Chapter 2

Free and Finitely Presented Lattices

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This is an extended version of our chapter in the book *LatticeTheory: Special Topics and Applications*, vol 2, edited by George Grätzer and Friederich Wehrung. This version has one additional section, The generalized word problem and automorphisms. All of the numbering remains the same except this version has its own bibliography.

2-1. Introduction

Since free lattices are covered in Section 1-5 of LTF and in great detail in our book with Ježek [11], in this chapter we present the theory of finitely presented lattices including some new results, and then specialize to the case of free lattices. The authors wish to thank Alejandro Guillen for several helpful suggestions.

2-2. Preliminaries

Terms (or *lattice terms*) are defined in Section 4.1 of Chapter I of LTF. Recall that

$$x \vee y \vee z \qquad x \vee (y \vee z) \qquad (x \vee y) \vee z$$

are all terms in $\{x, y, z\}$. We define the *rank* of a term slightly differently from the definition given in LTF. A variable has rank 1, and if t_i is a term of rank r_i, then $t_1 \vee \cdots \vee t_k$ and $t_1 \wedge \cdots \wedge t_k$ both have rank $1 + r_1 + \cdots + r_k$. This gives preference to terms with unnecessary parentheses removed: the first term above has rank 4, while the other two have rank 5.

We define the *depth* of a term as the depth of the term tree; that is, variables have have depth 0, and if t_i has depth d_i then $t_1 \vee \cdots \vee t_k$ and $t_1 \wedge \cdots \wedge t_k$ both have depth $1 + \max\{d_1, \ldots, d_k\}$.

The set of subterms of a term is defined as usual: if t is a variable then $\{t\}$ is its set of subterms, and if $t = t_1 \vee \cdots \vee t_k$ or $t = t_1 \wedge \cdots \wedge t_k$ then the set of subterms is the union of $\{t\}$ and the subterms of t_i for $i = 1, \ldots, k$. Thus the subterms of the first term above are $\{x \vee y \vee z, x, y, z\}$. Note neither $x \vee y$ nor $y \vee z$ is a subterm. However, $y \vee z$ is a subterm of the middle term $x \vee (y \vee z)$.

2-2.1 Day's doubling construction

A useful construction for free lattice theory is Alan Day's doubling construction. We will use this construction in this chapter to derive one of the basic properties of free lattices, Whitman's condition, following Day's approach [2]. The doubling construction also plays a crucial role in the proof of Day's important result [1] that free lattices are weakly atomic.

Let L be a lattice. A subset C of L is *convex* if whenever a and b are in C and $a \le c \le b$, then $c \in C$. Of course an interval of a lattice is a convex set, as are lower and upper pseudo-intervals. A subset C of L is a *lower pseudo-interval* if it is a finite union of intervals, all with the same least element. An *upper pseudo-interval* is the dual concept.

Let C be a convex subset of a lattice L and let $L[C]$ be the disjoint union $(L - C) \cup (C \times 2)$. Order $L[C]$ by $x \le y$ if one of the following holds.

(i) x, $y \in L - C$ and $x \le y$ holds in L,

(ii) x, $y \in C \times 2$ and $x \le y$ holds in $C \times 2$,

(iii) $x \in L - C$, $y = (u, i) \in C \times 2$, and $x \le u$ holds in L,

(iv) $x = (v, i) \in C \times 2$, $y \in L - C$, and $v \le y$ holds in L.

There is a natural map λ from $L[C]$ back onto L given by

$$\lambda(x) = \begin{cases} x & \text{if } x \in L - C \\ v & \text{if } x = (v, i) \in C \times 2. \end{cases} \tag{2-2.1}$$

The next theorem shows that, under this order, $L[C]$ is a lattice.

Theorem 2-2.1. *Let C be a convex subset of a lattice L. Then $L[C]$ is a lattice and $\lambda\colon L[C] \to L$ is a lattice epimorphism.*

Proof. Routine calculations show that $L[C]$ is a partially ordered set. Let $x_i \in L - C$ for $i = 1, \ldots, n$ and let $(u_j, k_j) \in C \times 2$ for $j = 1, \ldots, m$. Let $v = \bigvee x_i \vee \bigvee u_j$ in L and let $k = \bigvee k_j$ in $\mathbf{2}$; if $m = 0$, then let $k = 0$. Then in $L[C]$,

$$x_1 \vee \ldots \vee x_n \vee (u_1, k_1) \vee \ldots \vee (u_m, k_m) = \begin{cases} v & \text{if } v \in L - C, \\ (v, k) & \text{if } v \in C. \end{cases} \tag{2-2.2}$$

To see this, let y be the right side of the above equation, i.e., let $y = v$ if $v \in L - C$ and $y = (v, k)$ if $v \in C$. It is easy to check that y is an upper bound for each x_i and each (u_j, k_j). Let z be another upper bound. First suppose $z = (a, r)$ where $a \in C$. Since z is an upper bound, it follows from the definition of the ordering that $a \geq v$ and $r \geq k$, and this implies $z \geq y$. Thus y is the least upper bound in this case. The case when $z \notin C$ is even easier. The formula for meets is of course dual. Thus $L[C]$ is a lattice, and it follows from equation (2-2.2) and its dual that λ is a homomorphism which is clearly onto L. □

H. Reppe [25], and T. Holmes and J. B. Nation [20], have shown that the doubling construction yields a lattice for subsets more general than convex sets and have characterized the finite lattices that can be obtained with this construction.

The next result follows easily from (2-2.2) and its dual.

Corollary 2-2.2. *Let L be a lattice generated by a set X, and let C be a convex subset of L with $X \cap C = \varnothing$. Let s be a term with variables in X whose evaluation in L is v. Then the evaluation of s in $L[C]$ is v if $v \notin C$, and either $(v, 0)$ or $(v, 1)$ otherwise.*

2-3. Finitely presented lattices and the word problem

Let X be a set (of variables). A lattice *relation* is a formal expression of the form $s \approx t$, where s and t are terms with variables from X. We also consider $s \leq t$ to be a relation, which in lattices is obviously equivalent to $s \approx s \wedge t$. A *presentation* is a pair (X, R) where X is a set and R is a set of relations with variables from X. We say that (X, R) is a *finite presentation* if both X and R are finite.

A lattice F is the *lattice finitely presented by* (X, R) if there is a map $\varphi\colon X \to F$ such that F is generated by $\varphi(X)$, F satisfies the relations R under the substitution $x \mapsto \varphi(x)$, for $x \in X$, and F satisfies the following mapping property: if L is a lattice and $\psi\colon X \to L$ is a map such that L satisfies R under the substitution $x \mapsto \psi(x)$, then there is a homomorphism $f\colon F \to L$ such that $f\varphi(x) = \psi(x)$ for all $x \in X$. Using the definition it is easy to see that the lattice finitely presented by (X, R) is unique up to isomorphism. This lattice is denoted $\mathrm{Free}(X, R)$.

The existence of Free(X, R) is easy to see: using the universal mapping property of the free lattice Free(X) one can easily verify that Free$(X)/\theta_R$, where θ_R is the congruence generated by R, is Free(X, R). The *word problem* for (X, R) is, given terms s and t with variables from X, to decide if the interpretations of s and t in Free(X, R) are equal. Equivalently, is $(s, t) \in \theta_R$? We shall see later that Free(X, R) is a subdirect product of finite lattices and this shows the word problem is decidable. The rough idea is to alternately try to prove $s \approx t$ and to try to find a finite lattice that witnesses that it is not true. This is the approach of McKinsey [23] and Evans [5, 6].

Recently, Stan Burris discovered that Thoralf Skolem in his 1920 paper [26] gave a very simple and efficient algorithm for deciding the word problem, which we present now.

2-3.1 Skolem's solution to the word problem

Skolem viewed a lattice as a relational structure with one binary relation denoted $\leq$ and two ternary relations $\vee$ and $\wedge$. In this perspective, $a \vee b = c$ would be written as $(a, b, c) \in \vee$. His axioms are what one would expect *except* he only required $\leq$ to be a quasiorder. In this regard $\vee$ is not a function: it is possible for both $(a, b, c) \in \vee$ and $(a, b, c') \in \vee$ even though $c \neq c'$. (However, as we shall see, this implies $c \leq c'$ and $c' \leq c$.) To formalize this we define a *relational quasilattice* as follows.

Definition 2-3.1. A set U with three relations $\leq$, $\vee$ and $\wedge$ of arities 2, 3 and 3, respectively, is a *relational quasilattice* if it satisfies the following axioms and the duals of (iii)–(vi):

(i) $\leq$ is reflexive.

(ii) $\leq$ is transitive.

(iii) If $(x, y, z) \in \wedge$, then (z, x) and (z, y) are in $\leq$.

(iv) If $(x, y, z) \in \wedge$ and both (u, x) and (u, y) are in $\leq$ then (u, z) is in $\leq$.

(v) If $(x, y, z) \in \wedge$ and (x, x'), (x', x), (y, y'), (y', y), (z, z'), (z', z) are all in $\leq$, then $(x', y', z') \in \wedge$.

(vi) For all x, y there is a z with $(x, y, z) \in \wedge$.

Clearly if L is a lattice then one obtains a relational quasilattice by interpreting the join and meet operations as relations. The easiest example of a relational quasilattice that is not a lattice is $U = \{a, b\}$ and $\leq\, = U^2$ and $\vee = \wedge = U^3$.

It is easier to give Skolem's algorithm for deciding universal Horn sentences. This is slightly more general in that it gives a solution to the uniform word problem. Let

$$s_1 \approx t_1 \;\&\; \cdots \;\&\; s_k \approx t_k \longrightarrow s \approx t \tag{2-3.1}$$

be a Horn formula. The problem is to decide if this is true in every lattice under every substitution of the variables. The algorithm creates a relational structure $(U, \leq, \vee, \wedge)$, where U is the set of all subterms of the terms of s, s_i, t and t_i, $i = 1, \ldots, k$. All three relations are initially empty.

A minor technical problem arises, which can be resolved in various ways. The simplest is to write the Horn formula (2-3.1) regarding $\wedge$ and $\vee$ as binary operations. For example, we would write $(x \vee y) \vee z$ rather than $x \vee y \vee z$. In that way, our definition of *subterm* agrees with Skolem's. (Alternatively, one could use a different definition of subterm, or modify the definition of *relational quasilattice* and the algorithm of Theorem 2-3.2 to allow $(x_1, \ldots, x_k, z) \in \wedge$ and $\vee$.)

Theorem 2-3.2 (Skolem's Algorithm)**.** *The following polynomial time algorithm decides if the implication* (2-3.1) *is valid. Let U be the set of all subterms of the terms s, s_i, t and t_i, $i = 1, \ldots, k$.*

(a) *For $i = 1, \ldots, k$ add (s_i, t_i) and (t_i, s_i) to $\leq$.*

(b) *If $r = r_1 \vee r_2$ is a subterm, add (r_1, r_2, r) to $\vee$.*

(c) *If $r = r_1 \wedge r_2$ is a subterm, add (r_1, r_2, r) to $\wedge$.*

(d) *Close $(U, \leq, \vee, \wedge)$ under axioms* (i)–(iv) *and the duals of* (iii) *and* (iv).

(e) *If both (s, t) and (t, s) are in $\leq$ then* (2-3.1) *is true; otherwise not.*

Proof. Suppose the closure (d) shows that both (s, t) and (t, s) are in $\leq$. Then this derivation constitutes a proof that (2-3.1) is true.

For the other direction suppose either (s, t) or (t, s) is not in $\leq$ after completing part (d). Note that $\leq$ is a quasiorder by axioms (i) and (ii). Moreover, axiom (v) and its dual hold for $(U, \leq, \vee, \wedge)$. Let $\equiv$ be the equivalence relation associated with this quasiorder: $u \equiv v$ if (u, v) and (v, u) are in $\leq$. Let $\overline{u}$ be the equivalence class of u and $\overline{U} = \{\overline{u} : u \in U\}$. Of course $\overline{U}$ is an ordered set. Also note that if $(a, b, c) \in \vee$ then by axioms (iii) and (iv), $\overline{c}$ is the least upper bound of $\overline{a}$ and $\overline{b}$ under the order of $\overline{U}$.

Definition 2-3.3. A *partially defined lattice* is a partially ordered set $(P, \leq)$ together with two partial functions, $\bigvee$ and $\bigwedge$, from subsets of P into P such that if $p = \bigvee S$ then p is the least upper bound of S in $(P, \leq)$, and dually. We use $(P, \leq, \bigvee, \bigwedge)$ to denote this structure.

