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Notes on join semidistributive lattices

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A finite lattice may be regarded as a join semilattice with 0. Using this viewpoint, we give algorithms for testing semidistributivity, provide a new characterization of convex geometries, and characterize congruence lattices of finite join semidistributive lattices.

Keywords: Semilattice; semidistributive lattice; lower bounded lattice; convex geometry; congruence lattice.

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0. Introduction

A finite lattice may be regarded as a join semilattice with 0. In this paper, we discuss certain semidistributivity properties of a finite lattice in terms of its join semilattice presentation. This perspective allows us to prove some new results about finite join semidistributive (JSD) lattices.

If we think in terms of constructing finite lattices with specified properties, then join semilattice presentations are a good approach, but not the only approach.

- Formal Concept Analysis (FCA) practitioners would give very different answers [21]; see e.g. [22–24, 37] for results related to semidistributivity.
- All but the simplest examples should be verified by computer. The discussion following Theorem 13 relates an instance where this proved useful.

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For simplicity, we focus on the finite case, but there are infinite versions; see the last section. Historically, many of the ideas in this paper originated with Alan Day [12], Bjarni Jónsson [30] and Ralph McKenzie [33].

1. Preliminaries

Let $\mathbf{L}$ be a finite lattice. Then $\mathrm{J}(\mathbf{L})$ denotes the set of nonzero join irreducible elements and $\mathrm{M}(\mathbf{L})$ denotes the set of non-one meet irreducible elements. The unique lower cover of a join irreducible element x is denoted x_*. For subsets A, $B \subseteq L$ we say that A *refines* B, written $A \ll B$, if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. Note $A \ll B$ implies $\bigvee A \leq \bigvee B$.

A *join cover* of an element $a \in L$ is a subset B such that $a \leq \bigvee B$, and a join cover is *nontrivial* if $a \nleq b$ for all $b \in B$. A *minimal nontrivial join cover* (mntjc) of a is a join cover $a \leq \bigvee B$ such that if $a \leq \bigvee C$ and $C \ll B$, then $B \subseteq C$. A minimal nontrivial join cover is an antichain of join irreducible elements, with the property that $a \nleq b_* \vee \bigvee(B \backslash \{b\})$ for every $b \in B$. Every join cover in a finite lattice refines to a minimal join cover.

For $A \subseteq L$, let $A^\vee = \{\bigvee B : B \subseteq A\}$; note $0 = \bigvee \varnothing \in A^\vee$. Thus $\mathbf{A}^\vee$ is a join semilattice with 0, and hence a lattice (as $\mathbf{L}$ is finite).

Define a binary relation D on $\mathrm{J}(\mathbf{L})$ by $p \mathrel{\mathrm{D}} q$ if $q \in Q$ for some mntjc Q of p. The congruence relations on $\mathbf{L}$ are in one-to-one correspondence with D-closed subsets of $\mathrm{J}(\mathbf{L})$. The congruence γ corresponding to a D-closed subset C has $\mathbf{L}/\gamma \cong C^\vee$. This connection is explained more fully in Sec. 5.

A *join decomposition* of an element $a \in L$ is a nonempty subset $B \subseteq \mathrm{J}(\mathbf{L})$ such that $a = \bigvee B$. A join decomposition $a = \bigvee B$ is *minimal* if it is minimal as a join cover, i.e. no proper refinement of B joins to a. The decomposition is *irredundant* if no element of B can be omitted: $a \neq \bigvee B \backslash \{b\}$ for all $b \in B$. If a has a unique minimal join decomposition, $a = \bigvee B$, then B is called the *canonical* join decomposition of a.

2. Join Semilattices with 0

We think of a finite lattice as being given by a join semilattice presentation: $\mathbf{L} = \langle X^\vee \mid \mathcal{R} \rangle$ where

- $X = \mathrm{J}(\mathbf{L})$,
- $\mathcal{R}$ is a collection of inclusions $p \leq r$ and (minimal) nontrivial join covers $p \leq \bigvee Q$.

Thus, we are treating the join as a closure operator on $\mathrm{J}(\mathbf{L})$. With respect to the first item, you can tell from a presentation when some $x \in X$ is join reducible, and if so toss it out. Likewise, trivial or non-minimal join covers may be removed from $\mathcal{R}$ in the presentation.

