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Inequalities for Generalized Matrix Functions Based on Arbitrary Characters

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1. INTRODUCTION

If G is a subgroup of the symmetric group S_m of degree m, and χ is a character of G over the complex numbers, then the generalized matrix function d_χ^G associated with χ and G is defined on $m \times m$ complex matrices to be

$$d_\chi^G(A) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{m} a_{i,\sigma(i)}, \tag{1}$$

where a_{ij} denotes the (i, j)th entry of A.

Using the techniques of multilinear algebra Marcus [2] and others have been able to prove important inequalities for these generalized matrix functions in the case that the degree of χ is one. We extend these results to the case where χ is an arbitrary character. In doing so we obtain an interesting decomposition of the tensor space [Eq. (7)].

Also the proof of a key lemma is simplified by the use of algebraic derivations.

2. PRELIMINARIES

Let V be an n-dimensional vector space over the complex numbers. Let $\otimes^m V$ denote the tensor product of V with itself m times. Let G be a subgroup of S_m, and let $\chi_1, \ldots, \chi_k$ be all the irreducible characters of G over the complex numbers. Define linear transformations, $P(\sigma)$ and T_i, on $\otimes^m V$, by

$$P(\sigma)(x_1 \otimes \cdots \otimes x_m) = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(m)}, \tag{2}$$

and

$$T_i = \frac{\chi_i(e)}{g} \sum_{\sigma \in G} \chi_i(\sigma) P(\sigma) \tag{3}$$

where g is the order and e is the identity of G.

Using the relation

$$\sum_{\sigma \in G} \chi_i(\sigma)\chi_j(\sigma^{-1}\theta) = \begin{cases} 0, & \text{if } i \neq j, \\ g\,\dfrac{\chi_i(\theta)}{\chi_i(e)}, & \text{if } i = j, \end{cases} \tag{4}$$

it is easily seen that

$$T_i T_j = T_i \delta_{ij}, \tag{5}$$

and that

$$I_{\otimes^m V} = T_1 + \cdots + T_k. \tag{6}$$

Let χ be one of the χ_i's and T the corresponding T_i. Denote the image of $\otimes^m V$ under the map T by $V_\chi^m(G)$, i.e. $V_\chi^m(G) = T(\otimes^m V)$. The image of $x_1 \otimes \cdots \otimes x_m$ under T is denoted $x_1 * \cdots * x_m$.

The above remarks imply that

$$\otimes^m V = V_{\chi_1}^m(G) \oplus \cdots \oplus V_{\chi_k}^m(G). \tag{7}$$

Suppose V has an inner product, denoted by (,). Then this induces an inner product in $\otimes^m V$, again denoted by (,), whose restriction to $V_\chi^m(G)$ satisfies

$$(x_1 * \cdots * x_m, y_1 * \cdots * y_m) = \frac{\chi(e)}{g} d_\chi^G(A), \tag{8}$$

where $A = [(x_i, y_j)]$. Since each $P(\sigma)$ is Hermitian with respect to this inner product, each T_i is Hermitian. Also, the decomposition (7) is orthogonal with respect to this inner product.

As in [2] let $\Gamma_{m,n}$ be the set of all sequences $\omega = (\omega_1, \ldots, \omega_m)$ with $1 \leqslant \omega_i \leqslant n$. Define an equivalence relation, $\sim$, on $\Gamma_{m,n}$ by $\omega \sim \gamma$ if there exists $\sigma \in G$ such that $\gamma = \omega^\sigma = [\omega_{\sigma(1)}, \ldots, \omega_{\sigma(m)}]$. Let Δ be a system of distinct representatives of the equivalence classes. Define

$$\nu(\omega) = \sum_{\sigma \in G_\omega} \chi(\sigma), \tag{9}$$

where $\omega \in \Gamma_{m,n}$ and G_ω is the subgroup of G fixing ω. Define

$$\bar{\Delta} = \{\omega \in \Delta | \nu(\omega) \neq 0\}. \tag{10}$$

Now let $e_1, \ldots, e_n$ be an orthonormal basis of V. Let $e_\omega{}^* = e_{\omega_1} * \cdots * e_{\omega_m}$, $\omega \in \Gamma_{m,n}$. It follows easily from the definitions that if $\alpha, \beta \in \Gamma_{m,n}$

