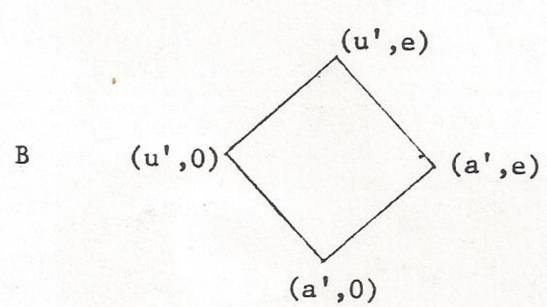


FIGURE 3

The lattice L is formed by repeated use of the Hall-Dilworth construction as follows. First form the lattice L_0 from L' and B by identifying u'/a' and $(u',0)/(a',0)$. L_1 is formed from L_0 and A_n by identifying $(u',e)/(a',e)$ and $r_i/s_i \wedge v_i$. L_n is formed from L_{n-1} and A_1 by identifying $r_2 \vee u_2/s_2$ and $r_1/s_1 \wedge v_1$. In L_n the quotient sublattice $r_1 \vee u_1/u'$ is isomorphic to $d/0$ in K_n'. L is the lattice obtained by identifying these quotients. The sublattice d/a' in L is schematically diagramed in Figure 4. Notice that L is generated by L' and the four generators of K_n and y_i, $i = 1,\ldots,n$ and a. Thus L is finitely generated.

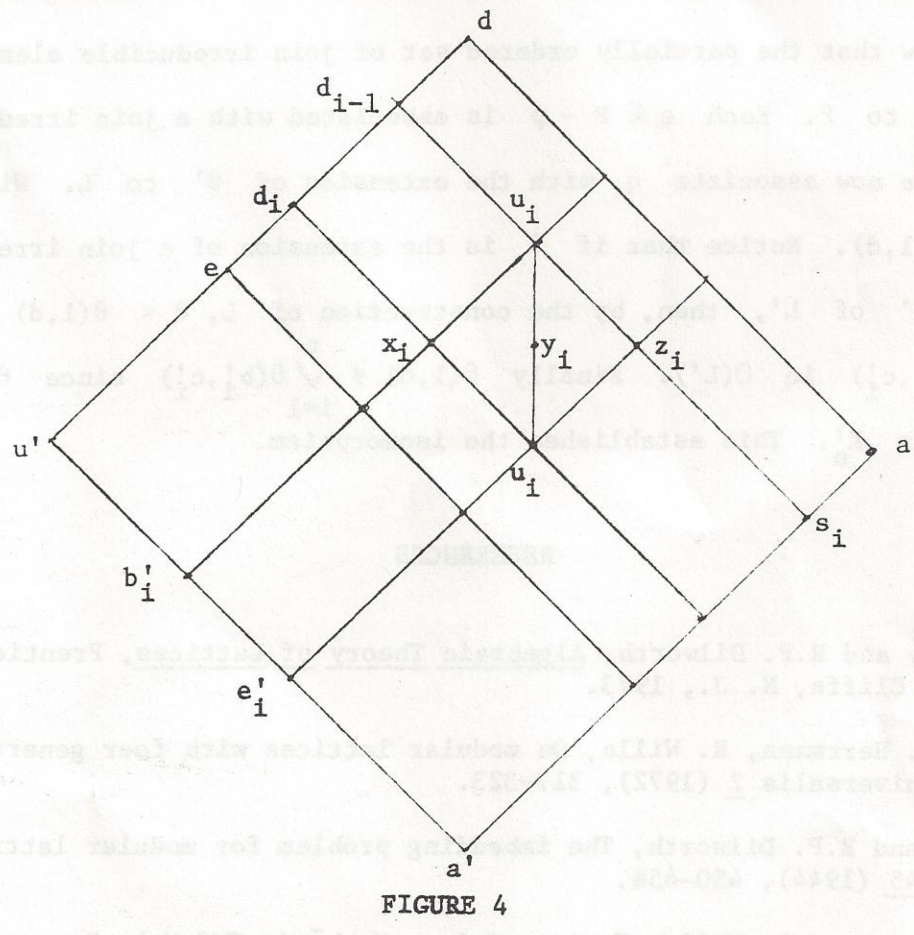


FIGURE 4

Every congruence of L is determined by its restriction to $1/a$. To see this let x/y be a quotient of L. Then $\theta(x,y) = \theta(x \vee d, y \vee d) \vee \theta((x \wedge d) \vee u', (y \wedge d) \vee u') \vee \theta(x \wedge u', y \wedge u')$. Now $x \vee d/y \vee d \subseteq 1/d \subseteq 1/a$. Since d/u' is projective to $1/d$ and $(x \wedge d) \vee u'/(y \wedge d) \vee u' \subseteq d/u'$, $\theta((x \wedge d) \vee u', (y \wedge d) \vee u')$ is determined by its restriction to $1/d$. The last quotient lies in L' and hence is determined by its restriction to u'/a' which projects d/a.

The quotients d_{i-1}/d_i in K'_n are independent in the sense that $\theta_{K'_n}(d_{i-1}, d_i)$ restricted to d_{j-1}/d_j is the identity relation for $i \neq j$. From these facts it follows that every congruence of L' has an extension to L (whose restriction to L' is the original congruence). Moreover, every congruence on L is an extension of a congruence on L' except $\theta(1,d)$. In particular every congruence of L is a join of join irreducible congruences. Hence to show that $\Theta(L) \cong D$ we need only show that the partially ordered set of join irreducible elements of $\Theta(L)$ is isomorphic to P. Each $q \in P - p$ is associated with a join irreducible θ' of $\Theta(L')$. We now associate q with the extension of θ' to L. With p we associate $\theta(1,d)$. Notice that if θ is the extension of a join irreducible congruence θ' of L', then, by the construction of L, $\theta < \theta(1,d)$ if and only if $\theta' \leq \theta(b'_i, c'_i)$ in $\Theta(L')$. Finally $\theta(1,d) \neq \bigvee_{i=1}^{n} \theta(b'_i, c'_i)$ since $\theta(1,d) > \bigvee_{i=1}^{n} \theta(d_{i-1}, d_i)$ in K'_n. This establishes the isomorphism.

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