Different bases for a presentation are discussed in [4]. A reasonable choice for the current application would be the D-basis, which includes the covering relation from the order on $\mathrm{J}(\mathbf{L})$ and all minimal nontrivial join covers.

3. Semidistributivity

The properties we are concerned with are lower boundedness, join semidistributivity, meet semidistributivity (MSD), and being a convex geometry.

The *JSD law* is

$$a \vee b = a \vee c \to a \vee b = a \vee (b \wedge c).$$

In terms of our semilattice presentation, the following characterization from Jónsson and Kiefer [29] is especially useful.

Lemma 1. *A finite lattice* $\mathbf{L}$ *is JSD if and only if every element of* $\mathbf{L}$ *has a canonical join decomposition* (*i'.e. a unique non-refinable join decomposition*).

The dual of JSD is MSD. A lattice is *semidistributive* (SD) if it is both JSD and MSD.

A lattice homomorphism $h : \mathbf{K} \to \mathbf{L}$ is *lower bounded* if every congruence class of the kernel $\ker h$ has a least element. A lattice $\mathbf{L}$ is said to *lower bounded* (LB) if it is the image of a lower bounded homomorphism $h : \mathrm{FL}(X) \twoheadrightarrow \mathbf{L}$ where $\mathrm{FL}(X)$ is a finitely generated free lattice.

Lemma 2. *If* $\mathbf{K}$ *is a JSD lattice and* $h : \mathbf{K} \twoheadrightarrow \mathbf{L}$ *is a lower bounded, surjective homomorphism, then* $\mathbf{L}$ *is JSD.*

The proof is straightforward; see [18, Theorem 2.20]. Thus a lower bounded lattice inherits JSD from the free lattice.

Corollary 3. *If* $\mathbf{L}$ *is lower bounded, then it is JSD.*

Since any homomorphism between finite lattices is lower bounded, the same argument gives that finite JSD lattices are closed under homomorphic images.

Corollary 4. *If* $\mathbf{K}$ *is a finite JSD lattice and* $h : \mathbf{K} \twoheadrightarrow \mathbf{L}$ *is a surjective homomorphism, then* $\mathbf{L}$ *is JSD.*

Moreover, there is an easy criterion for a finite lattice to be lower bounded [12].

Lemma 5. *A finite lattice* $\mathbf{L}$ *is lower bounded if and only if it contains no* D*-cycle* $x_0 \mathrel{\mathrm{D}} x_1 \mathrel{\mathrm{D}} x_2 \dots x_{n-1} \mathrel{\mathrm{D}} x_0$.

Sometimes it is easy to see that a semilattice presentation contains no D-cycle. However, not every finite JSD lattice is lower bounded; for example, the lattice of convex subsets of a 4-element chain is JSD but not LB. Therefore, we pursue another criterion, which with variation will apply to MSD and SD lattices as well.

Standard arguments give $\bigcup_{x \in \mathrm{J}(\mathbf{L})} K(x) = \mathrm{M}(\mathbf{L})$. Now, let $X = \mathrm{J}(\mathbf{L})$, so that $\mathbf{L} = X^{\vee}$. For x join irreducible, define $x_{\dagger} = \bigvee\{y \in \mathrm{J}(\mathbf{L}) : y < x\}$, and define

$$K(x) = \{a \in L : a \text{ is maximal with respect to } a \geq x_{\dagger}, a \not\geq x\}.$$

Both $x_{\dagger}$ and $K(x)$ are easily computed from a semilattice presentation. Of course, $x_{\dagger}$ is the computed value of x_*. Once we have checked that $x_{\dagger} < x$, then $x_{\dagger} = x_*$ and no separate notation is needed. The next theorem gives the connections between the map K and semidistributivity.

Theorem 6. *Let* $\mathbf{L}$ *be a finite lattice.*

(1) $\mathbf{L}$ *is JSD if and only if* $x \neq x'$ *implies* $K(x) \cap K(x') = \varnothing$.
(2) $\mathbf{L}$ *is MSD if and only if* $|K(x)| = 1$ *for all* $x \in \mathrm{J}(\mathbf{L})$.
(3) $\mathbf{L}$ *is SD if and only if* K *is a bijection between* $\mathrm{J}(\mathbf{L})$ *and* (*one-element subsets of*) $\mathrm{M}(\mathbf{L})$.