$$(e_\alpha{}^*, e_\beta{}^*) = \begin{cases} 0, & \text{if} \quad \alpha \not\sim \beta, \\ \dfrac{\chi(e)}{g} \displaystyle\sum_{\sigma \in G_\beta} \chi(\sigma\theta), & \text{if} \quad \alpha = \beta^\theta. \end{cases} \tag{11}$$

In particular,

$$(e_\alpha{}^*, e_\alpha{}^*) = \frac{\chi(e)\nu(\alpha)}{g}. \tag{12}$$

3. A BASIC LEMMA

In this section the previous results are used to establish a quadratic relation between the inner product of tensors in $V_\chi{}^m(G)$ and vectors in V. This is a generalization of the Plücker relations for the Grassman space, which was first observed by Marcus [1] in the case of characters of degree one. We have been able to simplify the proof by using derivations instead of derivatives.

First we must introduce some additional notation. Let $v_1, \ldots, v_n$ be another orthonormal basis of V. Let $u_1{}^\omega, \ldots, u_{s_\omega}^\omega$ be an orthonormal basis of the space spanned by

$$W = \{v_{\omega\sigma}^* = v_{\omega_{\sigma(1)}} * \cdots * v_{\omega_{\sigma(m)}} | \sigma \in G\},$$

for all $\omega \in \bar{\Delta}$. We mention that

$$s_\omega = \frac{\chi(e)\nu(\omega)}{g_\omega}, \tag{13}$$

where g_ω is the order of G_ω. To see this let $W^\otimes = \{v_{\omega\sigma}^\otimes = v_{\omega_{\sigma(1)}} \otimes \cdots \otimes v_{\omega_{\sigma(m)}} | \sigma \in G\}$ and let $Q(\sigma)$ be the restriction of $P(\sigma)$ to $W^\otimes$. Let

$$S = \frac{\chi(e)}{g} \sum_{\sigma \in G} \chi(\sigma) Q(\sigma);$$

then S is a linear transformation from $W^{\otimes}$ onto W, and thus the rank of S is s_ω. Since $S^2 = S$, the rank of S is the trace of S. Letting $(\ ,\ )_G$ denote the inner product for group characters in G and viewing Q as a permutation representation of G on the set $\{\omega^\sigma | \sigma \in G\}$, we see that

$$\begin{aligned} s_\omega = \operatorname{tr}(S) &= \frac{\chi(e)}{g} \sum_{\sigma \in G} \chi(\sigma) \operatorname{tr}[Q(\sigma)], \\ &= \chi(e)(\chi, \operatorname{tr} Q)_G, \\ &= \chi(e)(\chi, 1^*_{G_\omega})_G, \\ &= \chi(e)(\chi|_{G_\omega}, 1_{G_\omega})_{G_\omega}, \\ &= \frac{\chi(e)}{g_\omega} \sum_{\sigma \in G_\omega} \chi(\sigma), \\ &= \frac{\chi(e)\nu(\omega)}{g_\omega} \end{aligned}$$

Where $1^*_{G_\omega}$ is the character in G induced from the identically one character in G_ω. We have used Theorem 8.3 from [8] and Frobenius reciprocity.

