SOME ORDER THEORETIC QUESTIONS
ABOUT FREE LATTICES AND FREE MODULAR LATTICES

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ABSTRACT

In this paper we look at some of the problems on free lattices and free modular lattices which are of an order theoretic nature. We review some of the known results, give some new results, and present several open problems.

Every countable partially ordered set can be order embedded into a countable free lattice [6]. However, free lattices contain no uncountable chains [25], so the above result does not extend to arbitrary partially ordered sets. The problem of which partially ordered sets can be embedded into a free lattice is open. It is not enough to require that the partially ordered set does not have any uncountable chains. In fact, there are partially ordered sets of height one, which cannot be embedded into any free lattice [23]. The importance of these concepts to projective lattices is discussed.

If $a > b$ and there is no $c$ with $a > c > b$ we say *a covers b* and write $a \succ b$. Covers in free lattices and free modular lattices are important to lattice structure theory. We discuss the connection between covers and structure theory and give some of the more important results about covers. Alan Day has shown that every quotient sublattice (i.e. interval) of a finitely generated free lattice contains a covering [9]. R.A. Dean, on the other hand, has some results on noncovers in free lattices. The analogous problems for free modular lattices are open.

Suppose $w(x_1,\ldots,x_n)$ is a lattice word and $L$ is a lattice. If we replace all but one of the variables with fixed elements from $L$ we obtain a function $f(x) = w(x,a_2,\ldots,a_n)$ from $L$ to $L$.



*I. Rival (ed.), Ordered Sets, 355–377.*

Call functions of this form unary polynomials; they always preserve order. For which lattices does every unary polynomial have a fixed point? We give some results on this new problem without completely solving it.

The word problem for free modular lattices with five or more generators is unsolvable [22]. The solvability of the word problem for FM(4) is open. There is evidence that four generated subdirectly irreducible lattices form a much more restricted class than five generated ones [26]. However, we will give a proof that there are $2^{\aleph_0}$ four generated simple modular lattices. We will also discuss Herrmann's result that the free modular lattice generated by two complemented pairs has a solvable word problem and the prospects for solving the word problem for FM(4). These problems are related to covers in FM(4) and various problems on covers in free modular lattices will be discussed.

We also give Wille's classification of those partially ordered sets *P* such that the free modular lattice generated by *P* is finite [50].

In this paper we examine some of the problems on free lattices and free modular lattices which are of an order theoretic nature. Many are combinatorial. We review some of the known results and show their importance to lattice structure theory. We also give some new results. Several open problems, both old and new, are scattered throughout the text.

## §1 PARTIALLY ORDERED SETS IN FREE LATTICES

If $X$ is a set, then the free lattice on $X$ is a lattice, denoted FL($X$), generated by $X$ such that any set map from $X$ to a lattice can be extended to a homomorphism. The free modular and distributive lattices are defined similarly and denoted FM($X$) and FD($X$). Free lattices have an interesting arithmetical and combinatorial structure as can be seen from the papers of Whitman, Dilworth, Dean, and Jónsson. Many of the important questions about free lattices have to do with their order theory.

The definition of free lattices guarantees that every lattice is a homomorphic image of a free lattice. However, not all lattices can be embedded into a free lattice. Free lattices satisfy the following conditions [32]:

(W) $u = \bigwedge u_i \leq \bigvee v_j = v$ implies either $u_i \leq v$ for some $i$ or $u \leq v_j$ for some $j$,

(SD) $u \vee v = u \vee w$ implies $u \vee v = u \vee (v \wedge w)$
$u \wedge v = u \wedge w$ implies $u \wedge v = u \wedge (v \vee w)$.

These are rather restrictive conditions. Very recently, J.B. Nation has shown that *a finite lattice can be embedded into a free lattice if and only if it satisfies* (W) *and* (SD), proving a long-standing conjecture of Bjarni Jónsson. The problem of characterizing infinite sublattices of free lattices remains open. A. Kostinsky does have a characterization of finitely generated sublattices (see [37]).