Theorem 6 originated in antiquity, as lattice theory goes, and the proof is elementary. It is given in [18, Theorems 2.54–2.56], see also [15, 16, 19, 20].

Finite JSD lattices form a proper subclass of UC-closure systems, introduced in [3]. UC stands for closure systems with *unique critical* sets, and they are easy to recognize in specific join semilattice representations. On the other hand, determining whether a lattice is JSD from its canonical basis [28] is an open problem, see [3, Problem 51(B)].

4. Convex Geometries

There are many equivalent characterizations of a finite *convex geometry*. These are surveyed, and shown to be equivalent, in [5, Secs. 5-2], or [8, Proposition 2.1], or [39, Theorem 7.2.27]. Let us choose as the definition the most lattice theoretic characterization: *a finite lattice is* (*the lattice of closed sets of*) *a convex geometry if and only if it is JSD and lower semimodular* (*LSM*). We want to relate this definition to the original version due to Dilworth [13], which is directly related to semilattice presentations.

Theorem 7. *A finite lattice* $\mathbf{L}$ *is a convex geometry if and only if every element of* $\mathbf{L}$ *has a unique irredundant join decomposition.*

Having unique irredundant join decompositions is a stronger property than having canonical join decompositions, which by Lemma 1 is equivalent to JSD. The pentagon is a natural example of a lattice that is JSD but not a convex geometry, since its largest element has 2 irredundant decompositions. While a convex geometry can contain many pentagon sublattices, the next result restricts how they can occur.

Lemma 8. *Let* $\mathbf{L}$ *be a finite JSD lattice. Then* $\mathbf{L}$ *is a convex geometry if and only if it has no mntjc* $p \leq \bigvee Q$ *with* $p \in \mathrm{J}(\mathbf{L})$ *and* $q < p$ *for some* $q \in Q$.

In particular, you can tell from its D-basis whether a finite semilattice presentation gives a convex geometry. Note that [40] provides an algorithm for recognizing a join semilattice representation of a convex geometry based on the anti-exchange property of its associated closure operator.

Proof. First, suppose $\mathbf{L}$ contains such a mntjc. Then $\bigvee Q$ is canonical for that element, and we have two distinct irredundant decompositions:

$$\bigvee Q = p \vee \bigvee (Q\backslash \downarrow p).$$

(The second join is irredundant because $Q \ll R$ whenever $\bigvee Q = \bigvee R$.) So $\mathbf{L}$ does not have unique irredundant join decompositions.

Conversely, suppose that in $\mathbf{L}$ we have $a = \bigvee Q$ canonically and also $a = \bigvee P$ irredundantly. Then $Q \ll P$. If $p \in P\backslash Q$, then there exists q_0 with $q_0 < p$ and $q_0 \nleq s$ for all $s \in P\backslash\{p\}$. (Every $q \in Q$ is below some $p \in P$ and $Q \not\ll P\backslash\{p\}$ by the irredundancy of $a = \bigvee P$.) So $q_0 < p \leq \bigvee Q$ nontrivially. Refine the cover of p to a mntjc: $p \leq \bigvee R$ with $R \ll Q$. Now, $a = \bigvee R \vee \bigvee (P\backslash\{p\})$ whence $Q \ll R \cup (P\backslash\{p\})$. Thus $q_0 \leq r$ for some $r \in R$ by the choice of q_0. As $R \ll Q$ and Q is an antichain, this implies $q_0 = r$. Thus, the mntjc $p \leq \bigvee R$ has $r = q_0 < p$. □

This immediately gives a well-known consequence.

Corollary 9. *A finite atomistic JSD lattice is a convex geometry.*

Lemma 8 has a geometrical interpretation via a result of Kashiwabara *et al.* [32], see also [38]. Finite convex geometries can be represented as convex sets of points in some space $\mathbb{R}^n$, extending some convex set of points S. There is a one-to-one correspondence between join irreducibles and points in the configuration outside S and $q < p$ in such representation means that point q is in the convex hull of $S \cup \{p\}$. For $p \leq \bigvee Q$ to be a mntjc means that p is in the convex hull of $S \cup Q$ and all points in Q are *extreme* points of the configuration $S \cup Q$. Clearly, q cannot be an extreme point if $q < p$.

Lemma 8 provides a companion for other lattice-theoretic characterizations of convex geometries. Recall that JSD for finite lattices is characterized by the exclusion of 5 sublattices, one of which is $\mathbf{M}_3$ [31].