Lemma 3.1. *Let $v_1, \ldots, v_n$ and $e_1, \ldots, e_n$ be orthonormal bases of V. Let $u_1^\omega, \ldots, u^\omega_{s_\omega}$ be as above. Let $m_t(\omega)$ be the number of times t appears in ω. Then*

$$\sum_{\omega \in \bar{\Delta}} m_t(\omega) \sum_{i=1}^{s_\omega} \left| \left(\left(\frac{g}{\chi(e)\nu(\alpha)} \right)^{1/2} e_\alpha^*, u_i^\omega \right) \right|^2$$

$$= \sum_{i=1}^{n} m_i(\alpha) |(e_i, v_t)|^2, \qquad t = 1, \ldots, n, \qquad \alpha \in \bar{\Delta}.$$

Proof. Let $T: V \to V$ be defined by $Tv_t = v_t$, $Tv_i = 0$, $i \neq t$. Let $L(T)$ denote the restriction of the mth part of the derivation induced by T in the tensor algebra generated by V. That is

$$L(T) = \sum_{i=1}^{m} I * \cdots * \overset{i\text{th}}{T} * \cdots * I,$$

where I is the identity of V and $I * \cdots * \overset{i\text{th}}{T} * \cdots * I$ is the restriction of $I \otimes \cdots \otimes T \otimes \cdots \otimes I$ to $V_\chi^m(G)$.

Suppose $\omega_j = \omega_{\sigma(j)}$ for all j except possibly for $j = i$. Then, since $m_t(\omega) = m_t(\omega^\sigma)$ holds for $t = 1, \ldots, n$, we have $\omega_i = \omega_{\sigma(i)}$ as well; and thus $\omega = \omega^\sigma$. Using this and (8) and (1) we have

$$\begin{aligned}(L(T)e_\omega{}^*, e_\omega{}^*) &= \left(\sum_{i=1}^{m} e_{\omega_1} * \cdots * Te_{\omega_i} * \cdots * e_{\omega_m}, e_{\omega_1} * \cdots * e_{\omega_m}\right),\\ &= \sum_{i=1}^{m} \frac{\chi(e)}{g} \sum_{G} \chi(\sigma) \prod_{j \neq i} (e_{\omega_j}, e_{\omega_{\sigma(j)}})(Te_{\omega_i}, e_{\omega_{\sigma(i)}}),\\ &= \frac{\chi(e)}{g} \sum_{\sigma \in G_\omega} \chi(\sigma) \sum_{i=1}^{m} (Te_{\omega_i}, e_{\omega_i}),\end{aligned}$$

but

$$\begin{aligned}(Te_{\omega_i}, e_{\omega_i}) &= \left(T \sum_{k=1}^{n} (e_{\omega_i}, v_k)v_k, e_{\omega_i}\right),\\ &= |(e_{\omega_i}, v_t)|^2.\end{aligned}$$

It follows that

$$\begin{aligned}(L(T)e_\omega{}^*, e_\omega{}^*) &= \frac{\chi(e)}{g} \sum_{i=1}^{m} |(e_{\omega_i}, v_t)|^2 \sum_{\sigma \in G_\omega} \chi(\sigma),\\ &= \frac{\chi(e)\nu(\omega)}{g} \sum_{j=1}^{n} m_j(\omega)|(e_j, v_t)|^2.\end{aligned}$$

Now

$$\begin{aligned}L(T)v_{\gamma_{\sigma(1)}} * \cdots * v_{\gamma_{\sigma(m)}} &= \sum_{i=1}^{m} v_{\gamma_{\sigma(1)}} * \cdots * Tv_{\gamma_{\sigma(i)}} * \cdots * v_{\gamma_{\sigma(m)}},\\ &= \sum_{\substack{i \text{ with}\\ \gamma_{\sigma(i)}=t}} v_{\gamma\sigma}^* = m_t(\gamma)v_{\gamma\sigma}^*.\end{aligned}$$

Thus $\langle v_{\gamma\sigma}^* | \sigma \in G\rangle$ is an eigenspace of $L(T)$ corresponding to the eigenvalue $m_t(\gamma)$.

