The Role of Gluing Constructions in Modular Lattice Theory

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In the 1930's and 1940's lattice theory was often broken into three subdivisions: distributive lattice theory, modular lattice theory, and the theory of all lattices. A question about lattices could usually be formulated for each of these subdivisions. Of the three resulting questions, the one about modular lattices almost always proved to be the most difficult. The problem of embedding a lattice into a complemented lattice was an example of such a problem. It is trivial to see that every lattice can be embedded into a complemented lattice, and Birkhoff's representation theorem [1] shows that every distributive lattice can be embedded in a complemented distributive lattice. However the problem of embedding modular lattices into complemented modular lattices remained open for some time. R. P. Dilworth and Marshall Hall addressed this problem in their 1944 paper [23], showing, in fact, that there are finite modular lattices which cannot be embedded into a complemented modular lattice.

This paper used a construction that has become known as Hall-Dilworth gluing, but is now being called Dilworth gluing since it actually originated in an earlier paper of Dilworth, see below. With this construction Dilworth and Hall produced three examples of modular lattices, none of which can be embedded into a complemented modular lattice. Although other papers of Dilworth (and also Hall) contain deeper results, this paper has proved extremely important in the subsequent development of modular lattice theory. The examples themselves have proved useful in refuting various conjectures. The gluing technique used in constructing these lattices has turned out to be useful in settling some of the deeper questions of modular lattice theory. This gluing technique was the origin of more general gluing, which in turn has proved to be especially fruitful in solving some of the most stubborn problems of modular lattice theory.

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The Dilworth gluing is simply this: if a nonempty filter $\mathbf{F}$ of a lattice $\mathbf{L}_0$ is isomorphic to an ideal $\mathbf{I}$ of a lattice $\mathbf{L}_1$, let L be the union of L_0 and L_1 with the elements of F and I identified via the isomorphism. L can be ordered with the transitive closure of the union of the orders on $\mathbf{L}_0$ and $\mathbf{L}_1$. It is easy to see that under this order L is a lattice. A schematic representation of this situation is given in Figure 2 of the Background for this chapter.

It is more difficult to see that the lattice $\mathbf{L}$ is modular if both $\mathbf{L}_0$ and $\mathbf{L}_1$ are. This was established by Dilworth in [12]. Since this paper preceded the Hall-Dilworth paper, we now use the term *Dilworth gluing* for this construction. The Dilworth gluing does not preserve equations in general. (However the distributive law is preserved.) Jónsson's Arguesian law (discussed below) is an example of an equation which is not preserved.

The examples and complemented modular lattices. The Hall-Dilworth paper used gluing to construct three types of examples of modular lattices which cannot be embedded into complemented modular lattices. The basic idea behind all of them is that a projective plane can be embedded into a projective geometry of higher dimension if and only if it satisfies Desargues' theorem. Now the subspaces of a projective geometry form a complemented modular lattice and this lattice determines the geoemetry, see Chapter 13 of [2] and [35]. Since a projective geometry is determined by its lattice of subspaces, we identify a projective geometry with its lattice of subspaces. The first example is constructed by gluing the lattice of subspaces of a projective geometry which fails Desargues' theorem and M_3 (the five element modular, nondistributive lattice) over the two element lattice. A schematic representation of this lattice is given in Figure 3 of the Background. By an argument similar to the proof that non-Desarguesian projective planes cannot be embedded into a higher dimensional projective geometry, Hall and Dilworth showed that this lattice could not be embedded into a complemented modular lattice.

The second example was formed by gluing the lattices of subspaces of two finite Desarguesian projective planes over the two element lattice. These planes were coordinatized by fields with different characteristics.

The third example, which was somewhat more subtle, was constructed by gluing two isomorphic copies of a Desarguesian projective plane over a two dimensional interval. The two dimensional intervals in a projective plane are all isomorphic to $\mathbf{M}_n$, where n is the number of points on a line in the plane. There are $n!$ automorphisms of $\mathbf{M}_n$, and hence that many ways of gluing the two planes together over a two dimensional quotient (some of which will be isomorphic as lattices). With the aid of classical coordinatization techniques it can be shown that only some of these lattices can be embedded into complemented modular lattices. The ones that cannot are the third type of Hall-Dilworth example.

