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AN APPLICATION OF DILWORTH'S LATTICE OF MAXIMAL ANTICHAINS*

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Abstract. Recently there has been a good deal of interest in the maximal sized antichains of a partially ordered set [1–8] A theorem of Dilworth states that under the natural ordering these antichains form a distributive lattice. This paper outlines a proof of this theorem and applies it to strengthen the results and shorten the proofs of [4].

An antichain in a finite partially ordered set P is a set of pairwise non-comparable elements of P. Let L be the set of all maximal sized antichains ordered by $A \leq B$ in L, if for every element a in A, there exists an element b in B such that $a \leq b$ in P. Then we have the following theorem of Dilworth [2].

Theorem 1. *L is a lattice.*

Proof. If A and B are elements of L, define

$$A_B = \{a \in A : a \geq b \text{ for some } b \in B\} ,$$

(1) $$A^B = \{a \in A : a \leq b \text{ for some } b \in B\} .$$

$A \leq B$ in L is equivalent to $A^B = A$ which is equivalent to $B_A = B$. Now define

(2) $$A \vee B = A_B \cup B_A , \qquad A \wedge B = A^B \cup B^A .$$

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It is easy to see that $A \vee B$ and $A \wedge B$ are the least upper bound and greatest lower bound of A and B in L provided that $A \vee B$ and $A \wedge B$ are in L. That is, provided $A \vee B$ and $A \wedge B$ are maximal sized antichains. This is seen to be the case by observing that

$$\begin{aligned} &A_B \cup A^B = A, \quad B_A \cup B^A = B, \\ (3) \qquad &A_B \cap B_A = A^B \cap B^A = A \cap B, \\ &A_B \cap A^B = B_A \cap B^A = A \cap B \end{aligned}$$

Now since

$$\begin{aligned} |A_B \cup B_A| + |A^B \cup B^A| &= |A_B| + |B_A| - |A_B \cap B_A| + |A^B| + |B^A| - |A^B \cap B^A| \\ &= |A_B \cap A^B| + |A_B \cup A^B| + |B_A \cap B^A| + |B_A \cup B^A| - 2|A \cap B|, \end{aligned}$$

we have

$$(4) \qquad |A \vee B| + |A \wedge B| = |A| + |B| .$$

Let n be the number of elements in a maximal sized antichain. Then $|A| = |B| = n$. Since $A \vee B$ and $A \wedge B$ are clearly antichains, $|A \vee B| \leq n$ and $|A \wedge B| \leq n$. It now follows from the above equation that $|A \vee B| = |A \wedge B| = n$ and hence $A \vee B$ and $A \wedge B$ are maximal sized antichains.

Since L is finite, Theorem 1 says in particular that L has a least and a greatest element. This fact has the following corollary, which is somewhat stronger than the theorem in [4]. Recall that automorphism of a finite partially ordered set P is an order-preserving permutation of the elements of P.

Corollary. *Let P be a finite partially ordered set and G a group of automorphisms of P. Then there exist maximal sized antichains A and B such that A and B are each unions of orbits of G and if $c \in C$, where C is a maximal sized antichain, there exists $a \in A$ and $b \in B$ such that $a \leq c \leq b$.*

Proof. If $\sigma \in G$, then σ acts on L in the following natural way

$$\sigma(A) = \{\sigma(a) : a \in A\}, \quad A \in L.$$

It is easily checked that $\sigma(A) \in L$ and that σ preserves the ordering in L. Hence σ is a lattice automorphism of L. In particular, the least element A and the greatest element B of L are fixed under G, and hence are unions of orbits of G. This proves the corollary.

Dilworth was able to show that L is distributive. We give a short proof of this.

Theorem 2. *L is distributive.*

Proof. Following a suggestion of J.B. Nation, let D be the set of all antichains of P (including $\emptyset$) ordered by $I \leq J$ if for all $i \in I$, there is a $j \in J$ such that $i \leq j$ If $I, J \in D$, then the least upper bound in D of I and J is the set of maximal elements of $I \cup J$. Thus since D is finite and has a least element, D is a lattice. The inclusion map injects L into D. It is not difficult to show that this map is a lattice isomorphism and thus L can be thought of as a sublattice of D.

Now it is easy to see that D is a distributive lattice. In fact, D is isomorphic to the lattice of all order ideals of P. The isomorphism is obtained by mapping any antichain of P onto the order ideal generated by it. The inverse map maps an order ideal onto its maximal elements. It is well-known that the lattice of order ideals is distributive.

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