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Free-lattice functors weakly preserve epi-pullbacks

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Abstract. Suppose $p(x, y, z)$ and $q(x, y, z)$ are terms. If there is a common "ancestor" term $s(z_1, z_2, z_3, z_4)$ specializing to p and q through identifying some variables

$$p(x, y, z) \approx s(x, y, z, z)$$
$$q(x, y, z) \approx s(x, x, y, z),$$

then the equation

$$p(x, x, z) \approx q(x, z, z)$$

is a trivial consequence. In this note we show that for lattice terms, and more generally for terms of lattice-ordered algebras, a converse is true, too. Given terms p, q, and an equation

$$p(u_1, \ldots, u_m) \approx q(v_1, \ldots, v_n) \qquad (*)$$

where $\{u_1, \ldots, u_m\} = \{v_1, \ldots, v_n\}$, there is always an "ancestor term" $s(z_1, \ldots, z_r)$ such that $p(x_1, \ldots, x_m)$ and $q(y_1, \ldots, y_n)$ arise as substitution instances of s, whose unification results in the original equation $(*)$. In category theoretic terms the above proposition, when restricted to lattices, has a much more concise formulation: *Free-lattice functors weakly preserve pullbacks of epis.* Finally, we show that *weak* preservation is all that we can hope for. We prove that for an arbitrary idempotent variety $\mathcal{V}$ the free-algebra functor $F_{\mathcal{V}}$ will not *preserve* pullbacks of epis unless $\mathcal{V}$ is trivial (satisfying $x \approx y$) or $\mathcal{V}$ contains the "variety of sets" (where all operations are implemented as projections).

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1. Introduction

The motivation of this study arose from coalgebra. When studying F-coalgebras for a type functor F, the limit preservation properties of F are known to exert a crucial influence on the structure theory of the class of all F-coalgebras. In particular, *weak preservation of pullbacks* is a condition which is assumed in numerous studies of coalgebras. In Rutten's seminal paper [19], theorems were flagged with an asterisk if they were proved under the assumption that the type functor F weakly preserved pullbacks. Although many of these asterisks could later be discarded (see [5,20]), for several important results such a condition on the type functor remains essential.

Fortunately, many familiar Set-endofunctors enjoy the mentioned property, however, there are notable exceptions, such as e.g. the bounded finite powerset functor or the neighborhood functor, which both are relevant for describing certain transition systems. The first one of these preserves *preimages* (pullbacks along injective maps) and the second one preserves *kernel pairs* (pullbacks of two equal maps), but neither of both preserves all weak pullbacks.

It could later be shown [21,11], that in general these two simpler conditions, (weak) preservation of preimages and weak preservation of kernel pairs, combine to be equivalent to weak preservation of pullbacks. Finally, it was also discovered that the second condition, weak preservation of kernel pairs, is equivalent to weak preservation of pullbacks of epis (see [9]).

From a universal algebraic point of view it is of interest to study the functors $F_{\mathcal{V}}$ which, for a given variety of algebras $\mathcal{V}$, send a set X to the free algebra $F_{\mathcal{V}}(X)$ and a map $\sigma : X \to Y$ to its homomorphic extension $\bar{\sigma} : F_{\mathcal{V}}(X) \to F_{\mathcal{V}}(Y)$. Note that for an arbitrary term $p(x_1, \ldots, x_n)$ we have that

$$\bar{\sigma}\, p(x_1, \ldots, x_n) = p(\sigma x_1, \ldots, \sigma x_n), \tag{1.1}$$

where on the left hand side of the equation p is understood to be evaluated in $F_{\mathcal{V}}(X)$ and on the right hand side in $F_{\mathcal{V}}(Y)$.

In this context, weak preservation of kernel pairs translates into an interesting algebraic condition, asserting that given an equation

$$p(u_1, \ldots, u_m) \approx q(v_1, \ldots, v_n), \tag{1.2}$$

where $\{u_1, \ldots, u_m\} = \{v_1, \ldots, v_n\}$, then the terms p and q are in fact both obtained from a common "ancestor" term s by identifying some of its variables, so that the equality (1.2) trivially follows from this representation.

Example 1.1. Assume that $F_{\mathcal{V}}$ weakly preserves kernel pairs, then for any $\mathcal{V}$-equation

$$p(x, x, y) \approx q(x, y, y) \tag{1.3}$$

there exists a quaternary term s such that

$$\begin{aligned} p(x, y, z) &\approx s(x, y, z, z) \\ q(x, y, z) &\approx s(x, x, y, z). \end{aligned}$$

This representation then trivially entails (1.3), since the most general unifier of $s(x,y,z,z)$ with $s(x,x,y,z)$ is $s(x,x,z,z)$, resulting in the original equation

$$p(x,x,z) \approx s(x,x,z,z) \approx q(x,z,z).$$

The "ancestor condition" in the previous example has been introduced in [9]. Applying it to the description of n-permutable varieties given by Hagemann and Mitschke [12], we could show that for an n-permutable variety $\mathcal{V}$, the functor $F_{\mathcal{V}}$ weakly preserves kernel pairs if and only if $\mathcal{V}$ is congruence permutable, which in turn holds, according to Mal'tsev [15], if and only if there exists a term $m(x,y,z)$ such that the equations

$$m(x,y,y) \approx x \tag{1.4}$$

$$m(x,x,y) \approx y \tag{1.5}$$

are satisfied.

In this note we are going to show that for any variety $\mathcal{L}$ of lattices, or more generally, of lattice-ordered universal algebras, the free algebra functor $F_{\mathcal{L}}$ weakly preserves kernel pairs. Therefore, any pair of terms p, q which combine to a valid equation (1.2) are instances of a common ancestor term s, so that the equation (1.2) trivially results from instantiations of s resulting in a syntactically identical term.

From the mentioned paper [9] it follows that for no congruence modular variety $\mathcal{V}$ the functor $F_{\mathcal{V}}$ preserves preimages. Hence the functor $F_{\mathcal{L}}$ studied in this note will not preserve preimages, which shows that the variable condition $\{u_1, \ldots, u_m\} = \{v_1, \ldots, v_n\}$ cannot be dropped.

2. Preliminaries

We need only elementary category theoretic notions as can be found in the first chapters of any introductory text, such as e.g. [14] or [1]. Most of the time we shall remain in the category Set of sets and mappings. In particular, unless otherwise said, all functors we consider will be Set-endofunctors. Even the free-algebra functors $F_{\mathcal{V}}$, which are in the focus of this paper, will be considered as Set-endofunctors; in this case we refrain from explicitly mentioning the forgetful functor U taking an algebra to its underlying set.