Some of these ideas were clarified by the introduction of the *Arguesian* law by Bjarni Jónsson. This is a lattice equation which reflects Desargues' Theorem of projective geometry. In particular, the lattice of subspaces of a projective geometry satisfies this equation if and only if the projective geometry satisfies Desargues' Theorem. It can be shown that a subdirectly irreducible modular lattice of length at least four which can be embedded into a complemented modular lattice satisfies the Arguesian equation, see Chapter 13 of [2]. The first and the third Hall-Dilworth examples are non-Arguesian and so not embeddable into a complemented modular lattice. In fact, since Arguesian lattices are defined by an equation, it follows that the first and third examples are not even in the variety $\mathcal{K}$ generated by all complemented modular lattices. (It is conceivable that $\mathcal{K}$ equals the class of lattices embeddable into a complemented modular lattice. In fact this is a good unsolved problem: *Is the class of lattices embeddable into a complemented modular lattice closed under the formation of homomorphic images?*)

One might wonder to what extent the nonembeddability of modular lattices into complemented modular lattices is dependent on the failure of the Arguesian law. Indeed the modular law is satisfied by most of the lattices associated with classical algebraic systems. In fact these lattices satisfy stronger equations: Freese and Jónsson [16] have shown that if all the algebras in a variety of algebras have modular congruence lattices, these lattices satisfy the Arguesian equation. Thus the first and the third Hall-Dilworth examples can never lie in such a modular *congruence variety*. In [15] it is shown that the second Hall-Dilworth example also cannot lie in any modular congruence variety.

Is it true that in some restricted class of modular lattices, closer to the class of lattices associated with classical algebraic systems, embedding into complemented lattices might be possible? This was shown not to be the case by Herrmann and Huhn in [28]. They showed that the lattice of subgroups of $(\mathbf{Z}/4\mathbf{Z})^3$ cannot be embedded into any complemented modular lattice.

Some applications of the examples. The Hall-Dilworth examples have been used often in producing counter-examples. In this section we present a few of the important examples. C. Herrmann and W. Poguntke [29] used the second kind of example to show that *the class of all lattices embeddable into the lattice of normal subgroups of a group cannot be defined by finitely many first order axioms.* The idea is to let $\mathbf{L}_p$ be the lattice obtained by gluing (the lattice of subspaces of) a projective plane of characteristic p to a projective plane of characteristic p^+ (the next prime after p) over a 1-dimensional quotient. $\mathbf{L}_p$ is not embeddable into the lattice of normal subgroups of a group In fact it lies in no variety generated by the congruences lattices of a variety of algebras with modular congruence lattices, see [15]. On the other hand it is not hard to see that a nonprincipal ultraproduct of the $\mathbf{L}_p$'s is also one of the Hall-Dilworth examples of the second kind, but the two projective planes have characteristic 0. From this it follows that the whole lattice can be embedded into the lattice of subspaces of a vector space over the rationals. Hence it can be embedded into the lattice of subgroups of an Abelian group. This result also proves that many other classes of modular lattices cannot be defined by finitely many axioms. For example the class of all lattices embeddable into the lattice of subgroups of an Abelian group. Also it shows that the variety generated either of the above two classes cannot be finitely defined.

In [31] Jónsson made a careful investigation of the third type of Hall-Dilworth example. He found necessary and sufficient conditions for this type to be non-Arguesian. These conditions coincided with the Hall-Dilworth conditions for nonembeddability into complemented modular lattices and were equivalent to the lattice not having a representation as a lattice of permuting equivalence relations. This reinforced the idea that the Arguesian equation reflected Desargues' law in geometry.