By the above remarks, $(\overline{U}, \leq, \bigvee, \bigwedge)$ is a partially defined lattice where $\bigvee$ is given by the rule: for each $(a, b, c) \in \vee$ we have $\bigvee\{\overline{a}, \overline{b}\} = \overline{c}$, and dually for $\bigwedge$. As we show in the proof of Dean's Theorem below, partially defined lattices are embedded into their ideal lattice, $\mathrm{Idl}_0(\overline{U}, \leq, \bigvee, \bigwedge)$. If $x_1, \ldots, x_n$ are the variables occurring in the terms of (2-3.1), we assign the variable x_i to $\mathrm{id}(\overline{x}_i)$ (the principal ideal generated by $\overline{x}_i$). A straightforward inductive argument shows that if $r \in U$ (so r is a subterm) then the evaluation in $\mathrm{Idl}_0(\overline{U}, \leq, \bigvee, \bigwedge)$ of r under the substitution $x_i \mapsto \mathrm{id}(\overline{x}_i)$ is $\mathrm{id}(\overline{r})$. Thus we can obtain axiom (vi) and its dual by embedding $(\overline{U}, \leq, \bigvee, \bigwedge)$ into its ideal lattice. Since $\overline{s}_i = \overline{t}_i$, $i = 1, \ldots, k$ and $\overline{s} \neq \overline{t}$, we see that this substitution into $\mathrm{Idl}_0(\overline{U}, \leq, \bigvee, \bigwedge)$ witnesses the failure of (2-3.1).

After initialization U never increases in size, so the relations are bounded in size by $|U|^3$. Since $|U|$ is the input size of the problem, it is not hard to see the algorithm is polynomial time. We leave the details to the reader. □

Some observations: first the connection between partially defined lattices, weak partial lattices and partial lattices. The latter two are defined in LTF. *Partially defined lattice* is the weakest of these notions, but all have an underlying (partial) order; see Lemma 80 of LTF. In [9, 11] partially defined lattices are just called partial lattices. Also the defined joins and meets in a partially defined lattice are not restricted to be binary. So, for example, $d = a \vee b \vee c$ is allowed (assuming d is the least upper bound in P, of course), while $a \vee b$ may not be defined and may not even exist in P.

The second observation is that given a finite presentation (X, R) we can form a partially defined lattice as in the proof of Theorem 2-3.2. First set U to be the union of X and all subterms of the terms occurring in R and perform steps (a)–(d). Then form $(P, \leq, \bigvee, \bigwedge)$ where $P = U/\equiv$. In this way $(P, \leq, \bigvee, \bigwedge)$ can be viewed as a finite presentation. Moreover, as the reader can show, $\mathrm{Free}(X, R) \cong \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. Consequently, in our study of finitely presented lattices we will study $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$.

2-3.2 Dean's Theorem

An *ideal* I in a partially defined lattice $(P, \leq, \bigvee, \bigwedge)$ is a subset of P such that if $a \in I$ and $b \leq a$ then $b \in I$, and if $a_1, \ldots, a_k$ are in I and $a = \bigvee a_i$ is defined then $a \in I$. It is worth pointing out that these two rules may have to be applied repeatedly to find the ideal generated by a set. The set of all ideals of $(P, \leq, \bigvee, \bigwedge)$ including the empty ideal forms a lattice denoted $\mathrm{Idl}_0(P, \leq, \bigvee, \bigwedge)$ or just $\mathrm{Idl}_0(P)$. The map $p \mapsto \mathrm{id}(p)$ embeds P into $\mathrm{Idl}_0(P)$, preserving the order (and its negation) and all the defined joins and meets. This is easy to see: if $a < b$ in P then $\mathrm{id}(a) \subsetneq \mathrm{id}(b)$, and if $a = a_1 \vee \cdots \vee a_k$ is a defined join then a is in the ideal I generated by the union of the $\mathrm{id}(a_i)$'s, whence it follows that $I = \mathrm{id}(a)$. If b is the greatest lower bound in $(P, \leq)$ of $\{a_1, \ldots, a_k\}$ then $\mathrm{id}(b) = \mathrm{id}(a_1) \cap \cdots \cap \mathrm{id}(a_k)$, so the meet relations are certainly preserved. Hence the map $p \mapsto \mathrm{id}(p)$ extends to a map

$$\psi\colon \mathrm{Free}(P, \leq, \bigvee, \bigwedge) \to \mathrm{Idl}_0(P, \leq, \bigvee, \bigwedge),$$

and this shows in particular that $(P, \leq)$ is embedded in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$.

If $w \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ we let

$$\underline{w} = \mathrm{id}_P(w) = \{a \in P : a \leq w\},$$

the ideal of P below w. Define $\overline{w}$, the filter above w, dually. If $w_1, \dots, w_k \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ let $\mathrm{id}_P(w_1, \dots, w_k)$ be the ideal of $(P, \leq, \bigvee, \bigwedge)$ generated by $\underline{w_1} \cup \cdots \cup \underline{w_k}$ which of course is the ideal $\underline{w_1} \vee \cdots \vee \underline{w_k}$. The filter $\mathrm{fil}_P(w_1, \dots, w_k)$ is defined dually. One can show by induction on the rank of w that, for the map ψ above,

$$\psi(w) = \mathrm{id}_P(w) = \underline{w}.$$

The next theorem is Dean's solution to the word problem [3].

Theorem 2-3.4. *Let s and t be terms with variables in P. Then $s \leq t$ holds in* $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ *if and only if one of the following holds:*

(i) *$s \in P$ and $t \in P$ and $s \leq t$ in $(P, \leq)$;*

(ii) *$s = s_1 \vee \cdots \vee s_k$ and $\forall\, i\ s_i \leq t$;*

(iii) *$t = t_1 \wedge \cdots \wedge t_k$ and $\forall\, j\ s \leq t_j$;*

(iv) *$s \in P$ and $t = t_1 \vee \cdots \vee t_k$ and $s \in \mathrm{id}_P(\{t_1, \dots, t_k\})$;*

(v) *$s = s_1 \wedge \cdots \wedge s_k$ and $t \in P$ and $t \in \mathrm{fil_P}(\{s_1, \dots, s_k\})$;*

(vi) *$s = s_1 \wedge \cdots \wedge s_k$ and $t = t_1 \vee \cdots \vee t_m$ and $\exists\, i\ s_i \leq t$ or $\exists\, j\ s \leq t_j$ or $\exists\, a \in P\ s \leq a \leq t$.*

Proof. Since $(P, \leq)$ is embedded in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$, if s and t are in P, then $s \leq t$ holds in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ if and only if it holds in $(P, \leq)$. A straightforward inductive argument shows that if (iv) or (v) holds then $s \leq t$ holds in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. Clearly if (ii), (iii) or (vi) holds then $s \leq t$ holds in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. Thus any of (i) to (vi) implies $s \leq t$.

For the converse suppose $s \leq t$ holds in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. Suppose $s \in P$ and $t = t_1 \vee \cdots \vee t_k$. Using the homomorphism ψ above

$$\begin{aligned} \mathrm{id}_P(s) \leq \mathrm{id}_P(t) &= \mathrm{id}_P(t_1 \vee \cdots \vee t_k) \\ &= \mathrm{id}_P(t_1) \vee \cdots \vee \mathrm{id}_P(t_k) \\ &= \mathrm{id}_P(\{t_1, \dots, t_k\}), \end{aligned}$$

and so $s \in \mathrm{id}_P(\{t_1, \dots, t_k\})$, as desired.

Now suppose $s \leq t$ and $s = s_1 \wedge \cdots \wedge s_k$ and $t = t_1 \vee \cdots \vee t_m$ and that there is no i with $s_i \leq t$, no j with $s \leq t_j$ and no $a \in P$ with $s \leq a \leq t$. Let C be the interval $[s, t]$ and let $F_P[C]$ be the lattice with C doubled, where $F_P = \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. Since $P \cap C = \varnothing$, $(P, \leq)$ is embedded in $F_P[C]$, and by (2-2.2) the image satisfies the join and meet relations. Hence there is a homomorphism $\varphi\colon F_P \twoheadrightarrow F_P[C]$. Let v be the interpretation of s in F_P and let u be the interpretation of t. By Corollary 2-2.2 the interpretation of s in $F_P[C]$ is either $(v, 0)$ or $(v, 1)$. But $s = s_1 \wedge \cdots \wedge s_k$ and each s_i is not in C and thus above $(v, 1)$. It follows that the interpretation of s in $F_P[C]$ is

$(v,1)$. Similarly, the interpretation of t in $F_P[C]$ is $(u,0)$. So $\varphi(s) = (v,1)$ and $\varphi(s) = (u,0)$. But $(v,1) \not\leq (u,0)$, which is a contradiction since $s \leq t$.

The remaining cases are straightforward. □

If there are no defined joins then $\mathrm{id}_P(\{t_1,\ldots,t_k\})$ is simply the set of elements below one of the t_i's. Hence condition (iv) of Dean's Theorem can be simplified to saying that if $s \in P$ and $t = t_1 \vee \cdots \vee t_k$, then $s \leq t$ if and only if $s \leq t_i$ for some i. In other words the elements of P are join prime. Also note that in this case condition (vi) simplifies to

(W) $s = s_1 \wedge \cdots \wedge s_k$ and $t = t_1 \vee \cdots \vee t_m$ implies $\exists\, i\ s_i \leq t$ or $\exists\, j\ s \leq t_j$

This is known as *Whitman's condition*.

If no joins and no meets are defined and the order on P is an antichain, then Dean's solution reduces to Whitman's solution to the word problem for free lattices.

Definition 2-3.5. If S and T are subsets of a lattice, we say that S *refines* T (or S *lower refines* T) if for all $s \in S$ there is a $t \in T$ with $s \leq t$. We denote this by $S \ll T$. The relation $S \gg T$ is defined dually and we say that S *upper refines* T in this case.

Lemma 2-3.6. *If $x \in P$ and $x \leq t_1 \vee \ldots \vee t_n$ in* $\mathrm{Free}(P,\leq,\bigvee,\bigwedge)$ *then there is a set $Y \subseteq P$ such that $Y \ll \{t_1,\ldots,t_n\}$ and $x \leq \bigvee Y$ in* $\mathrm{Free}(P,\leq,\bigvee,\bigwedge)$.

Proof. By (iv) of the Dean's Theorem, the hypotheses imply that x is in the ideal of $(P,\leq,\bigvee,\bigwedge)$ generated by $Y = \{y \in P : y \leq t_i \text{ for some } i\}$. Clearly $Y \ll \{t_1,\ldots,t_n\}$. The join of Y may not be defined in $(P,\leq,\bigvee,\bigwedge)$, but it is easy to see that every element of the ideal of $(P,\leq,\bigvee,\bigwedge)$ generated by Y, $\mathrm{id}_P(Y)$, is below $\bigvee Y$ in $\mathrm{Free}(P,\leq,\bigvee,\bigwedge)$, and hence $x \leq \bigvee Y$. □

2-4. Canonical form

Each element in a free lattice has a shortest term representing it, which is unique up to commutativity and associativity. This is called the *canonical form* of the element. This syntactical concept is closely related to the arithmetic of the free lattice. We will see that the elements of $\mathrm{Free}(P,\leq,\bigvee,\bigwedge)$ also have a canonical form and that there is a nice connection between this form and the arithmetic of the finitely presented lattice. A related but different canonical form is considered in Grätzer, Huhn and Lakser [19]. Our canonical form has the nice property that when applied to free lattices, it agrees with Whitman's.

As we mentioned above, the major difference between Dean's algorithm and Whitman's lies in conditions (iv), (v) and (vi). However if we are dealing with a certain kind of term, which we will call *adequate*, these difficult conditions can be replaced with the simple free lattice conditions.

Definition 2-4.1. Let $(P, \leq, \bigvee, \bigwedge)$ be a finite partially defined lattice. A term t with variables from P is called *adequate* if it is an element of P, or if $t = t_1 \vee \cdots \vee t_n$ is a formal join, each t_i is adequate, and if $p \leq t$ for $p \in P$ then $p \leq t_i$ for some i. If t is formally a meet the dual condition must hold.

Lemma 2-4.2. *Let s and t be adequate terms. Then $s \leq t$ in* $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ *if and only if $s \leq t$ in* $\mathrm{Free}(P, \leq)$.