Most of the time we shall omit parentheses when applying unary functions to arguments, and we assume that application associates to the right, so we write fx for $f(x)$ and fgx for $f(g(x))$.

For a map $f : X \to Y$ we denote the image of f by $f(X)$ or by $\operatorname{im} f$ and its preimage by $f^{-1}(Y)$. The *kernel* of f is

$$\ker f := \{(x_1, x_2) \in X \times X \mid fx_1 = fx_2\}.$$

Lemma 2.1. *Given a surjective map $f : X \twoheadrightarrow Y$ and an arbitrary map $g : X \to Z$, then there exists a (necessarily unique) map $h : Y \to Z$ with $h \circ f = g$ if and only if* $\ker f \subseteq \ker g$.

$$\begin{tikzcd} X \arrow[r, two heads, "f"] \arrow[dr, "g"'] & Y \arrow[d, dashed, "h"] \\ & Z \end{tikzcd}$$

Every surjective map $f : X \twoheadrightarrow Y$ is *right invertible*, i.e. has a *right inverse* which we shall denote by f^- and which obeys the equation $f \circ f^- = id_Y$. This general statement is equivalent to the axiom of choice, however we shall need it here only for finite sets X and Y.

3. Weak preservation of pullbacks

Recall that for morphisms $\alpha : X \to Z$ and $\beta : Y \to Z$ in a category, the *pullback* of α and β is a triple (P, π_1, π_2) consisting of an object P together with morphisms $\pi_1 : P \to X$ and $\pi_2 : P \to Y$ such that

- $\alpha \circ \pi_1 = \beta \circ \pi_2$, and
- for every *"competitor"*, i.e. for every other object Q with morphisms $\eta_1 : Q \to X$, $\eta_2 : Q \to Y$ also satisfying $\alpha \circ \eta_1 = \beta \circ \eta_2$, there is a *unique* $d : Q \to P$ such that $\eta_1 = \pi_1 \circ d$ and $\eta_2 = \pi_2 \circ d$.

$$\begin{tikzcd} & X \arrow[r, "\alpha"] & Z \\ & P \arrow[u, "\pi_1"] \arrow[r, "\pi_2"] & Y \arrow[u, "\beta"'] \\ Q \arrow[uur, bend left, "\eta_1"] \arrow[ur, dotted, "d"] \arrow[urr, bend right, "\eta_2"'] & & \end{tikzcd}$$

By dropping the *uniqueness* requirement, one obtains the definition of a *weak pullback*.

In the category Set of sets and mappings, the pullback of two maps α and β is, up to isomorphism, given by the set

$$\mathrm{pb}(\alpha, \beta) = \{(x, y) \in X \times Y \mid \alpha x = \beta y\}$$

and the coordinate projections π_1 and π_2.

- If $\alpha = \beta$ then $\mathrm{pb}(\alpha, \beta)$ is just $\ker \alpha$, the kernel of α, and $(\ker \alpha, \pi_1, \pi_2)$ is called a *kernel pair*.
- If β is injective, then $\mathrm{pb}(\alpha, \beta) \cong \alpha^{-1}(\beta(Y))$, hence such a pullback is called a *preimage*.

Weak pullbacks in Set are always of the shape (Q, η_1, η_2) where $d : Q \to \mathrm{pb}(\alpha, \beta)$ is right invertible and $\eta_i = \pi_i \circ d$ for $i = 1, 2$.

We say that a functor F *weakly preserves pullbacks* if applying F to a pullback diagram results in a weak pullback diagram. F is said to *preserve weak pullbacks*, if F transforms weak pullback diagrams into weak pullback diagrams.

Fortunately, it is easy to see that a Set-functor *preserves weak pullbacks* if and only if it *weakly preserves pullbacks*, see e.g. [6], and that it *preserves* preimages if and only if it *weakly preserves* preimages.

For Set-endofunctors F, weak preservation of pullbacks can be checked elementwise:

Proposition 3.1. *A* Set*-functor F weakly preserves the pullback of $\alpha : X \to Z$ and $\beta : Y \to Z$ iff for any $p \in F(X)$ and $q \in F(Y)$ with $r := (F\alpha)p = (F\beta)q$ there exists some $s \in F(\mathrm{pb}(\alpha, \beta))$ such that $(F\pi_1)s = p$ and $(F\pi_2)s = q$.*

$$\begin{tikzcd}
p \in F(X) \arrow[r, "F\alpha"] & F(Z) \ni r \\
s \in F(\mathrm{pb}(\alpha,\beta)) \arrow[u, "F\pi_1"] \arrow[r, "F\pi_2"'] & F(Y) \ni q \arrow[u, "F\beta"']
\end{tikzcd}$$

Since we are only concerned with finitary operations, the free-algebra functors $F_{\mathcal{V}}$ happen to be *finitary*. This means that given a set X and $p \in F(X)$, there is some finite subset $X_0 \subseteq X$ such that $p \in F(X_0)$. The following easy lemma allows us to restrict our consideration to finite sets and maps between them:

Lemma 3.2. *If F is finitary, and F weakly preserves pullbacks of maps between finite sets, then it weakly preserves all pullbacks.*

Weak preservation of pullbacks can be decomposed into two simpler preservation conditions. We recall from [11]:

Lemma 3.3. *A functor F weakly preserves pullbacks iff F weakly preserves kernel pairs and preimages.*

In this note, we shall consider pullbacks of maps α, β where $im\,\alpha = im\,\beta$. Therefore, the following result is relevant:

Lemma 3.4. *For a* Set*-functor F the following are equivalent*:

(1) *F weakly preserves kernel pairs*
(2) *F weakly preserves pullbacks of epis*
(3) *F weakly preserves the pullback of maps α and β for which $im\,\alpha = im\,\beta$.*

The equivalence of 1. and 2. is due to the first author with his student Ch. Henkel, see [13, 9]. The equivalence of 2. and 3. is easily seen by epi-mono-factorization of α and of β. We obtain $\alpha = m \circ \alpha'$ and $\beta = m \circ \beta'$ where m is mono and α' and β' are epi. Then $\mathrm{pb}(\alpha, \beta) = \mathrm{pb}(\alpha', \beta')$.