Thus

$$\begin{aligned}(L(T)e_\omega{}^*, e_\omega{}^*) &= \left(L(T) \sum_{\bar{\Delta}, i} (e_\omega{}^*, u_i{}^\gamma)u_i{}^\gamma, e_\omega{}^*\right),\\ &= \sum_{\gamma \in \bar{\Delta}} m_t(\gamma) \sum_{i=1}^{s_\gamma} |(e_\omega{}^*, u_i{}^\gamma)|^2,\end{aligned}$$

$$= \frac{\chi(e)\nu(\omega)}{g} \sum_{\gamma \in \bar{\Delta}} m_t(\gamma) \sum_{i=1}^{s_\gamma} \left| \left(\sqrt{\frac{g}{\chi(e)\nu(\omega)}}\, e_\omega^*, u_i^\gamma \right) \right|^2$$

This completes the proof of the lemma. ■

4. THEOREMS ON INEQUALITIES

In this section we list some theorems which may be obtained from the previous material, relegating a sample proof to the end.

THEOREM 4.1. *Let A be an $n \times n$ normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, and let G be a subgroup of S_n and χ an arbitrary character of G. Then*

$$|d_\chi^G(A)| \leqslant \frac{\chi(e)}{n} \sum_{i=1}^{n} |\lambda_i|^n. \tag{15}$$

THEOREM 4.2. *Let A be a positive semidefinite Hermitian $n \times n$ matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. Let $\omega \in \Gamma_{m,n}$ and $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_s$ be the multiplicities of the distinct integers in ω. Then*

$$d_\chi^G A[\omega|\omega] \geqslant \nu(\omega) \prod_{j=1}^{s} \lambda_{n-j+1}^{p_j}, \tag{16}$$

where $A[\omega|\omega]$ is the matrix whose (i, j)th entry is a_{ω_i,ω_j}. Moreover, if A is a real normal matrix with eigenvalues

$$\lambda_1 = |\lambda_1| \exp(i\theta_1), \qquad \lambda_2 = |\lambda_2| \exp(i\theta_2) \qquad ,\ldots, \qquad \lambda_n = |\lambda_n| \exp(i\theta_n),$$

numbered so that

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_n| \quad \textit{and} \quad \theta = \max_{1 \leqslant k \leqslant n} \theta_k \leqslant \pi/2m\ ;$$

and if χ is an arbitrary real valued character of G, then

$$d_\chi^G A[\omega|\omega] \geqslant \nu(\omega) \cos(m\theta) \prod_{j=1}^{s} |\lambda_{n-j+1}|^{p_j}. \tag{17}$$

THEOREM 4.3 (Marcus [6]). *Let A and B be $m \times n$ and $n \times m$ matrices respectively. Then*

$$|d_\chi^G(AB)|^2 \leqslant d_\chi^G(AA^*) d_\chi^G(B^*B). \tag{18}$$

THEOREM 4.4 (Schur [7]; also see [6]). *If A is $m \times m$ positive semidefinite Hermitian matrix then*

$$\chi(e) \det A \leqslant d_\chi^G(A).$$

THEOREM 4.5. *If A is an arbitrary $m \times m$ complex matrix with singular values $\alpha_1, \ldots, \alpha_m$, then*

$$|d_\chi^G(A)|^2 \leqslant \frac{\chi^2(e)}{m} \sum_{i=1}^m \alpha_i^{2m}.$$

THEOREM 4.6. *Let A and B be positive semidefinite Hermitian $n \times n$ complex matrices. Suppose $AB = BA$. Then, if $\omega \in \Gamma_{m,n}$, $m \leqslant n$,*

$$d_\chi^G(A + B)^{1/m}[\omega|\omega] \geqslant d_\chi^G(A^{1/m}[\omega|\omega]) + d_\chi^G(B^{1/m}[\omega|\omega]).$$

The proofs of these theorems are similar to the corresponding theorems for characters of degree one [2, 3]. To illustrate the technique of proof, we will prove the second part of Theorem 4.2. In order to do this we must establish formula (17). Since Δ is an arbitrary system of distinct representatives, we may assume $\omega \in \Delta$. If $\omega \in \Delta - \bar{\Delta}$ then both sides of (17) are zero. So assume $\omega \in \bar{\Delta}$. Let

$$\chi = c_1\xi_1 + \cdots + c_r\xi_r,$$

where each ξ_i is irreducible over the complex numbers and, of course, each c_i is a positive integer.