Here $(P, \leq)$ denotes P as a partially ordered set, with no nontrivial joins and meets defined. This lemma follows from Theorem 2-3.4 and the remarks which follow it.

An easy inductive argument shows that for every element w of the lattice $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ there is an adequate term representing w. Also every term is adequate in the case of free lattices. The next theorem will show that there is a shortest adequate term representing w, and that this term is unique up to commutativity. We call such a term the *canonical form* of w.

Theorem 2-4.3. *For each element of* $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ *there is an adequate term of minimal rank representing it, and this term is unique up to commutativity.*

Proof. Suppose that s and t are both shortest adequate terms that represent the same element w in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. If either s or t is in P, then clearly $s = t$.

Observe that if $t = t_1 \vee \cdots \vee t_n$ and some t_i is formally a join, we could lower the rank of t by removing the parentheses around t_i. Since t_i is adequate, the resulting term would still adequately represent w. But this would violate the minimally of t. Thus we conclude that each t_i is not formally a join.

Suppose that $t = t_1 \vee \cdots \vee t_n$ and $s = s_1 \vee \cdots \vee s_m$. Then $t_i \leq s_1 \vee \cdots \vee s_m$. This implies that either $t_i \leq s_j$ for some j, or $t_i = \bigwedge t_{ij}$ and $t_{ij} \leq s$ for some j, or there is an $x \in P$ with $t_i \leq x \leq s_1 \vee \cdots \vee s_m$. In the second case we have $t_i \leq t_{ij} \leq t$, and replacing t_i by t_{ij} in t produces a shorter term still representing w. It is easy to see that this term is still adequate, violating the minimality of the term t. If the third case holds then, by the adequacy of s, $x \leq s_j$ for some j. Hence in all cases there is a j such that $t_i \leq s_j$. Thus $\{t_1, \ldots, t_n\} \ll \{s_1, \ldots, s_m\}$. By symmetry, $\{s_1, \ldots, s_n\} \ll \{t_1, \ldots, t_m\}$. Since both are antichains (by the minimality) they represent the same set of elements of $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. Thus $m = n$ and after renumbering $s_i \approx t_i$. Now by induction s_i and t_i are the same up to commutativity.

If $t = t_1 \vee \cdots \vee t_n$ and $s = s_1 \wedge \cdots \wedge s_m$, then, since neither s nor t is in P, (W) implies that either $t_i = t$ for some i or $s_j = s$ for some j, violating the minimality.

The remaining cases can be handled by duality. □

Examining the proof of this theorem we see that an adequate term $t = t_1 \vee \cdots \vee t_n$ is a minimal adequate term if every proper subterm is a minimal adequate term, the t_i's form an antichain, and if $t_i = \bigwedge_j t_{ij}$, then $t_{ij} \not\leq t$ for every j. The next theorem is an easy consequence.

Theorem 2-4.4. *If a term $t = t_1 \vee \cdots \vee t_n$ with $n > 1$ is in canonical form then following conditions hold.*

(a) *The t_i's form an antichain.*

(b) *If $t_i = \bigwedge_{j=1}^{m} t_{ij}$ with $m > 1$, then $t_{ij} \not\leq t$ for all j.*

In free lattices the canonical form is associated with nonrefinable join representations, which in free lattices are unique. The next theorem will show that in a finitely presented lattice each element can have only finitely many nonrefinable join representations, and these can be easily found from the canonical form. We define the *canonical join representation* of $w \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ to be $w_1 \vee \cdots \vee w_m$ if the canonical form of w is $t_1 \vee \cdots \vee t_m$ and the interpretation of t_i in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ is w_i. It is useful to separate out the elements of P in such a representation. Thus let

$$
\begin{aligned}
\text{(2-4.1)} \qquad w &= w_1 \vee \cdots \vee w_n \vee x_1 \vee \cdots \vee x_k \\
&= \bigvee \bigwedge w_{ij} \vee \bigvee x_i
\end{aligned}
$$

be the canonical join representation of w where $x_i \in P$, $i = 1, \ldots, k$, and the canonical meet representation of w_i is $w_i = \bigwedge w_{ij}$.

Definition 2-4.5. A finite subset U of a lattice is said to be a *nonrefinable join representation* of an element w if $w = \bigvee U$, and whenever $w = \bigvee V$ for a finite subset V with $V \ll U$, then $U \subseteq V$.

Note that if U is a nonrefinable join representation of w then U is an antichain.

Theorem 2-4.6. *Let the canonical join representation for w be given by* (2-4.1). *Every join representation of w can be refined to a nonrefinable join representation of w. If $w = v_1 \vee \cdots \vee v_m$ in* $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ *then there exist $y_1, \ldots y_r \in P$ such that*

$$w = w_1 \vee \cdots \vee w_n \vee y_1 \vee \cdots \vee y_r$$

and

$$\{w_1, \ldots, w_n, y_1, \ldots, y_r\} \ll \{v_1, \ldots, v_m\}.$$

Every nonrefinable join representation of w contains $\{w_1, \ldots, w_n\}$ and also contains every x_i which is join irreducible.

Note that an element $x \in P$ is join irreducible in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ except when some $(z_1, \ldots, z_\ell, x) \in \bigvee$ is among the defining relations of $(P, \leq, \bigvee, \bigwedge)$ and $x \neq z_i$, $i = 1, \cdots, \ell$.

Proof. Assume $w = v_1 \vee \cdots \vee v_m$. Since, for $i = 1, \ldots, n$,

$$w_i \leq v_1 \vee \cdots \vee v_m = w$$

we have that either (i) $w_i \leq v_j$ for some j, (ii) $w_{ij} \leq w$, or (iii) $w_i \leq x \leq w$ for some $x \in P$. If either (ii) or (iii) held, we could produce a shorter adequate term representing w, violating the minimality of the representation $w = w_1 \vee \cdots \vee w_n \vee x_1 \vee \cdots \vee x_k$. Hence (i) must hold.

Since $x_i \leq v_1 \vee \cdots \vee v_m$, by Lemma 2-3.6 there is a set $\{z_1, \ldots, z_s\} \subseteq P$ such that $x_i \leq z_1 \vee \cdots \vee z_s$ in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ and

$$\{z_1, \ldots, z_s\} \ll \{v_1, \ldots, v_m\}.$$

Hence if we let $\{y_1, \ldots y_r\}$ be the union of the z's obtained from all of the x_i's,

$$w = w_1 \vee \cdots \vee w_n \vee y_1 \vee \cdots \vee y_r$$

and

$$\{w_1, \ldots, w_n, y_1, \ldots, y_r\} \ll \{v_1, \ldots, v_m\}.$$

This proves the first part of the theorem and also shows that every nonrefinable join representation of w must be a subset of $\{w_1, \ldots, w_n, y_1, \ldots, y_r\}$ for some $y_1, \ldots, y_r$ in P. The argument at the beginning of this proof shows that no w_i can be omitted from this subset and hence every nonrefinable join representation of w has the form $\{w_1, \ldots, w_n, y_1, \ldots, y_r\}$ for some $y_1, \ldots, y_r$ in P.

This proves everything except the statement about the join irreducible x_i's. First we claim that *each* x_i *in* (2-4.1) *is a maximal element of* $\mathrm{id}_P(w)$. If $x_i < y \leq w$ then we could replace x_i by y in (2-4.1). The resulting expression would still correspond to an adequate term, in violation of the uniqueness of the canonical form. Assume $\{v_1, \ldots, v_m\}$ is a nonrefinable join representation of w. By Theorem 2-3.4, $x_i \leq v_1 \vee \cdots \vee v_m$ means that x_i is in the ideal of P generated by $\bigcup_j \mathrm{id}_P(v_j)$. This ideal is obtained from this union by alternately taking joins of subsets of this union that are defined in $(P, \leq, \bigvee, \bigwedge)$ and adding all elements less than something in the set. Obviously all such elements will be less than or equal to w. But since x_i is a maximal element in P below w, the only way for a join of elements of P below w to contain (be greater than or equal to) x_i is for it to equal x_i. Thus, in the case that x_i is join irreducible, we must have $x_i \leq v_j$ for some j. We have shown that $\{v_1, \ldots, v_m\} = \{w_1, \ldots, w_n, y_1, \ldots, y_r\}$ for some y_j's. Since $x_i \leq w_k$ would violate the canonical form (2-4.1) of w, we must have $x_i \leq y_j$ for some j. But the maximality of x_i implies $x_i = y_j$, proving the last statement. □

Notice that this proof shows that *every nonrefinable join representation of* w *refines the canonical join representation.*

Recall that a lattice is *join-semidistributive* if for all a, b and c

$$a \vee b = a \vee c \quad \text{implies} \quad a \vee b = a \vee (b \wedge c).$$

This implication is denoted $\mathrm{SD}_\vee$. A lattice is *semidistributive* if satisfies $\mathrm{SD}_\vee$ and its dual $\mathrm{SD}_\wedge$.

2-4.1 Exercises

2.1. Let $(P, \leq)$ be a countable partially ordered set. Let $f\colon \mathrm{Free}(P) \twoheadrightarrow \mathrm{Free}(P, \leq)$ be the homomorphism from the free lattice to the free lattice over P induced by the identity map on P. Show there is a map $g\colon \mathrm{Free}(P, \leq) \rightarrowtail \mathrm{Free}(P)$ such that $f(g(x)) = x$. In particular $\mathrm{Free}(P, \leq)$ (and $(P, \leq)$ are embedded into $\mathrm{Free}(P)$.

2.2. Show there are no uncountable chains in free lattices; see [16]. Also show that the partially ordered set of of atoms and coatoms of the lattice of subsets of an uncountable set cannot be embedded into a free lattice; see [13, 24].

2.3. Use canonical form to show that free lattices are semidistributive.

2.4. Show that a finitely presented lattice is join-semidistributive if and only if every element has a unique nonrefinable join representation; see [10].

2.5. Show that every finite semidistributive lattice has a homomorphism onto the two-element lattice.

In the following exercises a partially defined lattice $(P, \leq, \bigvee, \bigwedge)$ is given. You should decide if $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ is finite. If it is finite you should draw it, and if it is infinite you should give infinitely many elements. To do this give infinitely many terms all in canonical form.

2.6. $P = \{a, b, c\}$ with $b \leq c$ and no defined joins or meets.

2.7. $P = \{a, b, c, d\}$ with $b \leq c \leq d$ and no defined joins or meets.

2.8. $P = \{a, b, c, d, e\}$ with $b \leq c \leq d \leq e$ and no defined joins or meets.

2.9. $P = \{a, b, c, d, e, 0, 1\}$ with order given in Figure 2-4.1 and defined joins $1 = a + e = b + d$ and $c = d + e$, and defined meet $0 = abc$.

2-5. A structure theorem of Grätzer, Huhn, and Lakser

In this section we prove the following theorem of G. Grätzer, H. Lakser and A. Huhn [19]. Our proof, though different in detail, is very much in the spirit of the original.

Theorem 2-5.1. *Every finitely presented lattice is a disjoint union of finitely many convex sublattices, each of which can be embedded into a free lattice.*

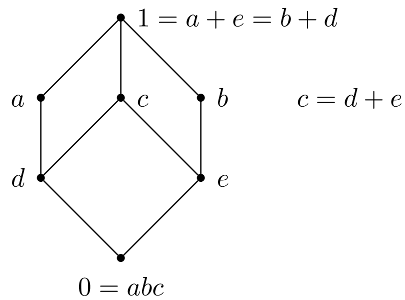

Figure 2-4.1: The diagram P

Let $(P, \leq, \bigvee, \bigwedge)$ be a partially defined lattice and consider the following algebras:

- $\mathrm{Term}(P)$, the (completely free) algebra of lattice terms generated by P,
- $\mathrm{Free}(P, \leq) = \mathrm{Free}(P, \leq, \varnothing, \varnothing)$, the free lattice generated by the ordered set $(P, \leq)$,
- $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$, a finitely presented lattice,
- the ideal lattice $L_0 = \mathrm{Idl}_0(P, \leq, \bigvee, \bigwedge)$,
- the filter lattice $L_1 = \mathrm{Fil}_1(P, \leq, \bigvee, \bigwedge)$,
- the direct product $L_0 \times L_1$.

Recall that for a finite (or even countable) ordered set P, the free lattice over the ordered set $\mathrm{Free}(P, \leq)$ embeds into the free lattice $\mathrm{Free}(P)$. In this proof, it will be more convenient to use the former.