Theorem 10. *The following are equivalent for a finite lattice* $\mathbf{L}$:

(1) $\mathbf{L}$ *is a convex geometry.*
(2) $\mathbf{L}$ *is LSM and JSD.*
(3) $\mathbf{L}$ *is LSM and does not contain* $\mathbf{M}_3$ *as a sublattice.*
(4) $\mathbf{L}$ *is LSM and does not contain a covering* $\mathbf{M}_3$ *as a sublattice.*
(5) $\mathbf{L}$ *is JSD and does not contain* $\mathbf{N}_5$ *as a sublattice with its critical quotient being* $[p_*, p]$ *for some* $p \in \mathrm{J}(\mathbf{L})$.

Note that a covering pentagon is not an option in an LSM lattice.

The history of the equivalences in Theorem 10 is recounted in Monjardet [35]; see also [14, 34]. Condition (3) is in Dilworth's original paper [13], while (2) is in Crawley and Dilworth [7], and (4) in Avann [6]. Condition (5) is a consequence of our Lemma 8.

Question. Is there a $K(x)$ version of convex geometries?

5. Congruence Lattices of Finite JSD Lattices

The congruence lattice of a lattice is distributive, and every finite distributive lattice is the congruence lattice of a finite lattice. In the 1990s, the authors observed that not every finite distributive lattice is the congruence lattice of a lower bounded lattice, and somewhat later Grätzer noticed the same thing for slim, planar, semimodular lattices (which are MSD) [26]. We strongly suspect there are similar results that we don't know about. In this section, we tie these together into one nice package.

Recall that a finite distributive lattice $\mathbf{D}$ is isomorphic to the lattice of order ideals of its join irreducible elements, $\mathbf{D} \cong \mathcal{O}(\mathrm{J}(\mathbf{D}))$. Moreover, since join irreducible elements are join prime in a distributive lattice, $\mathrm{J}(\mathbf{D}) \cong \mathrm{M}(\mathbf{D})$ as ordered sets, *via* the map $p \mapsto \kappa(p) = \bigvee\{a \in D : a \ngeq p\}$. This is of course the map K of Sec. 3.

Let us recall some basic notions, which will be applied with $X = \mathrm{J}(\mathbf{L})$ and $\delta = \mathrm{D}$. Given a binary relation δ on a set X, the reflexive, transitive closure of δ yields a quasi-order $\overline{\delta}$. We then form the equivalence relation $\equiv = \overline{\delta} \cap \overline{\delta}^{-1}$, so that $x \equiv y$ if and only if $x \overline{\delta} y$ and $y \overline{\delta} x$. Thus $\overline{\delta}$ becomes a partial order on $X/\equiv$.

Given a quasi-ordered set $\mathbf{Q} = \langle Q, \leq \rangle$, a subset $I \subseteq Q$ is an *ideal* of $\mathbf{Q}$ if $x \leq y$ and $y \in I$ implies $x \in I$. We allow the empty set as an ideal. The set of all ideals of $\mathbf{Q}$, ordered by set inclusion, forms a (completely distributive) lattice, traditionally denoted $\mathcal{O}(\mathbf{Q})$. Dually, $F \subseteq Q$ is a *filter* if $x \leq y$ and $x \in F$ implies $y \in F$. Again the empty set is allowed. The set of all filters of $\mathbf{Q}$, ordered by reverse set inclusion, forms a lattice $\mathcal{F}(\mathbf{Q})$. Moreover, $\mathcal{F}(\mathbf{Q}) \cong \mathcal{O}(\mathbf{Q})$ *via* the set complementation map. Note that $\mathcal{O}(\mathbf{Q}/\equiv)$ is the usual lattice of order ideals of an ordered set.