4. The free-algebra functor

For any nontrivial variety $\mathcal{V}$ of algebras of fixed signature τ, and for a set X of variables, we denote by $F_{\mathcal{V}}(X)$ the $\mathcal{V}$-algebra freely generated by the set X. Its defining property is:

Proposition 4.1. *Given any algebra A of type τ and given any set map $\varphi : X \to A$, there is a unique homomorphism $\tilde{\varphi} : F_{\mathcal{V}}(X) \to A$ such that $\tilde{\varphi} \circ \iota_X = \varphi$ where ι_X denotes the inclusion of variables X in $F_{\mathcal{V}}(X)$.*

$$\begin{tikzcd} F_{\mathcal{V}}(X) \arrow[r, "\tilde{\varphi}"] & A \\ X \arrow[u, hook, "\iota_X"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

In particular, starting with a map $f : X \to Y$, and extending it with ι_Y we obtain a map $\bar{f} : F_{\mathcal{V}}(X) \to F_{\mathcal{V}}(Y)$ as homomorphic extension of $\iota_Y \circ f$.

$$\begin{tikzcd} F_{\mathcal{V}}(X) \arrow[r, "\bar{f}"] & F_{\mathcal{V}}(Y) \\ X \arrow[u, hook, "\iota_X"] \arrow[r, "f"] & Y \arrow[u, hook, "\iota_Y"] \end{tikzcd}$$

It is easy to check that $\overline{id_X} = id_{F_{\mathcal{V}}(X)}$ and $\overline{g \circ f} = \bar{g} \circ \bar{f}$ for a map $g : Y \to Z$, hence $F_{\mathcal{V}}$ with object map $X \mapsto F_{\mathcal{V}}(X)$ and morphism map $f \mapsto \bar{f}$ is a functor. We shall consider $F_{\mathcal{V}}(X)$ as a set and $\bar{f}$ as a set map when considering $F_{\mathcal{V}}$ as a set functor.

If f in the above picture is surjective, then so is $\bar{f}$. This is in fact so for any Set-functor F. Namely, if $f : X \to Y$ is surjective it has a right-inverse f^- such that $f \circ f^- = id_Y$, from which the functor properties yield $F(f) \circ F(f^-) = id_{F(Y)}$, demonstrating that $F(f^-)$ is a right inverse to Ff, which therefore is surjective.

It is interesting to observe, even though it will not be needed for the proof of our main result, that for a free-lattice functor $F_{\mathcal{L}}$, with $\mathcal{L}$ a variety of lattices (without further operations), the converse is almost true:

Proposition 4.2. *If $\mathcal{L}$ is a (quasi-)variety of lattices and $\varphi : F_{\mathcal{L}}(X) \twoheadrightarrow F_{\mathcal{L}}(Y)$ is a surjective homomorphism, then there is a subset $X_0 \subseteqq X$ and a surjective map $f : X_0 \twoheadrightarrow Y$ such that φ restricted to $F_{\mathcal{L}}(X_0)$ is $\bar{f}$.*

$$\begin{tikzcd} F_{\mathcal{L}}(X) \arrow[r, two heads, "\varphi"] & F_{\mathcal{L}}(Y) \\ F_{\mathcal{L}}(X_0) \arrow[u, hook, "\iota"] \arrow[ur, two heads, "\bar{f}"'] & \end{tikzcd}$$

Proof. Each element $y \in Y$ must have a φ-preimage in X, since the free generators in any lattice free in $\mathcal{L}$ are both $\vee$- and $\wedge$-irreducible, see [3]. Collecting these preimages of Y into a subset X_0 of X, let f be the restriction of φ to X_0. By construction, $f : X_0 \to Y$ is surjective, and φ agrees with $\bar{f}$ on $F_{\mathcal{V}}(X_0)$. □

Here we are interested in $F_{\mathcal{L}}$ where $\mathcal{L}$ is any (quasi-)variety of lattices, but we allow additional operations in the signature, as long as the axioms of $\mathcal{L}$

force those operations to be monotonic with respect to the lattice ordering. In short: we assume that $\mathcal{L}$ is a quasi-variety of *lattice-ordered universal algebras.*

If $a, b \in A$ for such a lattice-ordered algebra, we denote by $[a, b]$ the interval

$$[a, b] := \{x \in A \mid a \leq x \leq b\},$$

which is, of course, nonempty iff $a \leq b$. With $F_{\mathcal{L}}(X)$ we continue to denote the free lattice-ordered algebra in $\mathcal{L}$ generated by X.

For the rest of this section, assume that $g : X \twoheadrightarrow Y$ is a surjective map such that for each $y \in Y$ the fibre $g^{-1}(\{y\})$ is finite, and let $\bar{g} : F_{\mathcal{L}}(X) \to F_{\mathcal{L}}(Y)$ be the homomorphic extension of g.

Lemma 4.3. *There are homomorphisms $\breve{g}, \hat{g} : F_{\mathcal{L}}(Y) \to F_{\mathcal{L}}(X)$ such that*

1. $\bar{g} \circ \breve{g} = id = \bar{g} \circ \hat{g}$,
2. $(\hat{g} \circ \bar{g})p \leq p \leq (\breve{g} \circ \bar{g})p$ *for each* $p \in F_{\mathcal{L}}(X)$.

Proof. Let $\hat{g}, \breve{g} : F_{\mathcal{L}}(Y) \to F_{\mathcal{L}}(X)$ be the unique homomorphisms which for all $y \in Y$ are defined by

$$\hat{g}(y) := \bigwedge \{x \in X \mid gx = y\}$$

and dually by

$$\breve{g}(y) := \bigvee \{x \in X \mid gx = y\}.$$

(1) Given $y \in Y$ then

$$\begin{aligned} \bar{g}\hat{g}y &= \bar{g}\left(\bigwedge \{x \mid x \in X, gx = y\}\right) \\ &= \bigwedge \{\bar{g}x \mid x \in X, gx = y\} \\ &= \bigwedge \{gx \mid x \in X, gx = y\} \\ &= \bigwedge \{y\} \\ &= y, \end{aligned}$$

hence $\bar{g} \circ \hat{g}$ (and similarly $\bar{g} \circ \breve{g}$) is the identity.
(2) For each variable $x \in X$ we have $gx \in Y$, hence

$$(\hat{g} \circ \bar{g})x = \hat{g}(gx) = \bigwedge \{x' \in X \mid gx' = gx\} \leq x.$$