For lattice terms in P, the expresion *canonical form* refers to the finitely presented lattice canonical form as presented in the previous section, while *free lattice canonical form* designates the canonical form for $\mathrm{Free}(P, \leq)$.

Lemma 2-5.2. *If $t \in \mathrm{Term}(P)$ is in canonical form, then it is in free lattice canonical form.*

Proof. This follows from an easily from induction using Theorem 2-4.4. $\square$

There are natural homomorphisms

$$\mathrm{Term}(P) \xrightarrow{\varphi} \mathrm{Free}(P, \leq) \xrightarrow{\pi} \mathrm{Free}(P, \leq, \bigvee, \bigwedge) \xrightarrow{h} L_0 \times L_1 \ .$$

Moreover, there is the canonical form map in the other direction

$$\mathrm{Term}(P) \underset{\gamma}{\longleftarrow} \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$$

which takes an element a in the finitely presented lattice to its canonical form, i.e., $\gamma(a)$ is the adequate term of minimal rank such that $\pi\varphi\gamma(a) = a$. Then, by Lemma 2-5.2, $\varphi\gamma(a)$ is just the evaluation of $\gamma(a)$ in the free lattice $\mathrm{Free}(P, \leq)$.

To emphasize the distinction between canonical form and canonical form for free lattice, for terms s and t we write $s \leq_1 t$ if $\varphi(s) \leq \varphi(t)$ in $\mathrm{Free}(P, \leq)$, and $s \leq_2 t$ if $\pi\varphi(s) \leq \pi\varphi(t)$ in $\mathrm{Free}(P, \leq, \bigwedge, \bigvee)$. Clearly $\leq_1 \subseteq \leq_2$, i.e., $s \leq_1 t$ implies $s \leq_2 t$.

Theorem 2-5.3. *Let K be a nonempty class of* $\ker h$. *Then $\varphi\gamma|_K$ is an embedding of K into* $\mathrm{Free}(P, \leq)$.

We need to show that $\varphi\gamma|_K$ is a homomorphism; it will then be an embedding since $\pi\varphi\gamma = id$. Thus each such K is embedded into $\mathrm{Free}(P, \leq)$ which in turn is embedded into a free lattice by Exercise 2.1.

Lemma 2-5.4. *Fix an ideal $I_0 \in L_0$ and a filter $F_0 \in L_1$. Let $K = h^{-1}(I_0, F_0)$. Let $t = t_1 \vee \ldots \vee t_n$ with $n > 1$ be a term in canonical form. Then $\pi\varphi(t)$ is in K if and only if for x, $y \in P$,*

(i) *$x \in I_0$ implies $x \leq_2 t_i$ for some i,*

(ii) *$y \in F_0$ implies $y \geq_2 t_i$ for all i.*

Proof. Part (i) is the definition of adequate, while (ii) holds in all lattices. □

Theorem 2-5.5. *Let $s = s_1 \vee \cdots \vee s_m$ and $t = t_1 \vee \cdots \vee t_n$ be terms in canonical form such that $h\pi\varphi(s) = h\pi\varphi(t)$. (We allow $m = 1$ and/or $n = 1$.) Then the canonical form of $s \vee t$ is the free lattice canonical form of $s \vee t$.*

In other words, if the c_i all come from K, then the canonical form of their join in $\mathrm{Free}(P, \leq, \bigwedge, \bigvee)$ is the free lattice canonical form of the join of their canonical forms in $\mathrm{Free}(P, \leq)$.

Proof. Say $h\pi\varphi(s) = h\pi\varphi(t) = (I_0, F_0)$. Then $h\pi\varphi(s \vee t) = (I_0, F_0)$ as well. Moreover, since t is adequate, each $x \in I_0$ satisfies $x \leq_2 t_i$ for some i and similarly for s. Hence the term $s \vee t$ is adequate. We show that the procedure to put $s \vee t$ into canonical form results in the same term as putting it into free lattice canonical form.

One of the steps in putting $s \vee t$ into canonical form is to replace

$$\{s_1, \ldots, s_m, t_1, \ldots, t_n\} \cup \mathrm{id}_P(s \vee t)$$

with its maximal elements under $\leq_2$. But $\mathrm{id}_P(s \vee t) = I_0$. Since s and t are adequate, the maximal elements of $\{s_1, \ldots, s_m, t_1, \ldots, t_n\} \cup I_0$ are just the maximal elements of $\{s_1, \ldots, s_m, t_1, \ldots, t_n\}$ by Lemma 2-5.4. For free lattice canonical form of $s \vee t$ we want to replace $\{s_1, \ldots, s_m, t_1, \ldots, t_n\}$ with its maximal elements under $\leq_1$. But since the s_i's and the t_j's are adequate, we have, by Lemma 2-4.2, that $s_i \leq_1 t_j$ if and only if $s_i \leq_2 t_j$, and vice versa. Thus the result of this step is the same for canonical and free lattice canonical form.

Let u be the formal join of the maximal elements from the previous step. Then u may not be in canonical form, but it is adequate, and $\pi\varphi(s \vee t) = \pi\varphi(u)$. Hence we have $t_{ij} \leq_2 u$ if and only if $t_{ij} \leq_1 u$. So, if $t_{ij} \leq_2 u$, the step that replaces t_i by t_{ij} gives the same result in both kinds of canonical form. □

Of course, the dual result holds for meets, and together these prove Theorem 2-5.3, which in turn proves Theorem 2-5.1.

2-6. Covers

In [15] the authors gave an effective proceedure for determining if an element of a free lattice had a lower cover, and finding it if it did. Here we prove a similar result for finitely presented lattices.

Definition 2-6.1. A *join cover* of $x \in L$, where L is a (partially defined) lattice, is a finite set $S \subseteq L$ such that $x \leq \bigvee S$. We use the term *meet cover* for the dual notion, although this terminology is less than ideal. A subset S of the join irreducible elements of L is said to be *closed* if for every $u \in S$, every join cover of u can be refined to a join cover of u consisting of elements of S. We call a join cover S of x a *nonrefinable join cover* of x if whenever T is a join cover of x and $T \ll S$ then $S \subseteq T$. A join cover S of x is *nontrivial* if there is no $s \in S$ with $x \leq s$.

Lemma 2-6.2. *Every join irreducible element* $w \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ *is contained in a finite closed set.*

Proof. Define a set $\mathrm{J}'(w)$ of join irreducibles associated with w as follows. If $w \in P$ then $\mathrm{J}'(w) = \varnothing$. If the canonical meet representation of w is given by

$$w = \bigwedge \bigvee w_{ij} \wedge \bigwedge x_k \tag{2-6.1}$$

then we let

$$\mathrm{J}'(w) = \{w\} \cup \bigcup_{ij} \mathrm{J}'(w_{ij}).$$

We claim that $\mathrm{T}(w) = P' \cup \mathrm{J}'(w)$ is closed, where P' consists of those elements of P that are join irreducible in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. If $w \in P'$ then this follows from Lemma 2-3.6. Clearly if $v \in \mathrm{T}(w)$ then $\mathrm{T}(v) \subseteq \mathrm{T}(w)$. Hence it suffices to show that any join cover of w refines to one in $\mathrm{T}(w)$. Suppose $w \leq \bigvee U$. Since $w \in \mathrm{T}(w)$, the claim is obvious if $w \leq u$ for some $u \in U$. Otherwise, $w_i \leq \bigvee U$ for some i, or $x_j \leq \bigvee U$ for some j. In the latter case the claim follows from Lemma 2-3.6 again. In the former case $w_{ij} \leq \bigvee U$ for all j. Since each $\mathrm{T}(w_{ij}) \subseteq \mathrm{T}(w)$, an inductive argument shows that there is a refinement of U to a join cover V_{ij} of w_{ij}. If $V = \bigcup_j V_{ij}$, then V is a join cover of w_i, and hence w, refining U. □

Lemma 2-6.3. *The intersection of a finite closed set with an arbitrary closed set is closed.*

Proof. Suppose that both S and T are closed sets with S finite and let $w \in S \cap T$. If $w \leq \bigvee U$ then there is a join cover $V \subseteq S$ refining U. Moreover, since S is finite, we may assume that if $V' \subseteq S$ is a join cover refining V, then $V \subseteq V'$. Since T is also closed, it has a subset V_1 with $V_1 \ll V$ which is a join cover of w. But since S is closed, it has a subset $V_2 \ll V_1$ which is a join cover of w. By the choice of V, we have

$$V \subseteq V_2 \ll V_1 \ll V.$$

This implies $V = V_1$. Thus $V \subseteq S \cap T$, showing that $S \cap T$ is closed. □

Using the last two lemmas we can show that if w is a join irreducible element of $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ then there is a unique smallest closed set containing w, which is denoted $\mathrm{J}(w)$. In fact, an induction argument shows that a finite closed set of minimum cardinality containing w will be $\mathrm{J}(w)$. The lemmas also show that $\mathrm{J}(w)$ can be characterized as the smallest set S containing w such that if $u \in S$ and V is a nonrefinable join cover of u, then $V \subseteq S$. We extend the definition of $\mathrm{J}(w)$ to include all elements of $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ by defining $\mathrm{J}(w) = \bigcup_u \mathrm{J}(u)$, where the union is over all elements u which lie in a nonrefinable join representation of w. Notice that all the elements of $\mathrm{J}(w)$ are join irreducible. A argument similar to the proof of the last two lemmas shows that this extended $\mathrm{J}(w)$ is the smallest closed set such that any join cover of w can be refined to one contained in $\mathrm{J}(w)$. We record this as a theorem.

Theorem 2-6.4. *If w is join reducible, then $\mathrm{J}(w) = \bigcup_u \mathrm{J}(u)$, where the union is over all elements u which lie in a nonrefinable join representation of w. Moreover, every join cover of w can be refined to one contained in $\mathrm{J}(w)$.*

For $w \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$, we define the *rank* of w to be the rank of the canonical form of w. The *depth* of w is defined similarly. In the proof of Lemma 2-6.2, a closed set containing w was constructed as the union of $\mathrm{J}'(w)$ and P'. Since the elements of $\mathrm{J}'(w) - \{w\}$ all have depth less than the depth of w, we have the following theorem.

Theorem 2-6.5. *For each $w \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ there is a unique smallest closed set, $\mathrm{J}(w)$, with the property that every join cover of w can be refined to one whose elements lie in $\mathrm{J}(w)$. If $w \notin P$ then every element of $\mathrm{J}(w) - \{w\}$ has lower depth than w. If $w \in P$ then $\mathrm{J}(w) \subseteq P$.*

Let $P^\wedge$ be the closure under meets of all subsets of P, including the empty set, in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. (The meet of the empty set is the greatest element.) An element with no nontrivial join cover is said to be *join prime*.

Corollary 2-6.6. *If $w \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ then $\mathrm{J}(w) \cap P^\wedge \neq \varnothing$. Furthermore, if $w \in P^\wedge$ then $\mathrm{J}(w) \cap P \neq \varnothing$ unless w is join prime. Let $0 = \bigwedge P$ be the least element. Then $\mathrm{J}(0) = \{0\}$, and $0 \notin \mathrm{J}(w)$ if $w \neq 0$.*

Proof. We prove this by induction on the depth of w. If $w \in P$ then this corollary follows from the previous theorem. Thus we may assume $w \notin P$. If w is not join irreducible then $\mathrm{J}(w)$ is nonempty and all of its elements have depth lower than the depth of w, so the result follows easily by induction. Thus assume that w is join irreducible and that $w \notin P^\wedge$. If $v \in \mathrm{J}(w) - \{w\}$ then by the last theorem the depth of v is less that of w. By induction $\mathrm{J}(v) \cap P^\wedge \neq \varnothing$. Since $\mathrm{J}(v) \subseteq \mathrm{J}(w)$, we are done unless $\mathrm{J}(w) = \{w\}$. However, if this were the case then w would be join prime. Since w is not in $P^\wedge$, then one of the canonical meetands, say w_1, of w is not in P. If the canonical joinands of w_1 are w_{1j}, $j = 1, \ldots, n$, then by an easy application of part (b) of Theorem 2-4.4, $\{w_{11}, \ldots, w_{1n}\}$ is a nontrivial join cover of w. Thus w is not join prime. If $w \in P^\wedge - P$ and is not join prime, then by the last theorem $\mathrm{J}(w) - \{w\} \subseteq P$. Since w is not join prime, this set is not empty.