If $f$: FL($X$) $\to$ L is an onto homomorphism, then $f^{-1}(a) = \{u \in \text{FL}(X): f(u) = a\}$ is a convex sublattice of FL($X$) for each $a \in L$; i.e., if $u, v \in f^{-1}(a)$ and $u \leq w \leq v$ then $w \in f^{-1}(a)$. $L$ is *projective* if there is a $u_a \in f^{-1}(a)$ such that the $u_a$'s form a sublattice isomorphic to $L$. Call a map $g$: $L \to \mathrm{FL}(X)$ a *transversal* for $f$ if $g(a) \in f^{-1}(a)$. Then $L$ is projective if $f$ has a transversal which is a homomorphism. In particular, if $L$ is projective, $f$ has an order-preserving transversal. *For which lattices does $f$: $\mathrm{FL}(X) \to L$ have an order-preserving transversal?* This property is important since it together with three other conditions form necessary and sufficient conditions for $L$ to be projective. If $L$ is countable, then $f$: $\mathrm{FL}(X) \to L$ does have an order-preserving transversal (Crawley and Dean [6]). In fact if $L = \{a_0, a_1, a_2, \ldots\}$ then define

$$g(a_n) = (g_0(a_n) \vee \bigvee(g(a_i): a_i < a_n \text{ and } i < n))$$
$$\wedge \bigwedge(g(a_j): a_j > a_n \text{ and } j < n),$$

where $g_0(a_n)$ is any element from $f^{-1}(a_n)$. On the other hand, if $L$ contains an uncountable chain, then $f$ has no order-preserving transversal. This follows from Galvin and Jónsson's theorem that $\mathrm{FL}(X)$ *has no uncountable chains*. Galvin and Jónsson gave the following ingenious proof. Let $G$ be the group of those automorphisms of $\mathrm{FL}(X)$ induced from permutations of $X$ which move only finitely many elements. The orbits of $G$ are antichains since if $u < u\sigma$ for $\sigma \in G$ then $\sigma^n = 1$ for some $n$ and hence $u < u\sigma < u\sigma^2 < \ldots < u\sigma^n = u$. Let $X_0$ be a countable subset of $X$. Each element $w$ of $\mathrm{FL}(X)$ is a word in finitely many letters. Clearly we can find a $\sigma \in G$ so that $w\sigma$ is a word in $X_0$. Since $\mathrm{FL}(X_0)$ is countable, it follows that $G$ has countably many orbits. *Thus* $\mathrm{FL}(X)$ *is a countable union of antichains and thus contains no uncountable chain.* These arguments work for every variety of lattices and also for Boolean algebras.

The problem of when $f$ has an order transversal is solved by the following theorem.

THEOREM 1. *Let $f$ be a homomorphism from* $\mathrm{FL}(X)$ *onto $L$. Then $f$ has an order-preserving transversal if and only if $L$ satisfies*

(*) *for each element $a$ in $L$ there is a finite subset $S(a)$ of $L$ such that if $a \le b$ in $L$ then there is a $c \in S(a) \cap S(b)$ with $a \le c \le b$.*

If $L$ satisfies (*), then $L$ can be arranged into a well-ordered sequence $(a_\alpha: \alpha < \kappa)$ such that each $b \in S(a)$ either comes before $a$ or at worst only finitely many places after $a$. We define $g$ on $a_\alpha$ inductively. First choose any inverse image $u_\alpha$ of $a_\alpha$. Then just as in Crawley's and Dean's proof above, adjust $u_\alpha$

so that its order is correct with respect to those elements of $S(a_\alpha)$ that come before $a_\alpha$ and the (finitely many) elements which directly precede $a_\alpha$ back to the last limit ordinal. It is easy to see that $g$ is an order-preserving transversal.

The proof of the converse is based on the fact that each element of FL($X$) is in a sublattice generated by a finite subset of $X$.