Now, let us apply these ideas to the set $\mathrm{J}(\mathbf{L})$ of join irreducible elements of a finite lattice and the relation D. Let $\overline{\mathrm{D}}$ be the reflexive, transitive closure of D and $\equiv$ the induced equivalence relation on $\mathrm{J}(\mathbf{L})$. Thus, $\langle \mathrm{J}(\mathbf{L}), \overline{\mathrm{D}} \rangle$ is a quasi-ordered set. Now, consider the map $\sigma : \mathrm{Con}\,\mathbf{L} \to \mathcal{P}(\mathrm{J}(\mathbf{L}))$ given by $\sigma(\theta) = \{p \in \mathrm{J}(\mathbf{L}) : (p, p_*) \notin \theta\}$. The crucial observation is that $\sigma(\theta)$ is a D-closed set: if $p \in \sigma(\theta)$ and $p \mathrel{\mathrm{D}} q$, then $q \in \sigma(\theta)$. On the other hand, if C is a D-closed subset of $\mathrm{J}(\mathbf{L})$, then $C = \sigma(\ker h)$ for the homomorphism $h : \mathbf{L} \to C^{\vee}$ with $h(x) = \bigvee({\downarrow}\, x \cap C)$. Thus $\mathrm{Con}\,\mathbf{L} \cong \mathcal{F}(\mathrm{J}(\mathbf{L})) \cong \mathcal{O}(\mathrm{J}(\mathbf{L}))$, where $\mathrm{J}(\mathbf{L})$ is quasi-ordered by $\overline{\mathrm{D}}$. The details are given in [36, Chap. 10], especially [18, Theorems 10.5 and 3.11 to Corollary 3.16]; see [17] for computational aspects.

In view of the above connection, for $p \in \mathrm{J}(\mathbf{L})$ let ψ_p denote the largest congruence separating p and p_*. The associated D-closed set is $\{q \in \mathrm{J}(\mathbf{L}) : p \mathrel{\overline{\mathrm{D}}} q\}$. Every join irreducible congruence relation is of the form ψ_p with $p \in \mathrm{J}(\mathbf{L})$.

We need a couple of facts about JSD lattices.

Lemma 11. (1) *In a finite JSD lattice, the canonical joinands of* 1 *are join prime.*

(2) *More generally, in any JSD lattice with a largest element* 1, *a maximal ideal not containing* 1 *is prime.*

Part (1) is from [29]; see also [2]. Part (2) is from [25]; its proof is an elementary exercise. Note that a lattice has a proper prime ideal if and only if it has **2** as a homomorphic image.

We are now in position to prove a characterization of congruences of finite JSD lattices.

Theorem 12. *The following are equivalent for a finite distributive lattice* $\mathbf{D}$*. Let* $\mathbf{D} \cong \mathcal{O}(\mathbf{P})$ *for an ordered set* $\mathbf{P}$ *(isomorphic to* $\mathrm{J}(\mathbf{D})$*) and let* M *denote the maximal members of* $\mathbf{P}$

(1) $\mathbf{D} \cong \mathrm{Con}\,\mathbf{L}$ *for a finite JSD lattice* $\mathbf{L}$*.*
(2) $\mathbf{D} \cong \mathrm{Con}\,\mathbf{S}$ *for a finite lower bounded lattice* $\mathbf{S}$*.*
(3) $\mathbf{D} \cong \mathrm{Con}\,\mathbf{G}$ *for a finite convex geometry* $\mathbf{G}$*.*
(4) $\mathbf{D} \cong \mathrm{Con}\,\mathbf{A}$ *for a finite, lower bounded, atomistic convex geometry* $\mathbf{A}$*.*
(5) *For all* $x \in P \backslash M$, $|\uparrow x \cap M| \geq 2$.

Proof. Of course (4) implies (2) and (3), each of which implies (1).

Assume (1), that $\mathbf{D} \cong \mathrm{Con}\,\mathbf{L}$ with $\mathbf{L}$ JSD and $|L| > 2$. Then $\mathbf{P} \cong \mathrm{J}(\mathbf{L})/\equiv$ where $\equiv$ is the equivalence relation associated with the quasi-order $\overline{\mathrm{D}}$. If φ is a maximal meet irreducible congruence on $\mathbf{L}$, then $\varphi = \psi_p$ for some join irreducible element p. Moreover, φ is a coatom in $\mathrm{Con}\,\mathbf{L}$, which means that L/φ is a simple JSD lattice by Corollary 4. Since every finite JSD lattice has $\mathbf{2}$ as a homomorphic image, $\mathbf{L}/\varphi \cong \mathbf{2}$, whence p is join prime.