The last sentence follows from the fact that 0 has no nontrivial join cover and it is not part of a nontrivial join cover of any other element. □

For computational purposes it is useful to have a more concrete description of $\mathrm{J}(w)$. First we show that $\mathrm{J}'(w)$ defined above is contained in $\mathrm{J}(w)$.

Lemma 2-6.7. *If w is join irreducible then*

$$\mathrm{J}'(w) \subseteq \mathrm{J}(w).$$

Proof. Let

$$w = w_1 \wedge \cdots \wedge w_n \wedge x_1 \wedge \cdots \wedge x_r = \bigwedge_{i=1}^{n} \bigvee_j w_{ij} \wedge \bigwedge_{k=1}^{r} x_k. \tag{2-6.2}$$

Now $w \leq w_i = \bigvee_j w_{ij}$, so $\{w_{i1}, w_{i2}, \ldots\}$ is a join cover of w. Suppose T is a join cover of w with $T \ll \{w_{i1}, w_{i2}, \ldots\}$. Then $w \leq \bigvee T \leq w_i$. We apply Dean's Theorem to $w \leq \bigvee T$. Certainly there is no $x \in P$ with $w \leq x \leq \bigvee T \leq w_i$ by canonical form (2-6.2). Similarly if there is a $t \in T$ with $w \leq t$, then since $t \leq w_{ij}$, $w \leq w_{ij}$, again contradicting canonical form. Finally if $w_{i'} \leq \bigvee T$ then clearly $i' = i$ and so $w_i = \bigvee T$. Thus T is a join representation of w_i. Assuming it is a nonrefinable join representation of w_i, Theorem 2-4.6 implies that it must contain each w_{ij} that is not in P (and every w_{ij} in P that is join irreducible). Hence $\mathrm{J}'(w) \subseteq \mathrm{J}(w)$. □

Note that if T is a nontrivial, nonrefinable join cover of w, then by canonical form $w_i \leq \bigvee T$, for some i, or $x_k \leq \bigvee T$. This leads to the question: if S is a nontrivial, nonrefinable join cover of x_k, is it a nontrivial, nonrefinable join cover of w? Not in general. However, we have the following.

Lemma 2-6.8. *Suppose S is a nontrivial, nonrefinable join cover of x_k such that $w_i \not\leq \bigvee S$ for $i = 1, \ldots, n$, and such that if $T \ll S$ for some nontrivial, nonrefinable join cover T of some $x_{k'}$, then $T = S$. Then S is a nontrivial, nonrefinable join cover of w.*

Theorem 2-6.9. *Let w be a join irreducible element with canonical form given by* (2-6.2). *To find* $\mathrm{J}(w)$ *we start with* $\mathrm{J}'(w)$. *For each nontrivial, nonrefinable join cover S of some x_k such that for all i, $w_i \not\leq \bigvee S$ we add* J *applied to each element of S. We repeat this for each $u \in \mathrm{J}'(w)$.*

Definition 2-6.10. Let L be a lattice and S a finite subsemilattice of L with a least element. Of course, this implies S is a lattice. Assume also that, for all $s \in S$ and a, $b \in L$,

(2-6.3) $s \leq a \vee b$ implies there is a $T \subseteq S$ with $T \ll \{a, b\}$ and $s \leq \bigvee T$.

The *standard homomorphism* (or *standard epimorphism*) is the map $f : L \to S$ defined by

$$f(u) = \bigvee \{v \in S : v \leq u\}. \tag{2-6.4}$$

Lemma 2-6.11. *The standard homomorphism is a homomorphism.*

Proof. Clearly f preserves order and satisfies $f(u) \leq u$. Hence, for $u \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ and $v \in S$, $v \leq u$ if and only if $v \leq f(u)$. The reader can check that f preserves meets. For joins, suppose that a, $b \in L$. Clearly $f(a \vee b) \geq f(a) \vee f(b)$. For the other direction, suppose that $v \leq a \vee b$, for some $v \in S$. Since $v \in S$, there is a set $T \subseteq S$ with

$$T \ll \{a, b\} \qquad v \leq \bigvee T.$$

Since $T \subseteq S$, $f(t) = t$ for $t \in T$. Thus

$$v \leq \bigvee T = \bigvee_{t \in T} f(t) \leq f(a) \vee f(b),$$

showing that $f(a \vee b) = f(a) \vee f(b)$. □

If S is a subset of a lattice, $S^\vee$ denotes the *join closure* of S; that is, the closure under joins of all finite subsets of S. This includes the join of the empty set, $\bigwedge S$, if that exists. Note that S is a join subsemilattice of the lattices, and, when S is finite, S is a lattice.

Let $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ is a finitely presented lattice. If we take $S = P^{\vee}$ (which, of course, is $\mathrm{Idl}_0(P, \leq, \bigvee, \bigwedge)$), or, more generally, if $S = P^{\vee(\wedge\vee)^n}$, then, by Theorem 2-6.5, $\mathrm{J}(w) \subseteq S$ for each $w \in S$. It follows that S satisfies (2-6.3) and so the standard homomorphism from $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ onto S exits.

We define $L(w)$ to be the join closure of $\mathrm{J}(w)$ in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. In symbols

$$L(w) = \mathrm{J}(w)^{\vee}.$$

By the remarks above, $L(w)$ is a lattice, which is a join subsemilattice of $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ and which satisfies (2-6.3). So the standard homomorphism $f\colon \mathrm{Free}(P, \leq, \bigvee, \bigwedge) \to L(w)$ exits and, as is easy to see, can be defined just using $\mathrm{J}(w)$:

$$f(u) = \bigvee\{v \in \mathrm{J}(w) : v \leq u\}.$$

This standard homomorphism played an important role in [15] and will also be important here.

We are interested in determining which elements of $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ have lower covers. Notice that a join irreducible element w has a lower cover if and only if it is completely join irreducible, that is, there is a greatest element, always denoted w_*, strictly less than w. We let $\kappa(w)$ denote the set of elements v maximal with the property that

$$v \geq w_* \quad \text{and} \quad v \ngeq w. \tag{2-6.5}$$

Theorem 2-6.12. *Let $w \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ be completely join irreducible. Then $\kappa(w)$ is finite and if v satisfies* (2-6.5)*, then $v \leq m$ for some $m \in \kappa(w)$. Moreover, either $|\kappa(w)| = 1$ or $\kappa(w) \subseteq P$.*

Proof. We will show that $\kappa(w)$ is the set of maximal elements of the members of non-upper refinable meet representations of w_* that are not above w. By the dual of Theorem 2-4.6 this set consists of the meet irreducible elements of the canonical meet representation of w_* that are not above w, together with those elements $p \in P$ maximal with the property $p \geq w_*$ but $p \ngeq w$. If $v \geq w_*$ but not $v \ngeq w$, then

$$w_* = v \wedge w.$$

By the dual of Theorem 2-4.6, this meet can be upper refined to a maximal one, proving the first part of the theorem.

Suppose that $\kappa(w)$ contains an element m not in P. Now if $w_* = v \wedge w$ then by the dual of Theorem 2-4.6 this meet representation can be upper refined to a nonrefinable one. Again by Theorem 2-4.6 this refinement must contain m. Since $m \ngeq w$ we must have $m \geq v$. This clearly implies $|\kappa(w)| = 1$. □

Exercise 2.10 gives an example of a partially defined lattice P and p and $q \in P$ such that $\kappa(p) = \kappa(q) = \{w\}$, where $w \notin P^{\vee}$.

The next theorem shows that the question of the existence of a lower cover of an arbitrary element can be reduced to the question for join irreducible elements.

Theorem 2-6.13. *An element* $w \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ *has a lower cover if and only if there is a completely join irreducible element in some nonrefinable join representation of* w.

Proof. Let u be a completely join irreducible element in a nonrefinable join representation of w. Let v be the join of the other elements of this representation. Then $u_* \vee v < w$ and hence

$$u_* = u \wedge (u_* \vee v).$$

Thus there is an $m \in \kappa(u)$ with $m \geq u_* \vee v$. Since $\kappa(u)$ is finite, we can choose an $m \in \kappa(u)$ such that $m \wedge w$ is a maximal element in the set

$$\{n \wedge w : n \in \kappa(u),\ n \geq v\}.$$

Then it is easy to check that $w \succ w \wedge m$.

Conversely, suppose that $w \succ v$ for some v. By Theorem 2-4.6 there are only finitely many elements involved in nonrefinable join representations of w. So we can choose u to be a minimal element of a nonrefinable join representation with the property $u \nleq v$. If

$$u \wedge v < a < u$$

then $a \vee v = w$. By Theorem 2-4.6 this join can be refined to a nonrefinable join, $w = \bigvee T$. There must be a $t \in T$ with $t \nleq v$. Hence $t \leq a < u$, contradicting the minimality of u. Thus $u \succ u \wedge v$, and so u is completely join irreducible. □

An epimorphism $f\colon K \to L$ is called *lower bounded* if each element $x \in L$ has a least preimage. This least preimage, when it exists, is denoted $\beta(x)$. *Upper bounded* is defined dually and the greatest preimage, when it exists, is denoted $\alpha(x)$. The map f is *bounded* if it is both upper and lower bounded. Notice that it follows immediately from the definition that *the standard homomorphism is lower bounded.*

Theorem 2-6.14. *A join irreducible element* $w \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ *is completely join irreducible if and only if the standard homomorphism*

$$f\colon\ \mathrm{Free}(P, \leq, \bigvee, \bigwedge) \to L(w)$$

is bounded.

Proof. First suppose that f is bounded. Clearly w is join irreducible in $L(w)$. Let $w_\dagger$ be its lower cover in $L(w)$, that is,

$$w_\dagger = \bigvee \{v \in \mathrm{J}(w) : v < w\}. \tag{2-6.6}$$

Also define $\kappa_{L(w)}(w)$ to be the set of all meet irreducibles $m \in L(w)$ satisfying $m \geq w_\dagger$, $m \ngeq w$. It is easy to check that $w \succ w \wedge \alpha(m)$ for $m \in \kappa_{L(w)}(w)$.

Conversely, suppose that w is completely join irreducible with lower cover w_*. Let $\psi(w, w_*)$ be the unique largest congruence separating w from w_*; such a congruence exists by Dilworth's characterization of lattice congruences; see [4]. Now $L(w)$ is an image of $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ separating w and w_*. It is not difficult to show that any image of $L(w)$ must identify w and $w_\dagger$ (see Theorem 4.1 of [15]). It follows that $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)/\psi(w, w_*) \cong L(w)$ and $\psi(w, w_*)$ is the kernel of f.

Recall that $P^\wedge$ is the meet closure of P in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. Consider $P^{\wedge(\vee\wedge)^n}$, the n-fold closure of $P^\wedge$ under joins and meets. This is a finite subset of $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ closed under meets and possessing a greatest element, hence a lattice. If n is large enough this lattice will satisfy the relations of P. The (dual) standard epimorphism (see Definition 2-6.10)

$$g\colon \mathrm{Free}(P, \leq, \bigvee, \bigwedge) \to P^{\wedge(\vee\wedge)^n}$$

is a homomorphism and trivially it is upper bounded. If we choose n large enough, g will separate w and w_*. Thus there will be an epimorphism $h\colon P^{\wedge(\vee\wedge)^n} \to L(w)$ such that $f = hg$. Since $P^{\wedge(\vee\wedge)^n}$ is finite, h is clearly bounded. Since g is upper bounded, f is upper bounded. Since f is the standard homomorphism, it is also lower bounded. □

So to test if w is completely join irreducible and find w_* if it is, we just need to test if the standard homomorphism is upper bounded and to find α if it is. The algorithm for this is presented in Section 5 of [9].

We close this section with a theorem which gives strong necessary conditions for a join irreducible element to have a lower cover. The definition of $w_\dagger$ is given in (2-6.6). Let $\mathrm{K}(w)$ be the set of maximal elements of the set

$$\{v \in L(w) : w_\dagger \leq v,\ w \nleq v\}.$$

Let S denote the maximal elements of the set

$$\{p \in P : w_\dagger \leq p,\ w \nleq p\}.$$

It is worth noting that, in the next theorem, the condition $w \leq \bigvee \mathrm{K}(w)$ is equivalent to $|\,\mathrm{K}(w)| > 1$. See Exercise 2.11 for additional information.