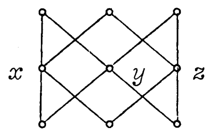
For an application of this theorem consider the ordinal sum of two free lattices FL($X$) and FL($Y$) (this is the lattice on FL($X$) $\cup$ FL($Y$) with the additional order relation $u \leq v$ for all $u \in$ FL($X$) and $v \in$ FL($Y$)). We denote this lattice FL($X$) $\dot{+}$ FL($Y$). It is a little surprising that FL($X$) $\dot{+}$ FL($Y$) is projective if and only if either $X$ or $Y$ is finite, or they are both countable. To see that FL($X$) $\dot{+}$ FL($Y$) is not projective when $Y$ is infinite and $X$ is uncountable suppose it satisfied (*). Let $X_i = \{x \in X: |S(x)| \leq i\}$. Then $X = \bigcup_{i<\omega} X_i$ and since $X$ is uncountable there is a $k$ with $X_k$ infinite.

An element in a free lattice can be below at most finitely many generators. Choose $y_1 \in Y$. Since each $a \in S(y_1) \cap$ FL($Y$) is below at most finitely many elements of $Y$, and $S(y_1) \cap$ FL($Y$) is finite, but $Y$ is infinite, there is a $y_2 \in Y$ such that $y_2 \not\leq a$ for all $a \in S(y_1) \cap$ FL($Y$). Continuing in this way we can find $y_1, \ldots, y_{k+1} \in Y$ such that $y_j \not\leq a$ for all $a \in$ FL($Y$) $\cap \bigcup_{i<j} S(y_i)$. For each $x \in X_k$ and for each $i$ there is an element $a_{xy_i} \in S(x) \cap S(y_i)$ with $x \leq a_{xy_i} \leq y_i$. Again, since $S(y_i)$ is finite, the dual of the above remark implies that $a_{xy_i} \in$ FL($X$) for at most finitely many $x \in X_k$. Since $X_k$ is infinite, we can choose $x \in X_k$ such that $a_{xy_i} \in$ FL($Y$) for $i = 1, \ldots, k + 1$. But for such an $x$ the $a_{xy_i}$ must be distinct. Hence $|S(x)| \geq k + 1$, a contradiction.

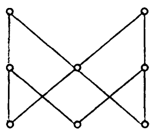
A related but unsolved problem is: *which partially ordered sets can be order embedded into a free lattice?* All of the lattices FL($X$) $\dot{+}$ FL($Y$) are sublattices of free lattices since they are sublattices of FL($X$) $\dot{+}$ $\underset{\sim}{1}$ $\dot{+}$ FL($Y$), which are in fact projective [23]. Thus the complete bipartite partially ordered set induced on $X \cup Y$ can be embedded into a free lattice.

Since every partially ordered set can be embedded into a lattice, Crawley's and Dean's result can be used to show that FL($X$), $|X| = \aleph_0$, *is an* $\aleph_0$*-universal partially ordered set.* That is, every countable partially ordered set can be order embedded into FL($X$). The same proof in fact shows that FD($X$) *is an* $\aleph_0$*-universal partially ordered set.* On the other hand the result of Galvin and Jónsson shows that if $P$ contains an uncountable chain then it cannot be embedded into a free lattice. With a more involved argument one can show that *the partially ordered set induced from the atoms and coatoms of the Boolean algebra of subsets of an uncountable set cannot be embedded into a free lattice* [23]. Thus not every bipartite partially ordered set can be embedded into a free lattice. In fact the problem of *which bipartite partially ordered sets can be embedded into a free lattice* is open.

Excluded partially ordered sets play a role in Nation's proof of Jónsson's conjecture cited above. If a lattice $L$ satisfies (W) and the following partially ordered set is embedded




in $L$, then $x \nleq y \vee z$ and cyclically and dually. It follows from [49] that the sublattice generated by $x$, $y$, and $z$ is free and hence infinite. Actually only five of the six relations of the form $x \nleq y \vee z$ are necessary to force $L$ to be infinite. In fact if $L$ contains



then $L$ must be infinite [35]. Ivan Rival has a similar result for lattices of breadth two which satisfy (W) [43].