For arbitrary terms $p = p(x_1, \ldots, x_n) \in F_{\mathcal{L}}(X)$ where $x_i \in X$, we conclude

$$\begin{aligned} (\hat{g} \circ \bar{g})p &= (\hat{g} \circ \bar{g})p(x_1, \ldots, x_n) \\ &= p(\hat{g}\bar{g}x_1, \ldots, \hat{g}\bar{g}x_n) \\ &\leq p(x_1, \ldots, x_n) \\ &= p, \end{aligned}$$

and dually $(\breve{g} \circ \bar{g})p \geq p$. □

Recall that a pair of order preserving maps $\phi : P \to Q$ and $\psi : Q \to P$ between posets P and Q are said to form a *monotone Galois connection*, if for all $p \in P$ and $q \in Q$ we have $\phi(p) \leq q \iff p \leq \psi(q)$. In this case, (ϕ, ψ) is called an *adjoint pair* with ϕ *left adjoint* to ψ and ψ *right adjoint* to ϕ. With $g, \bar{g}, \hat{g}, \check{g}$ as before, the previous lemma easily yields:

Lemma 4.4. *$\bar{g}$ has both a left adjoint $\hat{g}$ and a right adjoint $\check{g}$ with respect to a monotone Galois connection.*

Proof. What we claim is that $q \leq \bar{g}(p) \iff \hat{g}(q) \leq p$ and $\bar{g}(p) \leq q \iff p \leq \check{g}(q)$ for arbitrary $p \in F_{\mathcal{L}}(X)$, $q \in F_{\mathcal{L}}(Y)$ and $\hat{g}$, $\check{g}$ as defined before. This follows immediately from monotony of $\bar{g}$ and of $\hat{g}$ and claims 1. and 2. of the previous lemma. We need only check the first equivalence, the second one follows by duality:

$$\begin{aligned} \hat{g}q \leq p &\implies \bar{g}\hat{g}q \leq \bar{g}p \\ &\implies q \leq \bar{g}p \\ &\implies \hat{g}q \leq \hat{g}\bar{g}p \leq p. \end{aligned}$$

□

For our weak pullback preservation property we need to consider joint preimages of two points p and q with respect to two homomorphisms $\bar{g_1}$ and $\bar{g_2}$. We show now that they must constitute intervals whose borders are delineated using $\hat{g}_i$ and $\check{g}_i$. We state this in a slightly more general fashion, where $\{p\}$ and $\{q\}$ are generalized to intervals $[p_1, p_2]$ and $[q_1, q_2]$:

Lemma 4.5. *Let $g_1 : X \twoheadrightarrow Y$ and $g_2 : X \twoheadrightarrow Z$ be surjective maps with finite fibres and let $\bar{g}_1 : F_{\mathcal{L}}(X) \to F_{\mathcal{L}}(Y)$, $\bar{g}_2 : F_{\mathcal{L}}(X) \to F_{\mathcal{L}}(Z)$ their homomorphic extensions. For $p_1, p_2 \in F_{\mathcal{L}}(Y)$, $q_1, q_2 \in F_{\mathcal{L}}(Z)$ we have*

$$\bar{g}_1^{-1}[p_1, p_2] \cap \bar{g}_2^{-1}[q_1, q_2] = [\hat{g}_1 p_1 \vee \hat{g}_2 q_1, \check{g}_1 p_2 \wedge \check{g}_2 q_2].$$

Proof. For any $p_1, p_2 \in F_{\mathcal{L}}(Y)$, the equivalence resulting from the previous lemma,

$$p_1 \leq \bar{g}s \leq p_2 \iff \hat{g}p_1 \leq s \leq \check{g}p_2,$$

can be read as

$$\bar{g}^{-1}[p_1, p_2] = [\hat{g}p_1, \check{g}p_2].$$

Therefore

$$\begin{aligned} \bar{g}_1^{-1}[p_1, p_2] \cap \bar{g}_2^{-1}[q_1, q_2] &= [\hat{g}_1 p_1, \check{g}_1 p_2] \cap [\hat{g}_2 q_1, \check{g}_2 q_2] \\ &= [\hat{g}_1 p_1 \vee \hat{g}_2 q_1, \check{g}_1 p_2 \wedge \check{g}_2 q_2]. \end{aligned}$$

□

Specializing to the case $p_1 = p_2 =: p$ and $q_1 = q_2 =: q$ we obtain

$$\bar{g}_1^{-1}\{p\} \cap \bar{g}_2^{-1}\{q\} = [\hat{g}_1 p \vee \hat{g}_2 q \, , \, \check{g}_1 p \wedge \check{g}_2 q].$$

Lemma 4.6. *Given surjective maps with finite fibres $g_1 : X \twoheadrightarrow Y$ and $g_2 : X \twoheadrightarrow Z$ and given elements $p \in F_{\mathcal{L}}(Y)$, $q \in F_{\mathcal{L}}(Z)$, then the following equivalent conditions state that p and q share a common preimage under $\bar{g}_1$ and $\bar{g}_2$ in $F_{\mathcal{L}}(X)$:*

(1) $\bar{g}_1^{-1}\{p\} \cap \bar{g}_2^{-1}\{q\} \neq \emptyset$,
(2) $\hat{g}_1 p \leq \check{g}_2 q$ *and* $\hat{g}_2 q \leq \check{g}_1 p$
(3) $\bar{g}_1 \hat{g}_2 q \leq p$ *and* $\bar{g}_2 \hat{g}_1 p \leq q$.

Proof. Using Lemma 4.5 and Lemma 4.3

$$\begin{aligned} g_1^{-1}\{p\} \cap g_2^{-1}\{q\} \neq \emptyset &\iff \hat{g}_1 p \vee \hat{g}_2 q \leq \check{g}_1 p \wedge \check{g}_2 q \\ &\iff \hat{g}_1 p \leq \check{g}_2 q \text{ and } \hat{g}_2 q \leq \check{g}_1 p \\ &\iff \bar{g}_2 \hat{g}_1 p \leq \bar{g}_2 \check{g}_2 q = q \text{ and } \bar{g}_1 \hat{g}_2 q \leq \bar{g}_1 \check{g}_1 p = p. \end{aligned}$$ □

5. Weak preservation of kernel pairs

It is well known that the complete lattice of congruence relations of any lattice, hence of any lattice-ordered algebra, is distributive, so in particular, $\mathcal{L}$ is *congruence modular*. As a corollary to a result from [9] it therefore follows, that $F_{\mathcal{L}}$ will *not* preserve preimages, hence will *not* weakly preserve all pullbacks. Fortunately, though, this does not preclude $F_{\mathcal{L}}$ from preserving kernel pairs, or equivalently, pullbacks of maps whose images agree. This is in fact what we will prove now. Our main result is:

Theorem 5.1. *For any variety $\mathcal{L}$ of lattice-ordered algebras the functor $F_{\mathcal{L}}$ weakly preserves pullbacks of epis.*

From now on, whenever we denote terms p, q, and s as $p(x_1, \ldots, x_m)$, $q(y_1, \ldots, y_n)$, $s(z_1, \ldots, z_r)$ then we are implying that their variables are mutually different, i.e. $x_i \neq x_j$, $y_i \neq y_j$, $z_i \neq z_j$ unless $i = j$. An equation

$$p(u_1, \ldots, u_m) \approx q(v_1, \ldots, v_n)$$

arises from substituting variables u_i, v_j for x_i and y_j. For that purpose we are allowed to have $u_i = u_j$ or $v_i = v_j$ even when $i \neq j$. We denote the corresponding substitutions by u, resp. v, hence

$$u_i = u(x_i) \text{ and } v_j = v(y_j),$$

so $p(u_1, \ldots, u_m) = p(ux_i, \ldots, ux_m) = \bar{u}\, p(x_1, \ldots, x_m)$ and $q(v_1, \ldots, v_m) = \bar{v}\, q(y_1, \ldots, y_n)$.

An equation $p(u_1, \ldots, u_m) \approx q(v_1, \ldots, v_n)$ is called *balanced*, if the same variables occur on both sides, i.e. $\{u_1, \ldots, u_m\} = \{v_1, \ldots, v_n\}$. With these conventions and with the help of Proposition 3.1, we can equivalently express Theorem 5.1 in purely universal algebraic terms as follows:

Theorem 5.2. *Let $\mathcal{L}$ be a (quasi-)variety of lattice-ordered algebras, and let $p(x_1, \ldots, x_m)$ and $q(y_1, \ldots, y_n)$ be terms satisfying a balanced equation*

$$p(u_1, \ldots, u_m) \approx q(v_1, \ldots, v_n). \tag{5.1}$$

Then there is a term $s(z_1, \ldots, z_k)$ with $k \leq mn$, and variable substitutions $\sigma : \{z_1, \ldots, z_k\} \to \{x_1, \ldots, x_m\}$, and $\tau : \{z_1, \ldots, z_k\} \to \{y_1, \ldots, y_n\}$ so that

$$p(x_1, \ldots, x_m) \approx s(\sigma z_1, \ldots, \sigma z_k) \tag{5.2}$$

$$q(y_1, \ldots, y_m) \approx s(\tau z_1, \ldots, \tau z_k). \tag{5.3}$$

and

$$u \circ \sigma = v \circ \tau. \tag{5.4}$$

The following figure illustrates the situation. Given terms $p(x_1, \ldots, x_m)$, $q(y_1, \ldots, y_m)$ and a balanced equation $p(u_1, \ldots, u_m) \approx q(v_1, \ldots, v_n)$, we find a common ancestor term $s(z_1, \ldots, z_r)$ so that both $p(x_1, \ldots, x_m)$ and $q(y_1, \ldots, y_n)$ are instances modulo the equations of $\mathcal{V}$, of s by means of variable substitutions σ, resp. τ. Applying the substitutions u, resp. v, which defined the original equation $p(u_1, \ldots, u_m) \approx q(v_1, \ldots, v_n)$, we obtain $\gamma := u \circ \sigma = v \circ \tau$, and thus a common substitution instance of s from which the original equation follows trivially by $p(u_1, \ldots, u_m) \approx s(\gamma z_1, \ldots, \gamma z_r) \approx q(v_1, \ldots, v_n)$:

$$\begin{tikzcd}
 & & & & s(z_1, \ldots, z_r) \arrow[dll, mapsto, "\sigma"'] \arrow[dd, mapsto, "\gamma"] \arrow[drr, mapsto, "\tau"] & & & & \\
p(x_1, \ldots, x_m) \arrow[d, mapsto, "u"] & \approx & \overline{\sigma}\, s(z_1, \ldots, z_r) \arrow[d, mapsto, "u"] & \circ & & \circ & \overline{\tau}\, s(z_1, \ldots, z_r) \arrow[d, mapsto, "v"] & \approx & q(y_1, \ldots, y_n) \arrow[d, mapsto, "v"] \\
p(u_1, \ldots, u_m) & \approx & \bar{u}\bar{\sigma}\, s(z_1, \ldots, z_r) & = & \bar{\gamma} s(z_1, \ldots, z_r) & = & \bar{v}\bar{\tau}\, s(z_1, \ldots, z_r) & \approx & q(v_1, \ldots, v_n)
\end{tikzcd}$$

Formally:

$$\begin{aligned}
p(u_1, \ldots, u_m) &= p(ux_1, \ldots, ux_m) && \text{(def. of } u\text{)} \\
&= \bar{u}\, p(x_1, \ldots, x_m) && \text{(by 1.1)} \\
&\approx \bar{u}\, s(\sigma z_1, \ldots, \sigma z_k) && \text{(by 5.2)} \\
&= \bar{u}\, \bar{\sigma}\, s(z_1, \ldots, z_k) && \text{(by 1.1)} \\
&= \overline{u \circ \sigma}\, s(z_1, \ldots, z_k) && \text{(functor property)} \\
&= \overline{v \circ \tau}\, s(z_1, \ldots, z_k) && \text{(by 5.4)} \\
&\approx \cdots && \text{(same arguments in reverse)} \\
&= q(v_1, \ldots, v_n).
\end{aligned}$$

We now come to the proof of Theorem 5.1.

Proof. Let $X := \{x_1, \ldots, x_m\}$, $Y := \{y_1, \ldots, y_n\}$, $U := \{u_1, \ldots, u_m\}$ and $V := \{v_1, \ldots, v_n\}$ be sets of variables with $U = V$, $|X| = m$ and $|Y| = n$. Define $u(x_i) := u_i$ and $v(y_i) := v_i$.