Theorem 2-6.15. *The following are necessary conditions for a join irreducible element* $w \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ *to be completely join irreducible.*

(i) *Each element* $v \in \mathrm{J}(w)$ *is completely join irreducible.*

(ii) *If* $w \leq \bigvee \mathrm{K}(w)$ *then for each* $u \in \mathrm{L}(w)$ *with* $w \not\leq w_\dagger \vee u$*, there is a* $p \in S$ *such that* $w_\dagger \vee u \leq p$*. That is,* $K(w) = S$*.*

(iii) *If* $w \leq \bigvee \mathrm{K}(w)$ *then* $w \wedge s_1 = w \wedge s_2$ *for all* $s_1,\ s_2 \in S$ *(the meets are calculated in* $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$*).*

(iv) *If* $w \leq \bigvee \mathrm{K}(w)$ *then for all* $s_1,\ s_2 \in S$*,* $s_1 \vee (s_2 \wedge (s_1 \vee w))$ *equals* s_1 *or* $s_1 \vee w$*.*

Proof. Assume that w is completely join irreducible. Let $v \in \mathrm{J}(w)$, so that $\mathrm{J}(v) \subseteq \mathrm{J}(w)$. Let $f\colon Free(P, \leq, \bigvee, \bigwedge) \to L(w)$ and $g\colon Free(P, \leq, \bigvee, \bigwedge) \to L(v)$ be the standard epimorphisms. Just as in the proof of Theorem 2-6.11, we can show that there is a epimorphism $h\colon L(w) \to L(v)$ such that $g = hf$. By Theorem 2-6.14, f is bounded. Since h is obviously bounded, it follows that g is bounded. Thus v is completely join irreducible again by Theorem 2-6.14.

If $|\kappa(w)| = 1$ and m is the unique element, then $f(m)$ contains all elements of $L(w)$ which are above $w_\dagger$ but not above w, where f is the standard homomorphism from $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ onto $L(w)$. In this case $w \not\leq \bigvee \mathrm{K}(w)$. Thus for parts (ii), (iii), and (iv) we may assume that $|\kappa(w)| > 1$. This implies that $\kappa(w) \subseteq P$ by Theorem 2-6.12. Now if $u \in \mathrm{L}(w)$ satisfies $w \not\leq w_\dagger \vee u$ then there is an element $v \in \mathrm{K}(w)$ with $w_\dagger \vee u \leq v$. Since the standard homomorphism is upper bounded, $\alpha(v) \in \kappa(w) \subseteq P$, completing the proof of (ii).

For (iii), by using this same reasoning as above, we see that S is the set of maximal elements of the set

$$\{p \in P : w_* \leq p,\ w \not\leq p\}.$$

Thus $s \wedge w = w_*$ for each $s \in S$. Moreover, it follows that, for $s \in S$, s is completely meet irreducible with unique upper cover $s \vee w$. Property (iv) follows easily from this. ☐

Lemma 2-6.16. *Suppose* w *is a completely join irreducible element and let the canonical meet representation of* w *be* $w = \bigwedge w_i$*. Let* $u \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ *such that for some* i*,*

$$w \leq u \vee w_* \leq w_i \quad \textit{and} \quad w \not\leq u,$$

Then $u \vee w_* = w_i$*.*

Proof. By canonical form there is no $x \in P$ with $w \leq x < w_i$. So applying Dean's Theorem to

$$w = \bigwedge w_{i'} \leq u \vee w_* \leq w_i$$

gives $w_{i'} \leq u \vee w_* \leq w_i$, for some i'. This implies $i = i'$ and so $u \vee w_* = w_i$. ☐

Theorem 2-6.17. *Suppose w is a completely join irreducible element with lower cover w_* and suppose $|\kappa(w)| = 1$. Let the canonical meet representation of w be $w = \bigwedge w_i$. Then the canonical join representation of $w_i = \bigvee_j w_{ij}$ can be refined to a nonrefinable join representation $\{u_1, \dots, u_r\}$ of w_i with $u_j \le w_*$ for $j = 2, \dots, r$.*

Proof. If w_i is join irreducible, then $\{w_i\}$ is the only nonrefinable join representation and the theorem holds. Let $\kappa(w) = \{m\}$. If for all j, $w_{ij} \le m$, then $w_i \le m$ which is not possible because $w_i \ge w$ and $m \not\ge w$. Hence there is a j with $w_{ij} \not\le m$, and thus $w \le w_{ij} \vee w_*$. We take $j = 1$. By the lemma $w_i = w_* \vee w_{i1}$. This implies that there is a nonrefinable join representation $\{u_1, \cdots, u_r\}$ of w_i such that $\{u_1, \dots, u_r\} \ll \{w_{i1}, w_*\}$ Now if w_{i1} is join irreducible then it is one of the u_j's, say u_1, by Theorem 2-4.6. It follows that $u_j \le w_*$ for $j = 2, \dots, r$ since none of them can be below u_1.

So again by Theorem 2-4.6, $w_{i1} \in P$ and thus the set

$$S = \{x \in P : x \le w_{i1} \text{ and } x \vee w_* = w_i\}$$

is not empty. Let y be a minimal element of this set. Refine $w_i = y \vee w_*$ to a nonrefinable join representation $\{u_1, \dots, u_r\}$. Not all of the u_j's can be below m; so we may assume $u_1 \not\le m$. This implies $u_1 \le y$. If $u_1 \notin P$ then, by Theorem 2-4.6, $u_1 = w_{ij}$ for some j. But then $w_{ij} = u_1 \le y \le w_{i1}$. This forces j to be 1 and so $u_1 = w_{i1} \in P$, contrary to assumption. So $u_1 \in P$. By the lemma $u_1 \vee w_* = w_i$ and so $u_1 \in S$. By the minimality of y, $u_1 = y$. Of course none of $u_2, \dots, u_r$ can be below $u_1 = y$, and hence they are all below w_*, as desired. □

We let κ^{d} denote the notion dual to κ. We gather some additional notation.

$$\begin{aligned} w_\dagger &= \bigvee \{v \in \mathrm{J}(w) : v < w\} \\ K(w) &= \max\{v \in \mathrm{L}(w) : w_\dagger \le v,\ w \not\le v\} \\ M_0 &= \bigcup \{\kappa(v) : v \in \mathrm{J}(w) - \{w\}\} \\ k^\dagger &= \bigwedge \{u \in M_0 : u \ge \bigvee K(w)\} \end{aligned}$$

Theorem 2-6.18. *If a join irreducible element $w \in \mathrm{Free}(P, \le, \bigvee, \bigwedge)$ is completely join irreducible then*

(i) *$w \in P$ and is completely join irreducible in $\mathrm{Free}(P, \le, \bigvee, \bigwedge)$, or*

(ii) *$w \in \kappa^{\mathrm{d}}(p)$ for some $p \in P$, or*

(iii) *each $u \in \mathrm{J}(w) - \{w\}$ is completely join irreducible and*

$$w \not\le \bigvee \mathrm{K}(w).$$

Proof. Suppose w is completely join irreducible. If $w \in P$ then (i) holds. If $|\kappa(w)| > 1$ then $\kappa(w) \subseteq P$ by Theorem 2-6.12 and so (ii) holds. So $\kappa(w) = \{k\}$ for some completely meet irreducible element k. Now $\bigvee \mathrm{K}(w) \leq k$ and so $w \not\leq \bigvee \mathrm{K}(w)$. Each $u \in \mathrm{J}(w) - \{w\}$ is completely join irreducible by Theorem 2-6.15. □

2-6.1 Exercises

2.10. Let $P = \{p, p_0, q, q_0, r, c, d\}$ have order

$$p_0 < p < r \text{ and } q_0 < q < r$$

and defined joins

$$r = p_0 + q = q_0 + p$$

and no defined meets. Show that $\kappa(p) = \kappa(q) = \{w\}$, where

$$w = p_0 \vee q_0 \vee ((p \vee c \vee d) \wedge (q \vee c \vee d)).$$

This implies $\kappa^{\mathrm{dual}}(w) = \{p, q\}$, showing both $\kappa(p)$ is not necessarily in $P^{\vee}$ and that it is possible for $|\kappa(w)| > 1$. (The latter is not surprising: there are finite lattices that witness it.)

2.11. Suppose an element w of a finitely presented lattice is completely join irreducible. Use the fact that the standard homomorphism is bounded to show that $|\kappa(w)| = |K(w)|$. Use this to show that the following are equivalent.

(i) $|K(w)| > 1$,

(ii) $|\kappa(w)| > 1$,

(iii) $w \leq \bigvee K(w)$.

2-7. Weak atomicity, the derivative, and coverless lattices

A lattice is *weakly atomic* if every nontrivial interval contains a covering. As we mentioned earlier, one of the most important theorems on free lattices is Day's Theorem that they are weakly atomic. He proved it using his doubling construction. Two proofs are given in [11], so we shall omit the proof here.

Theorem 2-7.1 (Day [1])**.** *Every finitely generated free lattice is weakly atomic.*

This result naturally raised the question if every finitely presented lattice is weakly atomic. At the other extreme, is there a finitely presented lattice without any covers at all? Of course every finitely generated lattice without any covers is an image of a finitely generated free lattice with all of its covers collapsed. The congruence δ generated by all the covers of a lattice L is called the *derivation congruence*, and L modulo this congruence is called the *derivative* of L. It is denoted L'.

If we apply this to $L = \operatorname{Free}(X)$, where X is finite, two question arise. First, is the derivation congruence finitely generated; this is, compact? This is necessary (and sufficient) for L' to be finitely presented. Second, is L' coverless? In general this need not be the case. For example, the derivative of $\omega + 1$, a countable ascending chain with greatest element, is the two element lattice.

Theorem 2-7.2. *If X is finite, then the derivation congruence δ of* $\operatorname{Free}(X)$ *is finitely generated, namely*

$$\delta = \bigvee_{Y \subseteq X} \operatorname{con}(\bigwedge Y, \bigvee (X - Y))$$

Hence the derivative of $\operatorname{Free}(X)$ *is a finitely presented lattice.*

Proof. If $u \succ v$ in $\operatorname{Free}(X)$, then by Theorem 2-6.13 there is a completely join irreducible element w such that $[w_*, w]$ transposes up to $[v, u]$. Consequently $\psi(w, w_*) = \psi(u, v)$, where $\psi(u, v)$ is the unique largest congruence separating u from v.

So suppose w is completely join irreducible in $\operatorname{Free}(X)$. Then by Theorem 2-6.14 and its proof, $\operatorname{Free}(X)/\psi(w, w_*)$ is isomorphic to $L(w)$ and the map from $g\colon \operatorname{Free}(X) \twoheadrightarrow L(w)$ is the standard homomorphism and is bounded. By Exercise 2.3, free lattices are semidistributive. The reader can show that a bounded image of a semidistributive lattice is semidistributive, and so $L(w)$ is semidistributive. By Exercise 2.5, $L(w)$ has a homomorphism onto the two-element lattice. The composition of g with this map gives a homomorphism $f\colon \operatorname{Free}(X) \twoheadrightarrow 2$. Let $Y = \{y \in X : f(y) = 1\}$. Then f separates the cover

$$\bigwedge Y \succ \Big(\bigwedge Y\Big) \wedge \bigvee (X - Y).$$

Thus what we have shown is that any homomorphism separating w from w_* must also separate a cover of the form displayed above. Consequently a congruence that collapses all covers of the above form, must collapse every cover. This is the content of the theorem. □

Corollary 2-7.3. *If $|X| = 3$, then* $\operatorname{Free}(X)' \cong M_3$.

Proof. The description of δ from Theorem 2-7.2 shows that the derivative is defined by the relations saying any two members of X join above the other, and dually. It is easy to see that the lattice defined by these relations is M_3. □

Corollary 2-7.4. *Every nontrivial 3-generated lattice contains a cover.*

Now we come to a deeper theorem which is a nice application of our theory of coverings in finitely presented lattices.

Theorem 2-7.5. *If* $|X| > 3$*, then* $\operatorname{Free}(X)'$ *is coverless and infinite.*

Proof. If X is infinite, $\operatorname{Free}(X)$ has no cover and the result is trivial. So we assume X is finite.