## §2 COVERS IN FREE LATTICES

We say that $a$ *covers* $b$ in a lattice if $a > b$ and there is no element $c$ with $a > c > b$. We write $a \succ b$. If $X$ is infinite FL($X$) has no coverings. However, finitely generated free lattices have many coverings. Coverings in free lattices and free modular lattices are particularly important to lattice theory. If $a \succ b$ in a lattice $L$, then by Dilworth's characterization of lattice congruences there is a unique maximal congruence $\psi$ on $L$ with $(a, b) \notin \psi$. If $L$ is a free lattice, then $L/\psi$ is a finite, subdirectly irreducible lattice. Lattices of this form are called *splitting lattices* and were extensively studied by Ralph McKenzie [37]. By Jónsson's celebrated theorem, a subdirectly irreducible lattice cannot be in the join of finitely many lattice varieties without being in one of them. A splitting lattice cannot be in the join of an arbitrary collection of lattice varieties without being in one of them. Because of this property splitting lattices have played a very important role in lattice structure theory. Splitting modular lattices have even stronger structural properties (see §5).

McKenzie has shown that *one can recursively decide for lattice words* $u(x_1,\ldots,x_n)$ *and* $v(x_1,\ldots,x_n)$ *if* $u \succ v$ *in* $\mathrm{FL}(x_1,\ldots,x_n)$ [37]. Perhaps the most important result on coverings is that of Alan Day who shows that the *finitely generated free lattices are weakly atomic*. A lattice is *weakly atomic* if $a < b$ implies there exist $c$ and $d$ with $a \leq c \prec d \leq b$. Thus there is an abundance of coverings and thus of splitting lattices.

The next theorem contrasts these results by showing that free lattices do not satisfy stronger atomicity properties.

THEOREM 2 (R.A. Dean, unpublished). *The element* $x \vee (y \wedge z)$ *of* $\mathrm{FL}(x, y, z)$ *has no cover.*

To prove this we define the length of a lattice word to be the number of join and meet signs which appear in it and length of an element of $\mathrm{FL}(X)$ as the minimum of the lengths of the words which represent the element.

Now suppose $x \vee (y \wedge z) \prec w$ in $\mathrm{FL}(x, y, z)$. Then $w = x \vee (y \wedge z) \vee w$ and we can choose an element $u$ of minimal length such that $w = x \vee (y \wedge z) \vee u$. Since $x \vee y$ does not cover $x \vee (y \wedge z)$, $u$ cannot be $y$ or $z$ or $x$. By its minimality $u$ must be join-irreducible. Thus $u = \bigwedge u_i$ where each $u_i$ has smaller length than the length of $u$. Notice also that $u \leq y$ and $u \leq z$ cannot both hold. Thus we may assume $u \nleq y$. Now

$$x \vee (y \wedge z) \leq x \vee (y \wedge z) \vee (u \wedge y) \leq x \vee (y \wedge z) \vee u = w.$$

Since $x \vee (y \wedge z) \prec w$, we must have one equality. If $x \vee (y \wedge z) \vee (u \wedge y) = x \vee (y \wedge z) \vee u = w$, then $u \leq x \vee (y \wedge z) \vee (u \wedge y)$. Applying (W) we get that for some $i, u_i \leq x \vee (y \wedge z) \vee (u \wedge y) = w$. But $u \leq u_i$ and so

$w = x \vee (y \wedge z) \vee u_i$, violating the minimality of $u$.

If $x \vee (y \wedge z) = x \vee (y \wedge z) \vee (u \wedge y)$, then, since $x$, $y$, and $z$ are meet-prime, a few applications of (W) gives us $u \leq z$. Let $\sigma \in \mathrm{Aut}(\mathrm{FL}(x, y, z))$ be the automorphism with $x\sigma = x$, $y\sigma = z$, and $z\sigma = y$. Since $u \leq z$ and $u \nleq y$, $u\sigma \neq u$ and so $u\sigma$ and $u$ are incomparable. However $(x \vee (y \wedge z))\sigma = x \vee (y \wedge z)$ and thus $x \vee (y \wedge z) \prec w\sigma$. (W) implies that $x \vee (y \wedge z)$ is meet-irreducible and hence can have at most one cover. Thus $w\sigma = w$; that is, $x \vee (y \wedge z) \vee u = x \vee (y \wedge z) \vee (u\sigma)$. This implies $u \leq x \vee (y \wedge z) \vee (u\sigma)$. Since $u \nleq x$ or $y \wedge z$ or $u\sigma$, (W) yields that for some $i, u_i \leq x \vee (y \wedge z) \vee (u\sigma) = w$, which again violates the minimality of $u$.