M. Haiman [20] examined Jónsson's result and, using the correct skewfield, showed that the Arguesian equation required all of its six variables. Let H be the skewfield of all real quaternions and $\mathbf{L}_0 = \mathbf{L}_1 = \mathbf{L}(\mathsf{H}^3)$. Now H has a natural antiautomorphism, quaternion conjugation, that is $\mathbb{R}$-linear. Using this antiautomorphism, Haiman constructed a Dilworth gluing of $\mathbf{L}_0$ and $\mathbf{L}_1$ over a 2-dimensional interval that failed to be Arguesian by Jónsson's result which would require that the map be an automorphism. Moreover this lattice has the property that all 5-generated sublattices are Arguesian.

Freese used a modification of the third type of Hall-Dilworth example to settle some of the important previously unsolved problems of modular lattice theory. Let p and q be distinct prime numbers and let $\mathbf{F}$ and $\mathbf{K}$ be countably infinite fields with characteristics p and q, respectively. Let $\mathbf{L}_0$ be the lattice of subspaces of a 4-dimensional vector space over $\mathbf{F}$ and let $\mathbf{L}_1$ be the lattice of subspaces of a 4-dimensional vector space over $\mathbf{K}$. Every 2-dimensional interval in each of these lattices is isomorphic to $\mathbf{M}_\omega$. Let $\mathbf{L}$ be the lattice obtained by gluing these lattices over such an interval. Then $\mathbf{L}$ is not in the variety generated by the finite modular lattices [13]. In particular the variety of modular lattices is not generated by its finite members. The basic idea of the proof is this. If $\mathbf{F}$ and $\mathbf{K}$ were finite fields then $|F| = p^n$ and $|K| = q^m$ for some $n > 0$ and $m > 0$. Since $p^n \neq q^m$, it is impossible to construct $\mathbf{L}$ as above using finite fields. Of course to actually carry out the proof one needs to bring much of $\mathbf{L}$ into the free modular lattice.

Using a similar example, Freese [14] was able to show that the equational theory of modular lattices is undecidable, *i.e.*, there is no algorithm to determine if two lattice terms are equal in all modular lattices. A. Macintyre [34] constructed a skew field interpreting a finitely presented group with unsolvable word problem. If we construct $\mathbf{L}$ as above, but with Macintyre's field for $\mathbf{F}$, we can interpret this group with unsolvable word problem in $\mathbf{FM}(5)$, showing that its word problem is unsolvable. Herrmann, with the aid of his more general gluing construction, has shown that $\mathbf{FM}(4)$ has an unsolvable word problem, see below.

The lattice $\mathbf{L}$ constructed above can also be used to show that there are two lattice terms $v < u$ in five variables such that interval sublattice $[v, u]$ of $\mathbf{FM}(X)$ is distributive for every X which contains the variables of u and v. From this it follows that every free distributive lattice can be embedded into a free modular lattice. A related open problem is this: *Is the class of distributive sublattices of free modular lattices equal to the class of sublattices of free distributive lattices?*

Generalized gluing. In [25], Herrmann significantly generalized the notion of gluing. Herrmann's idea was to consider maximal complemented subintervals of a modular lattice of finite height. These "blocks" can be ordered by means of their least (or equivalently greatest) elements. With respect to this order, this system of blocks, called the *prime skeleton*, forms a lattice. Now if two blocks, $B = a/b$ and $C = c/d$, are comparable in this order, say with $B \leq C$ and $B \cap C \neq \emptyset$ (equivalently $b \leq d \leq a \leq c$) then $B \cup C$ is a sublattice which is isomorphic to the Dilworth gluing of B and C over the interval $[b, c]$ considered as an ideal in B and a filter in C. Thus the original modular lattice is then decomposed into a lattice of complemented modular blocks together with a system of Dilworth gluings between intersecting, comparable blocks. An example of this kind of decomposition is presented below.