Let $\mathrm{pb}(u, v) = \{(x, y) \in X \times Y \mid ux = vy\}$ be the pullback of u and v. The assumption $U = V =: W$ means that $\operatorname{im} u = \operatorname{im} v = W$, so

$$\forall x \in X. \exists y \in Y.\, ux = vy, \tag{5.5}$$

and symmetrically

$$\forall y \in Y. \exists x \in X.\, ux = vy. \tag{5.6}$$

These statements are equivalent to saying that the projections π_1 and π_2 from the pullback $\mathrm{pb}(u,v)$ to the components X and Y are surjective.

$$\begin{tikzcd} X \arrow[r, two heads, "u"] & W \\ \mathrm{pb}(u,v) \arrow[u, two heads, "\pi_1"] \arrow[r, two heads, "\pi_2"'] & Y \arrow[u, two heads, "v"'] \end{tikzcd}$$

In applying the free-algebra functor $F_{\mathcal{L}}$ to this pullback-diagram, we shall have to consider the elements of $\mathrm{pb}(u,v)$ as variables. To emphasize this, we set $Z := \mathrm{pb}(u,v)$ and write the elements of Z as follows:

$$Z = \{z_{x,y} \mid ux = vy\} = \{z_1, \ldots, z_k\}, \tag{5.7}$$

so we retain the identities

$$\pi_1 z_{x,y} = x \tag{5.8}$$

and

$$\pi_2 z_{x,y} = y. \tag{5.9}$$

In order to show that $F_{\mathcal{L}}$ weakly preserves this pullback, we must verify the conditions spelled out in Proposition 3.1. Thus given terms $p := p(x_1, \ldots, x_m) \in F_{\mathcal{L}}(X)$ and $q := q(y_1, \ldots, y_n) \in F_{\mathcal{L}}(Y)$ and an $\mathcal{L}$-equation $p(u_1, \ldots, u_m) \approx q(v_1, \ldots, v_n)$, we have $\bar{u}\, p(x_1, \ldots, x_m) = \bar{v}\, q(y_1, \ldots, y_n) =: r$ and must find some term $s(z_1, \ldots, z_k)$ such that

$$\bar{\pi}_1 s(z_1, \ldots, z_k) \approx p(x_1, \ldots, x_m) \tag{5.10}$$

and likewise

$$\bar{\pi}_2 s(z_1, \ldots, z_k) \approx q(y_1, \ldots, y_n). \tag{5.11}$$

Thus we are looking for a joint preimage $s(z_1, \ldots, z_k)$ of $p(x_1, \ldots, x_m)$ under $\bar{\pi}_1$ and of $q(y_1, \ldots, y_n)$ under $\bar{\pi}_2$.

This is where Lemma 4.6 comes into play. We shall establish the last of its 3 equivalent conditions, which means that we will prove that $\bar{\pi}_1 \hat{\pi}_2 q \leq p$ and $\bar{\pi}_2 \hat{\pi}_1 p \leq q$. By symmetry, it suffices to consider the first inequality, that is we need to check

$$\bar{\pi}_1 \hat{\pi}_2 q(y_1, \ldots, y_n) \leq p(x_1, \ldots, x_m).$$

Hence the following lemma will complete the proof: □

Lemma 5.3. *For $1 \leq i \leq m$ and $1 \leq j \leq n$ we have $q(\bar{\pi}_1 \hat{\pi}_2 y_1, \ldots, \bar{\pi}_1 \hat{\pi}_2 y_n) \leq p(x_1, \ldots, x_m)$.*

Proof. Let $g := \bar{\pi}_1 \circ \hat{\pi}_2$ then for any fixed $y \in Y$ we have:

$$\begin{aligned} g(y) &= \bar{\pi}_1 \bigwedge \{z_{x,y} \mid u\,x = v\,y\} \\ &= \bigwedge \{\bar{\pi}_1 z_{x,y} \mid u\,x = v\,y\} \\ &= \bigwedge \{x \mid u\,x = v\,y\}. \end{aligned} \tag{5.12}$$

Let v^- be a right inverse to v, which exists, as v is surjective. Observe that $\ker v \subseteq \ker g$, so by Lemma 2.1 there exists a map $h : W \to F_{\mathcal{L}}(X)$ with $h \circ v = g$.

$$\begin{tikzcd}
F_{\mathcal{L}}(X) & X \arrow[l, hook'] \arrow[r, two heads, "u"] & W \arrow[ll, bend right, "h"'] \arrow[d, shift right, "v^-"'] \\
& & Y \arrow[u, two heads, shift right, "v"'] \arrow[ull, bend left, "g"]
\end{tikzcd}$$

It follows that

$$h = h \circ v \circ v^- = g \circ v^-.$$

We now calculate, recalling from Proposition 4.1 that $\tilde{h} : F_{\mathcal{L}}(W) \to F_{\mathcal{L}}(X)$ denotes the homomorphic extension of h:

$$\begin{aligned}
q(\bar{\pi}_1\hat{\pi}_2 y_1, \dots, \bar{\pi}_1\hat{\pi}_2 y_n) &= q(gy_1, \dots, gy_n) \\
&= q(hvy_1, \dots, hvy_n) \\
&= \tilde{h}\, q(vy_1, \dots, vy_n) \\
&= \tilde{h}\, q(v_1, \dots, v_n) \\
&\approx \tilde{h}\, p(u_1, \dots, u_m) \\
&= \tilde{h}\, p(ux_1, \dots, ux_m) \\
&= p(hux_1, \dots, hux_m) \\
&= p(gv^-ux_1, \dots, gv^-ux_m) \\
&\leq p(x_1, \dots, x_m),
\end{aligned}$$

where in the last step we invoked the observation that according to (5.12):

$$\begin{aligned}
gv^-ux_i &= \bigwedge\{x \mid ux = vv^-ux_i\} \\
&= \bigwedge\{x \mid ux = ux_i\} \\
&\leq x_i
\end{aligned}$$

together with the fact that all terms, in particular $p(x_1, \dots, x_m)$, are monotonic in each argument. $\square$

So according to Theorem 5.2 one can always find an ancestor term $s(z_1, \dots, z_k) \in F_{\mathcal{L}}(Z)$ to p and q for any balanced equation $p(u_1, \dots, u_m) \approx q(v_1, \dots, v_n)$. By Lemma 4.5 we conclude:

Theorem 5.4. *The set of all ancestor terms of $p(x_1, \dots, x_m)$ and $q(y_1, \dots, y_n)$ with respect to the balanced equation $p(u_1, \dots, u_m) \approx q(v_1, \dots, v_n)$ is the nonempty interval $[s_0, s_1]$ in $F_{\mathcal{L}}(Z)$ whose bounds are given by*

$$s_0(z_1, \dots, z_k) = p(\hat{\pi}_1 x_1, \dots, \hat{\pi}_1 x_m) \vee q(\hat{\pi}_2 y_1, \dots, \hat{\pi}_2 y_n) \tag{5.13}$$

and

$$s_1(z_1, \dots, z_k) = p(\check{\pi}_1 x_1, \dots, \check{\pi}_1 x_m) \wedge q(\check{\pi}_2 y_1, \dots, \check{\pi}_2 y_n). \tag{5.14}$$

Here $\{z_1, \ldots, z_k\} = \{z_{x_i,y_j} \mid u_i = v_j\}$, *with* $\hat{\pi}_1 x_i = \bigwedge\{z_{x_i,y_j} \mid u_i = v_j\}$ *and* $\check{\pi}_1 x_i = \bigvee\{z_{x_i,y_j} \mid u_i = v_j\}$, *and similarly* $\hat{\pi}_2 y_j = \bigwedge\{z_{x_i,y_j} \mid u_i = v_j\}$ *and* $\check{\pi}_2 y_j = \bigvee\{z_{x_i,y_j} \mid u_i = v_j\}$.