## §3 LATTICES FREELY GENERATED BY PARTIALLY ORDERED SETS

The lattice freely generated by a partially ordered set $P$, denoted $\mathrm{FL}(P)$, is defined by the property that any order preserving map from $P$ into a lattice $L$ can be uniquely extended to a homomorphism from $\mathrm{FL}(P)$ to $L$. The structure of these lattices was described by Dilworth [16] and Dean [11]. They are projective if and only if $P$ satisfies (*) of Theorem 1. M.E. Adams and D. Kelly [1] have shown that *if $P$ has no chains of cardinality $\kappa$ then neither does* FL(P) ($\kappa$ an infinite cardinal). Moreover this result holds in any variety of lattices. In particular it holds for modular and distributive lattices.

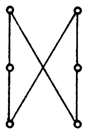
R. Wille has shown that *if $P$ is a finite partially ordered set then* $\mathrm{FL}(P)$ *is finite if and only if neither* $1 + 1 + 1$, $2 + 2$, *nor* $1 + 4$ *can be embedded in $P$.* Here, for example, $2 + 2$ is the partially ordered set consisting of two two-element chains.

In a related problem I. Rival and R. Wille [45] characterized those $P$ for which $\mathrm{FL}(P)$ can be "drawn."

THEOREM 3. *Let $P$ be a finite partially ordered set. The following are equivalent*:

(i) $\mathrm{FL}(x, y, z)$ *is not a sublattice of* $\mathrm{FL}(P)$;
(ii) $\mathrm{FL}(P)$ *has breadth at most* 3;
(iii) *neither* $1 + 1 + 1$, $2 + 3$, *nor* $1 + 5$ *can be order embedded into $P$*;
(iv) $\mathrm{FL}(P)$ *is a sublattice of* $\mathrm{FL}(H)$.

Here $H$ is the following partially ordered set:

H = 

One will readily agree that FL($x$, $y$, $z$) cannot be drawn. For example FL($X$), $|X| = \aleph_0$, is a sublattice of FL($x$, $y$, $z$). To show FL($H$) is "drawable," they draw it.

The lattices FL(2 + 2) and FL(1 + 4) are both infinite but have nice diagrams given by H. Rolf in [46].

Wille has also characterized those partially ordered sets $P$ with FM($P$) finite. (FM($P$) is the free modular lattice generated by $P$.)

THEOREM 4. *For a partially ordered set $P$ the following are equivalent:*

(i) FM($P$) *is finite;*
(ii) FM($P$) *is a subdirect product of copies of the two element lattice and of* $M_3$;
(iii) *neither* 1 + 1 + 1 + 1 *nor* 1 + 2 + 2 *can be embedded in* $P$.

In a similar vein, I. Rival, W. Ruckelshausen, and B. Sands have shown that a certain infinite partially ordered set they call the "herringbone" can be order embedded into every infinite, finitely generated lattice of finite width [41]. Dwight Duffus and Ivan Rival have shown that if a distributive lattice $D$ can be order embedded into a lattice $L$ then $D$ is a homomorphic image of a sublattice of $L$ [18].