$\operatorname{Free}(X)'$ is the finitely presented lattice with relations

$$\bigwedge Y \leq \bigvee (X - Y), \quad \varnothing \subsetneq Y \subsetneq X.$$

We make a partially defined lattice corresponding to these relations. Let r_Y be a symbol (representing $\bigvee Y$) and let q_Y be a symbol (representing $\bigwedge Y$). Define

$$R = \{r_y : Y \subseteq X \text{ and } 2 \leq |Y| \leq n - 2\}$$
$$Q = \{q_y : Y \subseteq X \text{ and } 2 \leq |Y| \leq n - 2\}$$

where $n = |X|$. Let $P = X \cup R \cup Q \cup \{0, 1\}$. Besides have 0 and 1 as the least and greatest elements, the order on P is

$$q_Y \leq x \leq r_Y \quad \text{if } x \in Y$$
$$r_Y \leq r_Z \quad \text{if } Y \subseteq Z$$
$$q_Y \leq q_Z \quad \text{if } Y \supseteq Z$$
$$q_Y \leq r_Z \quad \text{if } Y \cap Z \neq \varnothing \text{ or } Y \cup Z = X.$$

It is the last case of the last relation that captures the relations above defining $\operatorname{Free}(X)'$.

The defined joins are $\bigvee Y = r_Y$, for $Y \subseteq X$ with $2 \leq |Y| \leq n - 2$, and $\bigvee Y = 1$ if $|Y| = n - 1$. The defined meets are dual. Let $\operatorname{Free}(P, \leq, \bigvee, \bigwedge)$ be the finitely presented over this partially defined lattice. As discussed earlier in this chapter, $\operatorname{Free}(X)' \cong \operatorname{Free}(P, \leq, \bigvee, \bigwedge)$.

We shall show that $\operatorname{Free}(P, \leq, \bigvee, \bigwedge)$ has no cover in a series of steps:

- if $\operatorname{Free}(P, \leq, \bigvee, \bigwedge)$ has a cover, it has a completely join irreducible element;
- if it has a completely join irreducible element, it has one in $P^\wedge$;
- if it has a completely join irreducible element in $P^\wedge$, it has one in P; and
- if it has a completely join irreducible element in P, it has one in X.

We finish the proof by showing that no element of X is completely join irreducible.

The first step follows from duality and Theorem 2-6.13. Let w be completely join irreducible. By Theorem 2-6.6 $\mathrm{J}(w) \cap P^\wedge \neq \varnothing$. By Theorem 2-6.15(i) each element of $J(w)$ is completely join irreducible. Thus we may assume $w \in P^\wedge$. Theorem 2-6.6 again shows that $\mathrm{J}(w) \cap P \neq \varnothing$ if the elements of $P^\wedge - P$ are not join prime, which we now show is the case.

So assume $w \in P^\wedge$. Since $Q \cup \{0\}$ is closed under meets, the elements of $P^\wedge$ have one of the forms

$$r_{Z_1} \wedge \cdots \wedge r_{Z_k} \quad \text{or} \quad q_Y \wedge r_{Z_1} \wedge \cdots \wedge r_{Z_k} \quad \text{or} \quad y \wedge r_{Z_1} \wedge \cdots \wedge r_{Z_k}$$

(Note $0 \in P^\wedge$ but is neither completely join irreducible nor, by Theorem 2-6.6, in $\mathrm{J}(w)$.)

If $k = 0$ then $w = y$, for some y, or $w = q_Y$, for some $q_Y \in Q$. But both of these are already in P. So $k \geq 1$ and since $\bigvee Z_1 = r_{Z_1}$, Z_1 is a join cover of w. If this is nontrivial then w is not join prime and by Theorem 2-6.6 $\mathrm{J}(w)$ contains an element of P. So we may assume $w \leq z$ for some $z \in Z_1$. Since the only defined meets of $(P, \leq, \bigvee, \bigwedge)$ are of subsets of X, and since there is no $x \in X$ with $r_{Z_i} \leq x$, we must have by Dean's Theorem that $q_Y \leq z$ (or $y = z$ if w has the last form above). But then $q_Y \leq r_{Z_i}$ (or $y \leq r_{Z_i}$) and the form of w given above is redundant, which we may assume is not the case. This shows that if $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ has a cover, then there is a completely join irreducible element in P. The candidates are y, for some $y \in X$, or q_Y. Since

$$q_Y \leq \bigvee (X - Y)$$

is nontrivial, $X - Y \subseteq \mathrm{J}(q_Y)$. This completes the proof of our last bullet point.

To show that x is not completely join irreducible for $x \in X$, first note $\mathrm{J}(x) = X$ because $x \leq \bigvee (X - \{x\})$ is nontrivial and no other defined joins contain x. So $L(x) = X \cup R \cup \{0, 1\}$, $x_\dagger = 0$, and

$$K(x) = \{r_Y : x \notin Y \text{ and } |Y| = n - 2\}.$$

So $x \leq \bigvee K(w)$. The set S defined above Theorem 2-6.15 is $X - \{x\}$. By part (iii) of that theorem, if x were completely join irreducible, then $x \wedge y = x \wedge z$ for distinct elements of X. In other words $q_Y = q_Z$, where $Y = \{x, y\}$ and $Z = \{x, z\}$. This contradiction proves the theorem. □

2-7.1 Exercises

2.12. Show that M_n is a homomorphic image of $\mathrm{Free}(X)'$, where $|X| = n$. (Actually every n-generated, subdirectly irreducible, modular lattice is an image of $\mathrm{Free}(X)'$.)

2.13. Show that weakly atomic finitely presented lattices are atomic. That is, each nonzero element contains an atom.

Problem 2.1. Characterize weakly atomic finitely presented lattices.

2-8. When is a finitely presented lattice finite?

In [27] V. Slavík gave a technical characterization of when a finite lattice has a finite W-cover. A lattice $\hat{L}$ is the *W-cover* of a lattice L if $\hat{L}$ satisfies (W), and there is a epimorphism $f\colon \hat{L} \twoheadrightarrow L$ such that if K is a lattice satisfying (W) with a epimorphism g onto L, then there is a epimorphism $h\colon K \twoheadrightarrow \hat{L}$ such that $g = f \circ h$. The reader can verify that the W-cover of a lattice is unique and always exists.

If a lattice satisfies (W) then $\hat{L} = L$, of course. On the other hand the W-cover of the free distributive lattice on three generators is Free(3), as was shown by Alan Day. In fact he showed Free(3) could be constructed from the free distributive lattice on three generators using his doubling construction repeatedly to fix (W) failures. This played an important role in his proof that finitely generated free lattice are weakly atomic [1], Theorem 2-7.1.

Of course a finitely presented lattice may fail (W). Indeed, every finite lattice is finitely presented. However, looking at (vi) of Dean's Theorem 2-3.4 we see that failures of (W) in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ can only occur in intervals $[\bigwedge_i s_i, \bigvee_j t_j]$ which contain an element of P. Let A be a subset of a lattice L. We define an interval $I = [\bigwedge_i s_i, \bigvee_j t_j]$ to be a *(W,A)-failure* if no s_i or t_j is in I and $I \cap A = \varnothing$. The (W,A)-cover of L is defined analogously to the W-cover of L.

Let $(P, \leq, \bigvee . \bigwedge)$ be a partially defined lattice. As in Section 2-5 we let $L_0 = \mathrm{Idl}_0(P, \leq, \bigvee, \bigwedge)$ and $L_1 = \mathrm{Fil}_1(P, \leq, \bigvee, \bigwedge)$ and L be the sublattice of $L_0 \times L_1$ generated by $\{(x, x) : x \in P\}$. We identifify P with $\{(x, x) : x \in P\}$. Alan Day called L the *partial completion* of $(P, \leq, \bigvee, \bigwedge)$ and showed that $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ could be constructed from L by repeatedly using his doubling construction to fix (W,P)-failures and taking the inverse limit. The details of this construction are not hard and are presented in [1, 2] and in Section II.7 of [11].

Slavík showed that if the repeated doubling occurred too many times, the construction produced an infinite lattice. This in turn leads to the following theorem.

Theorem 2-8.1. *Let $F = \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ and let L be the partial completion of $(P, \leq, \bigvee, \bigwedge)$. Let d be the number of elements of $L - P$ that are both join and meet reducible. Then F is finite if and only if it has at most $43(|L| + d + |P|)$ elements. In particular, F is finite if and only if it has at most $86\,|L|$ elements.*

This yields the following algorithm to test if $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ is finite (and find it if it is).

- Find the size m of the partial completion of $(P, \leq, \bigvee, \bigwedge)$.
- Calculate

$$P^{\vee},\ P^{\vee\wedge},\ P^{\vee\wedge\vee},\ \cdots$$

Stop if the size of any of these exceeds $86m$; $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ is infinite in this case. Also stop if any of these equals the previous one. In this case $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ is finite and is the last one.

The second part is efficient since putting terms into canonical form is. The first step is potentially exponential since m is at least $|\,\mathrm{Idl}_0(P, \leq, \bigvee, \bigwedge)|$. Nevertheless, the procedure works well in many examples. Much of the work in [12] involved determining if certain finitely presented lattices are finite. In that paper we used a modification of the algorithm above based on the following fact: *If* $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ *contains a join irreducible element which is not completely join irreducible or the dual holds, then it is infinite.* Based on this observation we can modify the algorithm above as follows:

- Find the size m of the partial completion of $(P, \leq, \bigvee, \bigwedge)$.
- Sequentially calculate

$$P^{\vee},\ P^{\vee\wedge},\ P^{\vee\wedge\vee},\ \cdots$$

  (i) If one of these closures equals the one before it, stop. In this case $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ is this last closure.

  (ii) If the last closure calculated was the meet closure, check if any of the new elements is not completely join irreducible (and dually if the last closure was under joins). If a join irreducible element which is not completely join irreducible (or dually) is found, stop, $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ is infinite.

  (iii) If the size of any of these closures exceeds $86m$, stop and conclude that $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ is infinite.

In all of the examples from [12] and all other examples we have encountered, step (iii) of the above algorithm has never been reached. In other words, in all cases where $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ is infinite a join irreducible, not completely join irreducible element was found long before the size of the closure reached $86m$. The exercises give some examples.

The authors of this chapter have implemented the above algorithm in Lisp; write us if you would like a copy of the program, including the code.

There is an interesting open question here.

Problem 2.2. If $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ is infinite, must it have an element that is either join irreducible but not completely join irreducible, or is meet irreducible but not completely meet irreducible.

2-8.1 Exercises

2.14. Let $P = \{0, a, b, c, 1\}$ have the order of M_3 and defined joins and meets:

$$ab = ac = bc = 0 \text{ and } a + b = a + c = 1$$

Of course if we added the relation $b+c=1$, $\mathrm{Free}(P,\leq,\bigvee,\bigwedge)$ would be M_3, and so be finite. But without that relation the lattice is infinite. Show that the partial completion of $(P,\leq,\bigvee,\bigwedge)$ has 13 elements so in order to apply Theorem 2-8.1 we would need to show that the closure has more than $86\cdot 13 = 1118$ elements. Instead, use Theorem 2-6.17 to show that $a\wedge(b\vee c)$ is join irreducible but not completely join irreducible.

2.15. Let $P=\{0,a,b,c,d,1\}$ have order $a>b$ and $d>c$ and 0 and 1 are the least and greatest elements. The defined joins are only $1=b+c$ and the defined meets are only $0=bc$. Find the partial completion of $(P,\leq,\bigvee,\bigwedge)$. (It has 11 elements.) Show $\mathrm{Free}(P,\leq,\bigvee,\bigwedge)$ is infinite by using Theorem 2-6.18 to show $a\wedge d$ is join irreducible but not completely join irreducible.

2.16. Show that if we modify the partially defined lattice in the previous problem by adding the relation $bd=0$, then the partial completion has 10 elements and $|\,\mathrm{Free}(P,\leq,\bigvee,\bigwedge)|=12$.

2.17. Let $P=\{0,a,b,c\}$ with 0 below the others as the only nontrivial order relation. The meet relations are $ab=ac=bc=0$ and no defined joins. Find $P^{\wedge}$, $P^{\wedge\vee}$, $P^{\wedge\vee\wedge}$ and $P^{\wedge\vee\wedge\vee}$ and also find the partial completion.

2.18. Show that if P is just an unordered set of size n, then it partial completion has size $2^{n+1}+n-1$ and is distributive.