For $\mathbf{L}$ a modular lattice of finite length, Herrmann defined two mappings, $x \mapsto x^\sigma$ and $x \mapsto x^\pi$, of L into L by $0^\sigma = 0$ and $1^\pi = 1$ and

$$x^\sigma = \bigwedge\{y \in L : y \prec x\} \qquad x^\pi = \bigvee\{y \in L : y \succ x\}.$$

These mappings are isotone and form a Galois pair in that $x^\sigma \leq y$ if and only if $x \leq y^\pi$. Consequently, L^σ, the range of $x \mapsto x^\sigma$, is a join subsemilattice of $\mathbf{L}$, and L^π is a meet subsemilattice of $\mathbf{L}$, and they are isomorphic. Thus both $\mathbf{L}^\sigma$ and $\mathbf{L}^\pi$ are lattices, though not necessarily sublattices of L. More precisely, $\langle L^\sigma, +, \wedge\rangle$ and $\langle L^\pi, \vee, \cdot\rangle$ are lattices, where $\vee$ and $\wedge$ are the operations of $\mathbf{L}$, and

$$x \cdot y = (x \wedge y)^\sigma \qquad x + y = (x \vee y)^\pi.$$

A fundamental fact discovered by Herrmann was that the maximal complemented subintervals of $\mathbf{L}$ are precisely those of the form x^π/x, for $x \in L^\sigma$, or equivalently y/y^σ for $y \in L^\pi$. We choose the first format and define the *prime skeleton* of $\mathbf{L}$ to be $\mathrm{S}(\mathbf{L}) = \mathbf{L}^\sigma$ and for each $x \in \mathrm{S}(\mathbf{L})$, we let $\mathrm{L}(x) = x^\pi/x$. The intervals $\mathbf{L}(x)$ will be referred to as the blocks of $\mathbf{L}$.

Now if $x \leq y$ in $\mathrm{S}(\mathbf{L})$ and $x \leq y \leq x^\pi$, then $\mathrm{L}(x) \cap \mathrm{L}(y) = [y, x^\pi] = M$. The identity map on $\mathbf{M}$ can be viewed as a bijection φ_{xy} from a principal filter of $\mathbf{L}(x)$ to a principal ideal of $\mathbf{L}(y)$. In the case $x \not\leq y$ or $\mathrm{L}(x) \cap \mathrm{L}(y) = \emptyset$, we let $\varphi_{xy} = \emptyset$. These maps satisfy certain natural compatibility constraints, namely

(1) φ_{xx} is the identity map on $\mathbf{L}(x)$.
(2) If $\varphi_{xy} \neq \emptyset$, then it is an isomorphism of a filter of $\mathbf{L}(x)$ onto an ideal of $\mathbf{L}(y)$.
(3) If $x \prec y$ in $\mathbf{S}$, then $\varphi_{xy} \neq \emptyset$.
(4) If $x \leq z \leq y$ in $\mathbf{S}$, them $\varphi_{xy} = \varphi_{xz} \circ \varphi_{zy}$.
(5) $\mathrm{Im}\ \varphi_{x,x\vee y} \cap \mathrm{Im}\ \varphi_{y,x\vee y} = \mathrm{Im}\ \varphi_{x\wedge y,x\vee y}$.
(6) $\mathrm{Dom}\ \varphi_{x\wedge y,x} \cap \mathrm{Dom}\ \varphi_{x\wedge y,y} = \mathrm{Dom}\ \varphi_{x\wedge y,x\vee y}$.

Herrmann's gluing construction [25] (see also [26]) is essentially a converse of the above situation:

Let $\mathbf{S}$ *and* $\mathbf{L}(x)$, $x \in S$ *be lattices of finite length and for* $x \leq y$ *in* $\mathbf{S}$ *let* $\varphi_{xy} : \mathbf{L}(x) \to \mathbf{L}(y)$ *be partial bijections satisfying the above conditions. Let* L *be the disjoint union of the* $L(x)$*'s with elements identified under*