Now, let $\mu = \{p\} \succ S = \nu$ in $\mathrm{J}(\mathbf{L})/\equiv$. We want to show that ν has another upper cover besides $\{p\}$ in $\mathrm{J}(\mathbf{L})/\equiv$. Suppose not. Then for every $s \in S$, $s \leq \bigvee T$ a mntjc implies $T \subseteq S \cup \{p\}$. In other words, $S \cup \{p\}$ is a D-closed subset, whence $(S \cup \{p\})^{\vee}$ is a finite JSD lattice with only one join prime element. But in a finite JSD lattice the canonical joinands of 1 are join prime; and if 1 is join irreducible then the canonical joinands of 1_* are join prime. *Every finite JSD lattice except* $\mathbf{2}$ *contains at least* 2 *nonzero join prime elements.* So this would be a contradiction unless ν has another upper cover, giving (1) implies (5).

Finally let $\mathbf{P}$ be an ordered set with the property of (5). We will construct a finite, lower bounded, convex geometry $\mathbf{A} = P^{\vee}$ such that $\mathrm{Con}\,\mathbf{A} \cong \mathbf{D}$. Define the lattice $P^{\vee}$ with the antichain order on P and mntjc's $x \leq \bigvee\{y \in P : y \succ x\}$ for $x \notin M$, unless x has only one cover in $\mathbf{P}$. If x has only one upper cover y, then $y \notin M$, so choose $z \succ y$ and add the join cover $x \leq y \vee z$. There are no cycles, as $x \mathrel{\mathrm{D}} y$ implies $x \leq y$ in $\mathbf{P}$. So the lattice with this presentation is atomistic and lower bounded, whence $P^{\vee}$ is a convex geometry by Corollaries 3 and 9. By construction, the D-closed sets are filters of $\mathbf{P}$. Thus $\mathrm{Con}\,P^{\vee}$ is isomorphic to the lattice of filters of $\mathbf{P}$ ordered by reverse set inclusion, which is isomorphic to the lattice of order ideals $\mathcal{O}(\mathbf{P})$ ordered by inclusion. Therefore (5) implies (4). $\square$

Czédli and Kurusa [11] observed that finite slim, planar, semimodular lattices (in short, SPS lattices) are precisely the duals of convex geometries $\mathbf{G}$ with convex dimension $\mathrm{cdim}(\mathbf{G}) \leq 2$. Czédli and G. Grätzer found additional properties of congruence lattices of SPS lattices [9, 10, 27], which by Theorem 12 do not hold in congruence lattices of all convex geometries. On the other hand, the convex geometries constructed in the proof of the theorem are lower bounded lattices. This class of convex geometries is well-behaved. In particular, a finite convex geometry without D-cycles has a tractable optimum basis [1].

6. A Simple Semidistributive Lattice

Lemma 11 shows that every JSD lattice with 1 has **2** as a homomorphic image. The dual applies to any MSD lattice with 0. In particular, there is no finite, simple JSD lattice (or MSD lattice) except **2**. So Ralph McKenzie asked: *Is there a simple SD lattice besides* **2**?

In an earlier paper [20], the authors were able to adapt the methods discussed here to construct such an example, using the following notion. A lattice L is *strongly locally finite* if every interval $[u, v]$ is finite. Every strongly locally finite lattice is the union of a directed set of finite interval sublattices, and the methods described here apply to each of these. The final result is given as an infinite semilattice presentation $\langle X^\vee \mid \mathcal{R}\rangle$.

Theorem 13. *There is an infinite, simple SD lattice.*

Without reproducing the details, let us mention a couple of features of the construction. The lattice constructed for Theorem 13 is a union of intervals $[u_i, v_i]$ $(i \in \omega)$ with $u_{i+1} < u_i < v_i < v_{i+1}$. There are 6 types of join irreducible elements in X and besides order relations on X, 8 types of mntjc's in $\mathcal{R}$, generating multiple D-cycles. Theorem 6 allows us to check that each interval $[u_i, v_i]$ satisfies SD without calculating the entire interval: we need only find $K(x)$ for each join irreducible x in the interval. The calculations are complicated, but our lattice programs allow us to verify that the claimed values for $K(x)$ are correct, at least for "small" intervals.

For finite lattices, JSD is equivalent to the Jónsson–Kiefer Property [29]: every element a of a finite JSD lattice is a join of elements that are join prime in the ideal $\downarrow a$. The paper [2] gives an example of a dually algebraic, JSD lattice that contains no join prime nor meet prime element, and gives sufficient conditions for the Jónsson–Kiefer Property to hold in an infinite JSD lattice.

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