2.19. Let $P=\{a,b,c,d,e,0,1\}$ with the order inherited from Figure 2-8.1. The defined joins and meets are

$$d=a+b,\quad e=b+c,\quad 1=a+c,\quad b=de,\quad 0=de$$

Show that $\mathrm{Free}(P,\leq\bigvee,\bigwedge)$ is the lattice of that figure. Since $b=de$, this lattice is generated by the two-element chains $a<d$ and $c<e$. So one can use Rolf's lattice (see Figure 121 in Chapter VII of LTF or Firgue 5.3 of [11]) to verify the above claim. Note b is the only element that is join irreducible but not completely join irreducible, and there is no meet irreducible element that is not completely meet irreducible. This lattice was constructed as an example of a lattice that is weakly atomic but its partial completion is not a bounded homomorphic image of it, refuting one of Alan Day's conjectures.

2-9. McKenzie's example

Here is an example of finitely generated sublattice, L_1, of a finitely presented lattice, L_0, that is not finitely presented. Both the example and the proof are due to Ralph McKenzie.

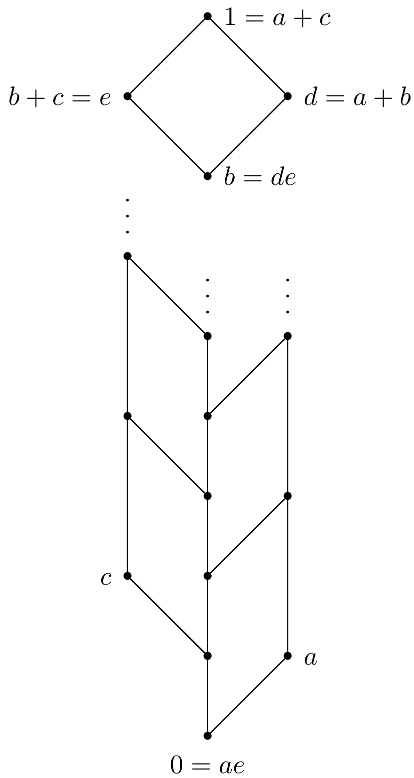

Figure 2-8.1: The diagram $\mathrm{Free}(P, \leq \bigvee, \bigwedge)$

Example 2-9.1. Define a partial lattice on $P = \{x, y, z, u, v, w\}$ whose order is given in Figure 2-9.1, with no defined joins and with meets $w = u \wedge x = x \wedge v = u \wedge v$. Let $L_0 = \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$.

Define $x_0 = x$, $y_0 = y$, $z_0 = z$ and $x_{n+1} = x_n \vee (y_n \wedge z_n)$, $y_{n+1} = y_n \vee (x_n \wedge z_n)$ and $z_{n+1} = z_n \vee (x_n \wedge y_n)$.

Now let L_1 be the lattice freely generated by x, y and z subject to the presentation $x_n = x$, $n = 0, 1, 2, \ldots$. We claim L_1 is isomorphic to the sublattice S of L_0 generated by x, y and z and that L_1 is not finitely presented. First it is easy to see that x, y and z in L_0 satisfy the relations of L_1 and so there is a homomorphism $f\colon L_1 \to L_0$ with $f(x) = x$, $f(y) = y$ and $f(z) = z$. In the ideal lattice, $\mathrm{Id}\, L_1$, let U be the ideal generated by $\{y_n : n = 0, 1, \ldots\}$ and V the ideal generated by $\{z_n : n = 0, 1, \ldots\}$. It is easy to check that $\mathrm{id}(x)$, $\mathrm{id}(y)$, $\mathrm{id}(z)$, U and V satisfy the relations defining L_0 so there is a homomorphism $g\colon L_0 \to \mathrm{Id}\, L_1$ with $g(x) = \mathrm{id}(x)$, $g(y) = \mathrm{id}(y)$, $g(z) = \mathrm{id}(z)$, $g(u) = U$, $g(v) = V$, and $g(w) = U \cap V$. Now $gf\colon L_1 \to \mathrm{Id}\, L_1$ sends x to

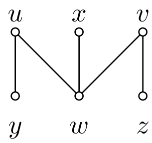

Figure 2-9.1: The diagram P

$\mathrm{id}(x)$, y to $\mathrm{id}(y)$ and z to $\mathrm{id}(z)$, and so is the natural embedding of L_1 into its ideal lattice (since they agree on the generators). Thus f is one-to-one and hence L_1 is isomorphic to the sublattice S.

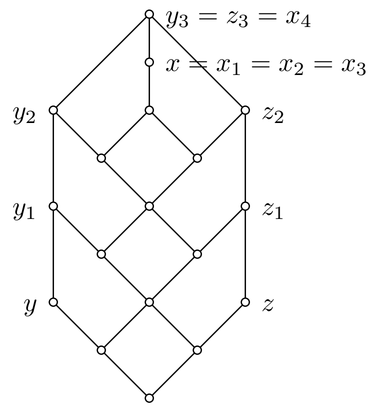

Figure 2-9.2: L_1 is not finitely presented

If L_1 were finitely presented then some finite subset of the relations $\{x_n = x : n = 0, 1, \ldots\}$ would define L_1. The lattice of Figure 2-9.2 and extensions of it show that this is not the case.

In contrast to McKenzie's theorem, every finitely generated sublattice of a free lattice is finitely presented, as we shall now show.

First a general lemma that applies to all algebras.

Lemma 2-9.2. *A finitely generated projective algebra B is finitely presented.*

Proof. Let $F(X)$ be a finitely generated free algebra with a retraction r such that $B \cong r(F(X))$. Let θ be the congruence generated by $\{(x, r(x)) : x \in X\}$, let $A = F(X)/\theta$, and let $f\colon F(X) \to A$ be the natural map. Since θ is compact, A is finitely presented by the relations

$$x \approx r(x).$$

By the definition of θ, $f(x) = f(r(x))$ and so f maps B onto A. But since $r^2 = r$, B also satisfies the relations above. Thus there is a homomorphism $g\colon A \to B$. Moreover, since $g(r/\theta) = r(x)$ and $f(r(x)) = r(x)/\theta = x/\theta$, we see that $f|_B$ and g are inverses. Thus $B \cong A$ is finitely presented. □

Theorem 2-9.3. *A finitely generated sublattice of a free lattice is finitely presented.*

Proof. The theorem follows from the Kostinsky-McKenzie result that finitely generated sublattices of free lattices are projective; see [22, 11]. □

2-10. The generalized word problem and automorphisms

This section is based primarily on the results of [14]. After the definition of standard homomorphism, Definition 2-6.10, and in the proof of Theorem 2-6.14 we introduced the lattice $P^{\wedge(\vee\wedge)^n}$. We denote this lattice L_1. It is a meet subsemilattice of $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. Moreover,

$$f_1 : \mathrm{Free}(P, \leq, \bigvee, \bigwedge) \to L_1 = P^{\wedge(\vee\wedge)^n}$$

is the (dual of) the standard homomorphism. We let $L_0 = P^{\vee(\wedge\vee)^n}$ and f_0 be the standard homomorphism. It is easy to see that f_0 is the identity on L_0 and that $f_0(w) \leq w$ for all $w \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. Similarly, f_1 is the identity on L_1 and $f_1(w) \geq w$. Let $f : \mathrm{Free}(P, \leq, \bigvee, \bigwedge) \to L_0 \times L_1$ be given by $f(w) = (f_0(w), f_1(w))$.

Lemma 2-10.1. *If $w \in L_0 \cap L_1$, then $f^{-1}(f(w)) = \{w\}$.*

Proof. Since $w \in L_0 \cap L_1$, $f(w) = (w, w)$. So, if $f(u) = f(w)$, then $w = f_0(u) \leq u$. Similarly, $w \geq u$. □

Standard results in universal algebra give the following corollary.

Corollary 2-10.2. *Finitely presented lattices are residually finite. Consequently, the variety of lattices is generated by its finite members.*

The *generalized word problem* for a finitely presented algebra A asks if there is an algorithm to determine, for an arbitrary element $d \in A$ and a finite set $U = \{u_1, \ldots, u_k\}$ of elements of A, if d is in the subalgebra generated by U.

Theorem 2-10.3. *The generalized word problem for lattices is (uniformly) solvable.*

Proof. Let $d \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ and let f be the homomorphism onto the finite lattice described above. By the lemma $f^{-1}(f(d)) = \{d\}$. Now let $u_1, \ldots, u_k$ be elements of $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. We claim d is in the sublattice generated by $u_1, \ldots, u_n$ if and only if $f(d)$ is in the sublattice generated by $f(u_1), \ldots, f(u_n)$. If the latter condition holds, then there is a term t such that

$$f(d) = t(f(u_1), \ldots, f(u_n)).$$

Since f is a homomorphism, $t(f(u_1), \ldots, f(u_n)) = f(t(u_1, \ldots, u_n)$. Thus, by the lemma, $d = t(u_1, \ldots, u_n)$. The other direction is obvious. This construction is effective so the theorem follows from the claim. □

The *isomorphism problem* for a variety $\mathcal{V}$ is to decide whether two finitely presented $\mathcal{V}$-algebras are isomorphic. Using the lemma we can bound the complexity of the images under an isomorphism. This yields a solution to the isomorphism problem. See [14] for the details.

Theorem 2-10.4. *The isomorphism problem for lattices solvable.*

These techniques also solve a problem raised by G. Grätzer in [17].

Theorem 2-10.5. *If L is a finitely presented lattice, then* $\mathrm{Aut}(L)$ *is finite.*

The method of this section of considering the standard homomorphism from $(P, \leq, \bigvee, \bigwedge)$ onto $P^{\vee(\wedge\vee)^n}$ gives a solution to the word problem and related problems that is conceptually easy. However, the results of Skolem and Dean give *much* more efficient solutions. For example, using our Lisp programs we calculate that if P is just a three element antichain, then $P^{\vee}$, $P^{\vee\wedge}$, $P^{\vee\wedge\vee}$, and $P^{\vee\wedge\vee\wedge}$ have sizes 8, 18, 44, and 677. But our programs found more that 10^6 elements in $P^{\vee\wedge\vee\wedge\vee}$ before we gave up.

In [18] Grätzer and Huhn gave an alternate proof Theorem 2-10.5. If, in the construction used at the beginning of this section, we take $n = 0$, then $L_0 = P^{\vee}$. Since $P^{\vee}$ is the join closure of all subsets, including the empty set, of P, it is easy to see $L_0 = P^{\vee} \cong \mathrm{Idl}_0(P, \leq, \bigvee, \bigwedge)$. Lemma 2-10.1 holds in this case for $d \in P$. Using this, the authors are still able to bound the complexity of the images of the generators under an automorphism, and thus prove the theorem.

Modular lattices

The above theorems do not hold for all varieties of lattices. Let $\mathcal{M}$ be the variety of all modular lattices. In [8] and [7], Freese shows that the word problem for free modular lattices is not solvable and that $\mathcal{M}$ is not generated by its finite members. A. Huhn [21] constructed a finitely presented modular lattice with an infinite automorphism group. Huhn defined a partially defined lattice $(P_n, \leq, \bigvee, \bigwedge)$ called an n-diamond. It consists of the Boolean lattice

B_{n+1} with $n+1$ atoms and an addition element d. All of the defined joins and meets of the Boolean lattice are part of the definition of $(P_n, \leq, \bigvee, \bigwedge)$. In addition we have the relations that w is a complement of each of the atoms of the Boolean lattice.

Theorem 2-10.6 (Huhn [21])**.** *Let L_n be the finitely presented modular lattice generated by $(P_n, \leq, \bigvee, \bigwedge)$ for $n > 1$. Then* $\mathrm{Aut}(L_n)$ *is infinite.*

2-10.1 Exercises

2.20. Show that if P is an ordered set with no defined joins or meets, then each element of P is both join and meet irreducible in $\mathrm{Free}(P, \leq)$. Consequently, if $Y \subseteq \mathrm{Free}(P, \leq)$ generates $\mathrm{Free}(P, \leq)$, then $P \subseteq Y$.

2.21. Let P_4 be a 4-crown; that is, the ordered set with elements a_i, b_i for $i = 0, 1, 2, 3$ and order $a_i < b_i$ and $a_i < b_{i+1}$ (indices are calculated modulo 4). The defined joins and meets are $a_i + a_{i+1} = b_{i+1}$ and $b_i b_{i+1} = a_i$. Find a subset $Y \subseteq \mathrm{Free}(P_4, \leq, \bigvee, \bigwedge)$ with $Y \cap P = \varnothing$ that generates $\mathrm{Free}(P_4, \leq, \bigvee, \bigwedge)$.

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