AN APPLICATION OF DILWORTH'S LATTICE OF MAXIMAL ANTICHAINS*

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Abstract. Recently there has been a good deal of interest in the maximal sized antichains of a partially ordered set \([1-8]\). A theorem of Dilworth states that under the natural ordering these antichains form a distributive lattice. This paper outlines a proof of this theorem and applies it to strengthen the results and shorten the proofs of [4].

An antichain in a finite partially ordered set \(P\) is a set of pairwise non-comparable elements of \(P\). Let \(L\) be the set of all maximal sized antichains ordered by \(A \leq B\) in \(L\), if for every element \(a\) in \(A\), there exists an element \(b\) in \(B\) such that \(a \leq b\) in \(P\). Then we have the following theorem of Dilworth [2].

**Theorem 1.** \(L\) is a lattice.

**Proof.** If \(A\) and \(B\) are elements of \(L\), define

\[
A_B = \{a \in A : a \geq b \text{ for some } b \in B\},
\]

(1)

\[
A^B = \{a \in A : a \leq b \text{ for some } b \in B\}.
\]

\(A \leq B\) in \(L\) is equivalent to \(A^B = A\) which is equivalent to \(B_A = B\). Now define

(2) \[
A \lor B = A_B \cup B_A, \quad A \land B = A^B \cup B^A.
\]

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It is easy to see that \( A \lor B \) and \( A \land B \) are the least upper bound and greatest lower bound of \( A \) and \( B \) in \( L \) provided that \( A \lor B \) and \( A \land B \) are in \( L \). That is, provided \( A \lor B \) and \( A \land B \) are maximal sized antichains. This is seen to be the case by observing that

\[
\begin{align*}
A_B \cup A^B &= A, & B_A \cup B^A &= B, \\
A_B \cap B_A &= A^B \cap B^A = A \cap B, \\
A_B \cap A^B &= B_A \cap B^A = A \cap B
\end{align*}
\]

Now since

\[
|A_B \cup B_A| + |A^B \cup B^A| = |A_B| + |B_A| - |A_B \cap B_A| + |A^B| + |B^A| - |A^B \cap B^A| = |A_B \cap A^B| + |A_B \cup A^B| + |B_A \cap B^A| + |B_A \cup B^A| - 2|A \cap B|,
\]

we have

\[
|A \lor B| + |A \land B| = |A| + |B|.
\]

Let \( n \) be the number of elements in a maximal sized antichain. Then \(|A| = |B| = n\). Since \( A \lor B \) and \( A \land B \) are clearly antichains, \(|A \lor B| \leq n\) and \(|A \land B| \leq n\). It now follows from the above equation that \(|A \lor B| = |A \land B| = n\) and hence \( A \lor B \) and \( A \land B \) are maximal sized antichains.

Since \( L \) is finite, Theorem 1 says in particular that \( L \) has a least and a greatest element. This fact has the following corollary, which is somewhat stronger than the theorem in [4]. Recall that automorphism of a finite partially ordered set \( P \) is an order-preserving permutation of the elements of \( P \).

**Corollary.** Let \( P \) be a finite partially ordered set and \( G \) a group of automorphisms of \( P \). Then there exist maximal sized antichains \( A \) and \( B \) such that \( A \) and \( B \) are each unions of orbits of \( G \) and if \( c \in C \), where \( C \) is a maximal sized antichain, there exists \( a \in A \) and \( b \in B \) such that \( a \leq c \leq b \).

**Proof.** If \( \sigma \in G \), then \( \sigma \) acts on \( L \) in the following natural way

\[
\sigma(A) = \{ \sigma(a) : a \in A \}, \quad A \in L.
\]
It is easily checked that $\sigma(A) \in L$ and that $\sigma$ preserves the ordering in $L$. Hence $\sigma$ is a lattice automorphism of $L$. In particular, the least element $A$ and the greatest element $B$ of $L$ are fixed under $G$, and hence are unions of orbits of $G$. This proves the corollary.

Dilworth was able to show that $L$ is distributive. We give a short proof of this.

**Theorem 2.** $L$ is distributive.

**Proof.** Following a suggestion of J.B. Nation, let $D$ be the set of all anti-chains of $P$ (including $\emptyset$) ordered by $I \leq J$ if for all $i \in I$, there is a $j \in J$ such that $i \leq j$. If $I, J \in D$, then the least upper bound in $D$ of $I$ and $J$ is the set of maximal elements of $I \cup J$. Thus since $D$ is finite and has a least element, $D$ is a lattice. The inclusion map injects $L$ into $D$. It is not difficult to show that this map is a lattice isomorphism and thus $L$ can be thought of as a sublattice of $D$.

Now it is easy to see that $D$ is a distributive lattice. In fact, $D$ is isomorphic to the lattice of all order ideals of $P$. The isomorphism is obtained by mapping any antichain of $P$ onto the order ideal generated by it. The inverse map maps an order ideal onto its maximal elements. It is well-known that the lattice of order ideals is distributive.

**References**