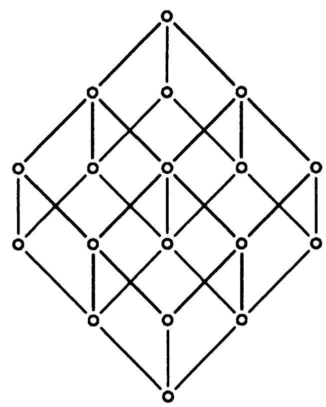

Figure 1

the φ_{xy}'s (i.e., a and b are identified if $\varphi_{xz}(a) = \varphi_{yz}(b)$). Then $\mathbf{L}$ *is a lattice. If each* $\mathbf{L}(x)$ *is modular then* $\mathbf{L}$ *is.*

Herrmann's gluing can be generalized in several ways. The finite length assumptions on $\mathbf{S}$ and $\mathbf{L}(s)$ can be relaxed. Moreover the blocks can be *loosely glued* rather than *tightly glued*. (If $\mathbf{F}$ is a filter of $\mathbf{L}_0$ which is isomorphic to an ideal $\mathbf{I}$ of $\mathbf{L}_1$, we form a lattice on the disjoint union of L_0 and L_1 whose order relation is the transitive closure of the order relations of $\mathbf{L}_0$ and $\mathbf{L}_1$ and the relation $x \leq y$ if x is mapped to y under the isomorphism of $\mathbf{F}$ to $\mathbf{I}$. Notice that this construction can be obtained by gluing $\mathbf{L}_0$, $\mathbf{F} \times \mathbf{2}$, and $\mathbf{L}_1$ using the ordinary Dilworth gluing.) A related type of gluing was developed by in Graczynska [17] and Gracyznska and Grätzer [18]. Further generalizations of these gluings and of Herrmann's gluing are developed in Day and Herrmann [5]. That paper gives various applications including applications to Maltsev products. Particular applications of this generalized gluing occur in Grätzer and Kelly [19] and Harrison [24].

Applications of generalized gluing. The following example, which is an unpublished result of Jónsson, illustrates how Herrmann's gluing can be used to produce some subtle examples. Let $S = \{0, a, b, 1\} \cong 2^2$ and $\mathbf{L}_0$, $\mathbf{L}_a$, $\mathbf{L}_b$, and $\mathbf{L}_1$ be four copies of an Arguesian projective plane of order n, $\mathbf{L} = \mathbf{L}(\mathbf{F}^3)$, the lattice of subspaces of the 3-dimensional vector space over the field with n elements. We can picture these lattices in the lattice, $\mathbf{L}(\mathbf{F}^5)$. Let $\mathbf{F}^5$ have as a basis $\{u_1, \ldots, u_5\}$ and consider the following length 3 intervals in $\mathbf{L}(\mathbf{F}^5)$,

$$\begin{aligned}
\mathbf{L}_0 &= [0, \mathbf{F}u_1 + \mathbf{F}u_2 + \mathbf{F}u_3] \\
\mathbf{L}_a &= [\mathbf{F}u_1, \mathbf{F}u_1 + \mathbf{F}u_2 + \mathbf{F}u_3 + \mathbf{F}u_4] \\
\mathbf{L}_b &= [\mathbf{F}u_3, \mathbf{F}u_1 + \mathbf{F}u_2 + \mathbf{F}u_3 + \mathbf{F}u_5] \\
\mathbf{L}_1 &= [\mathbf{F}u_1 + \mathbf{F}u_2 + \mathbf{F}u_3, \mathbf{F}^5]
\end{aligned}$$

It is easy to see that the union of these intervals is a sublattice of $\mathbf{L}(\mathbf{F}^5)$. A sublattice of this lattice is given in Figure 1.