As an exercise, the reader is invited to verify that the equation

$$p(x, x, y) \approx q(x, y, y) \tag{5.15}$$

discussed in the introductory Example 1.1, yields the common ancestor term

$$s_0(z_1, z_2, z_3, z_4) = p(z_1, z_2, z_3 \wedge z_4) \vee q(z_1 \wedge z_2, z_3, z_4).$$

From this we can obtain p and q by identification of variables

$$p(x, y, z) = s(x, y, z, z)$$
$$q(x, y, z) = s(x, x, y, z)$$

so that the original equation (5.15) trivially results from a further common identification:

$$p(x, x, y) = s(x, x, y, y) = q(x, y, y).$$

6. Extending the scope

Looking beyond lattices and lattice-ordered algebras, we find that a theorem analogous to Theorem 5.1 is also true for arbitrary *congruence permutable* varieties, often called *Maltsev varieties*, such as groups, rings, quasigroups, etc.. These varieties are also termed 2-*permutable* in order to emphasize that they belong to the more general class of *n-permutable* varieties. From [9] we quote:

Proposition 6.1. (1) *If $\mathcal{V}$ is a 2-permutable variety then $F_{\mathcal{V}}$ weakly preserves kernel pairs*

(2) *If $\mathcal{V}$ is n-permutable and $F_{\mathcal{V}}$ weakly preserves kernel pairs, then $\mathcal{V}$ is 2-permutable.*

This proposition also serves to document that there are indeed varieties $\mathcal{V}$ for which $F_{\mathcal{V}}$ fails to preserve weak pullbacks: The variety of *implication algebras* is 3-permutable, but not permutable, see [17], hence:

Corollary 6.2. *If $\mathcal{V}$ is the variety of implication algebras, the free-algebra functor $F_{\mathcal{V}}$ does not weakly preserve epi-pullbacks.*

Recall that by Mal'tsev's theorem [15,16] a variety $\mathcal{V}$ is permutable iff there exists a ternary term $m(x, y, z)$ satisfying the equations (1.4) and (1.5).

To a permutable variety $\mathcal{V}$ we can by Prop. 1 add arbitrary function symbols, yet $F_{\mathcal{V}}$ continues to weakly preserve pullbacks. We cannot guarantee this behavior in the case of lattice varieties, unless the operations added are monotonic. In that case we know that weak preservation of pullbacks is maintained. This may suggest that an extensional classification of all varieties $\mathcal{V}$ for which $F_{\mathcal{V}}$ weakly preserves pullbacks could be difficult.

One might try to extend the scope from varieties and free-algebra functors to a larger class of functors. One attempt would be to look at monads, as is done in [2] and in [8]. Every free-algebra functor is part of a monad $M = (F_{\mathcal{V}}, \iota, \mu)$ where $\iota_X : X \to F_{\mathcal{V}}(X)$ and $\mu_X : F_{\mathcal{V}}(F_{\mathcal{V}}(X)) \to F_{\mathcal{V}}(X)$ are obvious natural transformations.

A different perspective can be taken by considering $F_{\mathcal{V}}(-)$ as a *copower functor*. Given an object A in a concrete category $\mathscr{C}$ (with forgetful functor U) and a set X, let $A_{\mathscr{C}}[X]$ be the X-fold direct sum in $\mathscr{C}$ of A with itself, i.e.

$$A_{\mathscr{C}}[X] := \mathrm{U}\left(\coprod_{x\in X} A\right).$$

If $\mathcal{V}$ is a variety and $A \in \mathcal{V}$, then $\coprod_{x\in X} A$ exists in $\mathcal{V}$, as was shown by Sikorski [22], and it is in fact the same as the X-fold *free product* of A with itself, see [4], pp 184 ff.. As a special case, the free algebra with variables from X is the X-fold sum in $\mathcal{V}$ of $F_{\mathcal{V}}(1)$, i.e.

$$F_{\mathcal{V}}(X) \cong \coprod_{x\in X} F_{\mathcal{V}}(1),$$

so the free-algebra functor turns out to be a special instance of a copower functor.

Monoids $\mathcal{M}$ for which the functor $\mathcal{M}_{\mathscr{C}}[-]$ weakly preserves preimages or pullbacks of epis have been characterized with $\mathscr{C}$ being the variety $\mathfrak{M}$ of all monoids [10], the variety $\mathfrak{Mc}$ of all commutative monoids or the variety $\mathfrak{S}$ of all semigroups, see [7]. The relevance of $\mathcal{M}_{\mathfrak{Mc}}[-]$, for instance, arises from the fact that one can argue that this functor models multisets (bags) where the multiplicities of elements are counted by $\mathcal{M}$.

For lattices such an immediate Computer Science application is not yet known, nevertheless would it be interesting to consider $L_{\mathcal{L}}[-]$ where $\mathcal{L}$ is the variety of lattices and L an arbitrary lattice.

7. Uniqueness and pullback preservation

In category theoretical terms, uniqueness of the ancestor term would amount to the free-algebra functor *preserving* pullbacks of epis (not just weakly). However, in [2], the authors prove:

Proposition 7.1. *If $F_{\mathcal{V}}$ preserves pullbacks, then every binary commutative term $t(x, y)$ is a pseudo-constant, i.e. it satisfies $t(x, y) \approx t(z, z)$.*

The term $x \wedge y$ therefore witnesses that for every nontrivial variety $\mathcal{L}$ of lattices, the free-lattice functor $F_{\mathcal{L}}$ does not preserve pullbacks.