Let T be the linear transformation defined on V such that

$$Te_i = \sum_{j=1}^n a_{ij}e_j$$

then

$$A[\omega|\omega] = [(Te_{\omega_i}, e_{\omega_j})].$$

Let $v_1, \ldots, v_n$ be an orthonormal set of eigenvectors of T.

Let $K(T)$ denote the restriction of $\otimes^m T$ to $V_{\xi_k}^m(G)$, i.e.

$$K(T)x_1 * \cdots * x_m = Tx_1 * \cdots * Tx_m.$$

Then it follows from this definition and (8) that

$$
\begin{aligned}
d^G_{\xi_k}(A[\omega|\omega]) &= d^G_{\xi_k}[(Te_{\omega_i}, e_{\omega_j})], \\
&= \frac{g}{\xi_k(e)} (K(T)e_\omega{}^*, e_\omega{}^*), \\
&= \nu_k(\omega) \sum_{\gamma\in\bar{\Delta}} \sum_{i=1}^{s_\gamma} c_{\omega,\gamma,i} \left(\prod_{t=1}^{n} |\lambda_t|^{m_t(\gamma)} \right) (\cos \Theta_\gamma + i \sin \Theta_\gamma),
\end{aligned}
$$

where

$$
\Theta_\gamma = \sum_{t=1}^{n} m_t(\gamma)\theta_t, \qquad \nu_k(\omega) = \sum_{\sigma\in G_\omega} \xi_k(\sigma),
$$

and

$$
c_{\omega,\gamma,i} = \left| \left[\left(\frac{g}{\nu_k(\omega)\xi(e)} \right)^{1/2} e_\omega{}^*, u_i{}^\gamma \right] \right|^2 .
$$

Here the "star product," *, is the one obtained by using G and ξ_k.

It follows that

$$
\begin{aligned}
\text{Real part}\{d^G_{\xi_k}(A[\omega|\omega])\} &= \nu_k(\omega) \sum_{\gamma\in\bar{\Delta}} \sum_{i=1}^{s_\gamma} \left(c_{\omega,\gamma,i} \prod_{t=1}^{n} |\lambda_t|^{m_t(\gamma)} \right) \cos \Theta_\gamma, \\
&\geqslant \nu_k(\omega) \cos(m\theta) \prod_{\gamma\in\bar{\Delta}} \prod_{i=1}^{s_\gamma} \prod_{t=1}^{n} |\lambda_t|^{m_t(\gamma) c_{\omega,\gamma,i}}, \\
&= \nu_k(\omega) \cos(m\theta) \prod_{t=1} |\lambda_t|^{\sum_{\gamma\in\bar{\Delta}} \sum_{i=1}^{s_\gamma} m_t(\gamma) c_{\omega,\gamma,i}}, \\
&= \nu_k(\omega) \cos(m\theta) \prod_{t=1}^{n} |\lambda_t|^{\sum_{i=1}^{n} m_i(\omega)|(e_i,\nu_t)|^2}, \\
&\geqslant \nu_k(\omega) \cos(m\theta) \prod_{j=1}^{s} |\lambda_{n-j+1}|^{p_j},
\end{aligned}
$$

where we have used the arithmetic-geometric mean inequality, $\cos \Theta_\gamma \geqslant \cos(m\theta)$, and Lemma 3.1. The last step of the argument is the same as in [1] and [4].

Now, since χ and A are real, we have

$$
d_\chi{}^G(A[\omega|\omega]) = \sum_{k=1}^{r} c_k \text{ Real part}\{d^G_{\xi_k}(A[\omega|\omega])\}
$$

$$\geqslant \sum_{k=1}^{r} c_k \nu_k(\omega) \cos(m\theta) \prod_{j=1}^{s} |\lambda_{n-j+1}|^{p_j}$$

$$= \nu(\omega) \cos(m\theta) \prod_{j=1}^{s} |\lambda_{n-j+1}|^{p_j}.$$

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