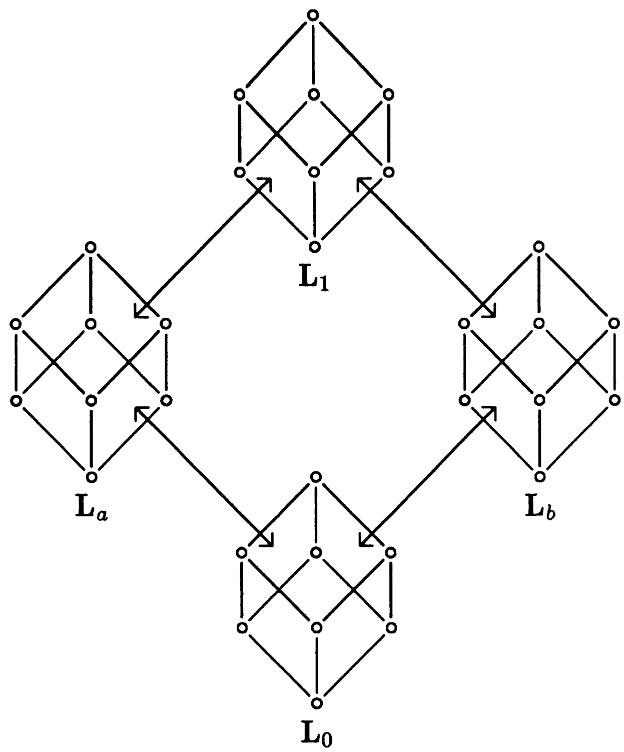

Figure 2

The situation with the blocks pulled apart is represented in Figure 2.

Of course the identification maps for these blocks are the identity maps. Assume that $\mathbf{F}$ has a nontrivial automorphism. What Jónsson did was to modify the map which connects $\mathbf{L}_0$ and $\mathbf{L}_a$ to be the map induced by the automorphism of $\mathbf{F}$. The resulting lattice fails the Arguesian law but the sublattice consisting of the union of $\mathbf{L}_0$ and $\mathbf{L}_a$ is Arguesian. In fact every proper sublattice of this lattice is Arguesian. Herrmann [unpublished] has shown that all non-Arguesian lattices with $S \cong 2^2$ are produced in this way.

Other interesting examples of minimal non-Arguesian lattices were produced by Day, Haiman, Herrmann, Jónsson, and Pickering. For each integer k, Pickering [36] was able to construct a minimal non-Arguesian lattice of length at least k. His construction used Herrmann's gluing with $\mathbf{S} = 2^3$. Using these lattices he was also able to show that *there is a variety of modular, non-Arguesian lattices all of whose finite members were Arguesian.* This guaranteed that obtaining a structural characterization of minimal non-Arguesian lattices would be difficult. Nevertheless some progress has been made. In a series of papers [6], [7], [8], [9], Day and Jónsson conducted a detailed analysis of the Arguesian law in an attempt to understand the structure of minimal non-Arguesian lattices. In analogy to the Desargues configuration, they define a certain lattice configuration which is projective in the class of modular lattices. Accordingly they are called *projectivity configurations.* If the lattice $\mathbf{L}$ fails the Arguesian equation, one of these configurations is non-Arguesian, and in the ideal lattice of $\mathbf{L}$ there will be a minimal, non-Arguesian projectivity configuration. Associated with these minimal configurations are several intervals which are projective planes which are glued together in a certain manner. The subsequent analysis split naturally into two cases. For the first case, called *Boolean*, there was a complete classification of models: each was the gluing of projective planes over a skeleton of the form 2^n for $n = 0, 1, 2, 3$. Examples of these have been given above. The results for the second case were not nearly so descriptive and it is clear that this problem is much more intractable.

Herrmann himself of course made frequent use of his gluing. One of his outstanding results was that $\mathbf{FM}(4)$ has an undecidable word problem, [26]. Although his proof used many ideas from [14], his construction was fundamentally different. He introduced the concept of a *skew frame* and with his gluing, modified the lattice of subgroups of a certain Abelian group producing a 4-generated modular lattice in which the group ring of a finitely presented group with unsolvable word problem could be interpreted. Using the fact that his skew frames are a projective configuration, he could pull this situation back into $\mathbf{FM}(4)$, proving his result.