Below, we shall need a stronger version of this proposition which, however, builds on the same proof idea. Given an equation $t(u_1, \dots, u_n) \approx t(v_1, \dots, v_n)$, we shall reuse our notation from the proof of Theorem 5.2 and introduce new variables z_{u_i,u_i} as well as z_{u_i,v_i} for $1 \le i \le n$.

Proposition 7.2. *If $F_{\mathcal{V}}$ preserves pullbacks of epis, then each term t satisfying a balanced equation $t(u_1, \dots, u_n) \approx t(v_1, \dots, v_n)$ also satisfies*

$$t(z_{u_1,u_1}, \dots, z_{u_n,u_n}) \approx t(z_{u_1,v_1}, \dots, z_{u_n,v_n}).$$

Proof. For $U = \{u_1, \ldots, u_n\} = \{v_1, \ldots, v_n\}$ consider the constant map $\alpha : U \to \{x\}$ then $\mathrm{pb}(\alpha, \alpha) = \ker \alpha = U \times U$ and

$$\bar{\alpha}\, t(u_1, \ldots, u_n) = t(x, \ldots, x) = \bar{\alpha}\, t(v_1, \ldots, v_n).$$

If $F_{\mathcal{V}}$ *preserves* the pullback of α with itself, there ought be *precisely one* term $s \in F_{\mathcal{V}}(\mathrm{pb}(\alpha, \alpha))$ with $\bar{\pi}_1 s = t(u_1, \ldots, u_n)$ and $\bar{\pi}_2 s = t(v_1, \ldots, v_n)$. However, we can present at least two candidates, namely $s_1 := t((u_1, u_1), \ldots, (u_n, u_n))$ as well as $s_2 := t((u_1, v_1), \ldots, (u_n, v_n))$, since

$$\bar{\pi}_1 s_1 \approx t(u_1, \ldots, u_n) \approx \bar{\pi}_1 s_2,$$

and also

$$\bar{\pi}_2 s_1 \approx t(u_1, \ldots, u_n) \approx t(v_1, \ldots, v_n) \approx \bar{\pi}_2 s_2.$$

Hence

$$s_1 = t((u_1, u_1), \ldots, (u_n, u_n)) \approx t((u_1, v_1), \ldots, (u_n, v_n)) = s_2.$$

Recall that the elements $(u_i, v_j) \in \mathrm{pb}(f, g)$ act as variables in $F_{\mathcal{V}}(\mathrm{pb}(f, g))$, which we emphasize by writing z_{u_i, v_j} for the variable (u_i, v_j) just like in the proof of Theorem 5.2. Thus we infer the equation

$$t(z_{u_1,u_1}, \ldots, z_{u_n,u_n}) \approx t(z_{u_1,v_1}, \ldots, z_{u_n,v_n}). \qquad \square$$

Full preservation of pullbacks seems to be an extremely strong condition in the realm of free-algebra functors. We first demonstrate this for permutable varieties. Given a Maltsev term m as in (1.4) and (1.5), then we can trivially infer the equation

$$m(x, y, y) \approx m(y, y, x).$$

Therefore, assuming that the free-algebra functor for a Maltsev variety $\mathcal{V}$ preserves pullbacks Proposition 7.2 yields the equation

$$m(z_{x,x}, z_{y,y}, z_{y,y}) \approx m(z_{x,y}, z_{y,y}, z_{y,x}),$$

which after renaming of variables can be written as

$$m(x, y, y) \approx m(u, y, v),$$

thereby expressing the fact that m must be independent of its first and third index. With the help of either (1.4) or (1.5), this implies $m(x, y, z) \approx m(y, y, y) \approx y$, showing that m is a projection operation, which, unless $\mathcal{V}$ is trivial, contradicts both (1.4) and (1.5). We conclude:

Corollary 7.3. *For each nontrivial Maltsev variety $\mathcal{V}$ the free-algebra functor $F_{\mathcal{V}}$ does not preserve epi-pullbacks, even though it does preserve them weakly.*

Finally, we test Proposition 7.2 on arbitrary idempotent varieties. Recall that a variety $\mathcal{V}$ is called *idempotent*, when each fundamental operation f satisfies $f(x, \ldots, x) \approx x$.

It has been shown in [8] that for idempotent varieties without constants the free-algebra functor $F_{\mathcal{V}}$ weakly preserves products and pullbacks of constant maps. The following theorem shows that nontrivial idempotent varieties will never (fully) preserve pullbacks:

Theorem 7.4. *If $\mathcal{V}$ is an idempotent variety for which $F_{\mathcal{V}}$ preserves pullbacks of epis, then $\mathcal{V}$ is either trivial (satisfying $x \approx y$) or it contains the "variety of sets" (where all operations are implemented as projections).*

Proof. We employ a result of Olšák [18] stating that any idempotent variety satisfying at least one nontrivial equation (not satisfied in the variety of sets) must have a six-ary term t satisfying the equations

$$t(x,y,y,y,x,x) \approx t(y,x,y,x,y,x) \approx t(y,y,x,x,x,y).$$

Applying Proposition 7.2, while renaming variables $z_{x,x}, z_{y,y}, z_{x,y}, z_{y,x}$ as x, y, z, u we obtain from the first equation the new

$$t(x,y,y,y,x,x) \approx t(z,u,y,u,z,x). \tag{7.1}$$

Hence also

$$t(y,y,x,x,x,y) \approx t(z,u,y,u,z,x),$$

from which a second application of Proposition 7.2 with variables $z_{y,z}, z_{y,u}, z_{x,u}, z_{x,z}$ renamed as a, b, c, d yields the equation

$$t(y,y,x,x,x,y) \approx t(a,b,z,c,d,u), \tag{7.2}$$

which clearly shows that t is independent of any of its arguments, so t defines a pseudo constant. Hence by idempotency

$$x \approx t(x,x,x,x,x,x) \approx t(y,y,y,y,y,y) \approx y. \qquad \square$$

8. Conclusion

We have shown that every balanced equation $p(u_1,\ldots,u_m) \approx q(v_1,\ldots,v_n)$ in free lattice-ordered algebras can be derived from the fact that p and q can be obtained by variable identification from a common ancestor term s, and the mentioned equation arises by further identifying variables until a syntactically identical term is achieved. In category theoretical language this means that the free algebra functor weakly preserves pullbacks of epis.

Finally, we demonstrated that *weak* preservation is all that we can hope for. In fact, the free-algebra functor $F_{\mathcal{V}}$ for any arbitrary idempotent variety $\mathcal{V}$ will not preserve pullbacks of epis unless $\mathcal{V}$ is trivial or contains the "variety of sets".

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