Another very important paper of Herrmann which uses gluing is [27]. Let $\mathcal{M}_0$ denote the variety of lattices generated by the subspace lattices of all vector spaces over the rationals. Then Herrmann's result states:

Every variety of modular lattices which contains $\mathcal{M}_0$ either is not generated by its finite dimensional members or does not have a finite equational basis.

This result has several important corollaries:

The variety of Arguesian lattices is not generated by its finite dimensional members. Neither the variety generated by all finite modular lattices nor the variety generated by all finite dimensional modular lattices have a finite equational basis.

As Jónsson noted in [31], lattices of permuting equivalence relations satisfy the Arguesian law. Moreover he showed in [33] that every complemented Argusian lattice has such a representation. It was an open problem for many years whether complementedness could be dropped from his result. In a beautiful result, M. Haiman [21], [22] used gluings to solve this problem in the negative. A *crown* is an ordered set of the form $C_n = \{a_0, \ldots, a_{n-1}\} \cup \{b_0, \ldots, b_{n-1}\}$ with $a_i < b_i$, b_{i+1} (indices computed modulo n). Let S_n be C_n with a least and greatest element added. A diagram of S_4 is given in Figure 3.

Let $\mathbf{V}$ be a $2n$-dimensional vector space over a prime field. For each element in S_n Haiman found an interval in $\mathbf{L}(\mathbf{V})$. The union of these intervals is a sublattice of $\mathbf{L}(\mathbf{V})$ whose prime skeleton is S_n. By gluing these intervals back together over S_n, but slightly modifying one of the intersections maps, Haiman produced a lattice which is Arguesian but cannot be represented as a lattice of equivalence relations. Moreover his proof shows in fact that the class of lattices having a representation as a lattice of permuting equivalence relations cannot be defined by finitely many first order axioms, although this class can be defined by an infinite set of Horn

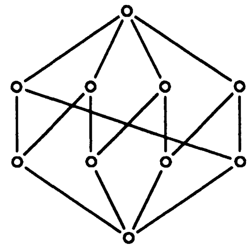

Figure 3

sentences, see Jónsson [32]. A good open problem: *Is the class of lattices having a representation as a lattice of permuting equivalence relations a variety?*

Conclusion. Certainly the results above show that the Hall-Dilworth paper was seminal. Dilworth's gluing and its generalizations together with the examples constructed by Dilworth and Hall are the basis of many of the most important results in modular lattice theory. Along with the papers of Dedekind [10], [11], and von Neumann on coordinatizing complemented modular lattices [35], the Hall-Dilworth paper is among the most influential in the field.

REFERENCES

1. G. Birkhoff, *On the combination of subalgebras*, Proc. Cambridge Phil. Soc. **29** (1933), 441–464.
2. P. Crawley and R. P. Dilworth, "Algebraic Theory of Lattices," Prentice-Hall, Englewood Cliffs, New Jersey, 1973.
3. A. Day, *Geometrical applications in modular lattices*, in "Universal Algebra and Lattice Theory," R. Freese and O. Garcia, eds., Lecture Notes in Mathematics, vol. **1004**, Springer-Verlag, New York, 1983, pp. 111–141.
4. A. Day, *Dimension equations in modular lattices*, Algebra Universalis **22** (1986), 14–26.
5. A. Day and Ch. Herrmann, *Gluings of modular lattices*, Order **5** (1988), 85–101.
6. A. Day and B. Jónsson, *The structure of non-Arguesian lattices*, Bull. Amer. Math. Soc. **13** (1985), 157–159.
7. A. Day and B. Jónsson, *A structural characterization of non-Arguesian lattices*, Order **2** (1986), 335–350.
8. A. Day and B. Jónsson, *Non-Arguesian configurations in a modular lattice*, Acta Sci. Math. (Szeged) **51** (1987), 309–318.
9. A. Day and B. Jónsson, *Non-Arguesian configurations and gluings of modular lattices*, Algebra Universalis (to appear).
10. R. Dedekind, *Über Zerlegungen von Zahlen durch ihre grössten gemeinsamen Teiler*, Festschrift der Herzogl. technische Hochschule zur Naturforscher-Versammlung, Braunschweig (1897). Reprinted in "Gesammelte mathematische Werke", Vol. 2, pp. 103–148, Chelsea, New York, 1968.
11. R. Dedekind, *Über die drei Moduln erzengte Dualgruppe*, Math. Annalen **53** (1900), 371–403.
12. R. P. Dilworth, *The arithmetical theory of Birkhoff lattices*, Duke Math. J. **8** (1941), 286–299. Reprinted in Chapter 3 of this volume.

13. R. Freese, *The variety of modular lattices is not generated by its finite members*, Trans. Amer. Math. Soc. **255** (1979), 277–300.
14. R. Freese, *Free modular lattices*, Trans. Amer. Math. Soc. **261** (1980), 81–91.
15. R. Freese, C. Herrmann and A. P. Huhn, *On some identities valid in modular congruence varieties*, Algebra Universalis **12** (1981), 322–334.
16. R. Freese and B. Jónsson, *Congruence modularity implies the Arguesian identity*, Algebra Universalis **6** (1976), 225–228.
17. E. Graczynska, *On the sums of double systems of lattices and DS-congruences of lattices*, in "Contributions to universal algebra," Colloq., Jozsef Attila Univ., Szeged,1975, 1977, pp. 161–178.
18. E. Graczynska and G. Grätzer, *On double systems of lattices*, Demonstratio Math. **13** (1980), 743–747.
19. G. Grätzer and D. Kelly, *Products of lattice varieties*, Algebra Universalis **21** (1985), 33–45.
20. M. Haiman, *Two notes on the Arguesian identity*, Algebra Universalis **21** (1985), 167–171.
21. M. Haiman, *Arguesian lattices which are not linear*, Bull. Amer. Math. Soc. **16** (1987), 121–124.
22. M. Haiman, *Arguesian lattices which are not type-1*, preprint.
23. M. Hall and R. P. Dilworth, *The imbedding problem for modular lattices*, Ann. of Math. **45** (1944), 450–456. Reprinted in Chapter 4 of this volume.
24. T. Harrison, *A problem concerning the lattice varietal product*, Algebra Universalis **25** (1988), 40–81.
25. Ch. Herrmann, *S-Verklebte Summen von Verbänden*, Math. Z. **130** (1973), 225–274.
26. Ch. Herrmann, *On the word problem for modular lattices with four generators*, Math Ann. **265** (1983), 513–527.
27. Ch. Herrmann, *On the arithmetic of projective coordinate systems*, Trans. Amer. Math. Soc. **284** (1984), 759–785.
28. Ch. Herrmann and A. P. Huhn, *Lattices of normal subgroups generated by frames*, Colloq. Math. Society János Bolyai **14** (1975), 97–136.
29. Ch. Herrmann and W. Poguntke, *The class of sublattices of normal subgroup lattices is not elementary*, Algebra Universalis **4** (1974), 280–286.
30. A. P. Huhn, *Schwach distributive Verbände, I*, Acta Sci. Math. (Szeged) **33** (1972), 297–305.
31. B. Jónsson, *Modular lattices and Desargues theorem*, Math. Scand. **2** (1954), 295–314.
32. B. Jónsson, *Representations of modular lattices and relation algebras*, Trans. Amer. Math. Soc. **92** (1959), 449–464.
33. B. Jónsson, *Representations of complemented modular lattices*, Trans. Amer. Math. Soc. **97** (1960), 64–94.
34. A. Macintyre, *The word problem for division rings*, J. Symbolic Logic **38** (1973), 428–436.
35. J. von Neumann, "Continuous Geometry," Princeton University Press, Princeton, N. J., 1960.
36. D. Pickering, "Minimal non-Arguesian lattices," Ph.D. Thesis, University of Hawaii, 1984.

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