## §4 FIXED POINTS FOR LATTICE POLYNOMIALS

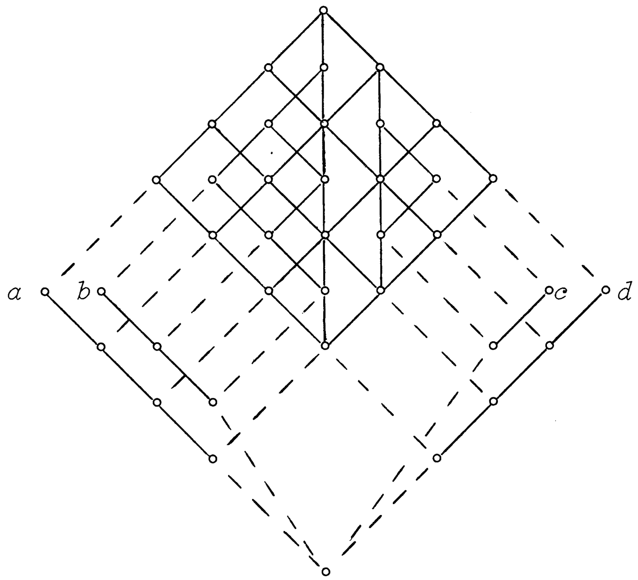
Let $L$ be a lattice and $w(x_1,\ldots,x_n)$ a lattice word (or term). If we replace all but one of the variables in $w$ with fixed elements of $L$, we obtain a function $f(x) = w(x, a_2,\ldots,a_n)$ from $L$ to $L$. Such a function is called a *unary algebraic function* or *unary polynomial* on $L$. Although such functions will not in general be homomorphisms, they will always preserve order. Recently there has been a great deal of interest in order preserving maps on a partially ordered set which have a fixed point; i.e., an element $x$ with $f(x) = x$. In this section we will address a similar, but new problem: *for which lattices does every unary* *algebraic function have a fixed point?* Let $\mathcal{F}$ denote those lattices which have this property. We will not completely answer this question but will give some partial results which are intended to draw interest to the problem. Recall the result of Anne Morel and A. Tarski [8], [47]: *every order-preserving map from a lattice L to itself has a fixed point if and only if L is complete.* Thus $\mathcal{F}$ *contains all complete lattices.* Not all lattices in $\mathcal{F}$ are complete. For example, $\omega \in \mathcal{F}$ ($\omega$ is the natural numbers under their usual order). In fact every distributive lattice is in $\mathcal{F}$. More generally, *every locally finite lattice is in* $\mathcal{F}$. (A lattice is locally finite if every finitely generated sublattice is finite.) To see this result let $f(x) = w(x, a_2, \ldots, a_n)$ be a unary polynomial on a locally finite lattice $L$. Let $L_1$ be the (finite) sublattice generated by $a_2, \ldots, a_n$. Then $f$ restricted to $L_1$ is a unary polynomial on $L_1$ and has a fixed point since $L_1$ is finite. This will be a fixed point for $L$.

On the other hand we can ask if there are any lattices not in $\mathcal{F}$. The answer is yes. In fact $f(x) = (((((x \wedge b) \vee c) \wedge a) \vee b) \wedge c) \vee a$ *has no fixed points in* $\mathrm{FL}(a, b, c)$. This is proved using some lemmas of Whitman [49]. We shall forego this proof and present instead a modular example. Let $L_1$ and $L_2$ be the modular lattices diagrammed in Figure 1. Let $f(x) = ((((x \wedge a) \vee b) \wedge c) \vee d) \wedge a$. Clearly if $f(x) = x$, then $x \leq a$. In $L_1$ one easily checks that $f(0) = 0$ and this is the only fixed point. In $L_2$ $f(a) = a$ and this is the only fixed point. Let $L$ be the sublattice of $L_1 \times L_2$ generated by $(a, a)$, $(b, b)$, $(c, c)$, and $(d, d)$. The unary polynomial $f$ acts on $L_1 \times L_2$ by $f(x, y) = (f(x), f(y))$ for $(x, y) \in L_1 \times L_2$. Thus $(0, a)$ is the only fixed point for $L_1 \times L_2$. The proof is completed by showing $(0, a) \notin L$.

Since $L$ is generated by $(a, a)$, $(b, b)$, $(c, c)$, and $(d,d)$, $(0, a) \in L$ if and only if there is a lattice word $w(a, b, c, d)$ such that $w^{L_1}(a, b, c, d) = 0$ and $w^{L_2}(a, b, c, d) = a$, where of course $w^{L_i}(a, b, c, d)$ is the evaluation of $w(a, b, c, d)$ in $L_i$.

In order to prove that this is not possible we need a lemma. In $\mathrm{FL}(a, b, c, d)$ define inductively $a_i$, $b_i$, $c_i$, $d_i$ by $a_0 = a$, $b_0 = b$, $c_0 = c$, $d_0 = d$ and $a_{i+1} = a_i \wedge (b_i \vee c_i) \wedge (b_i \vee d_i) \wedge (c_i \vee d_i)$ with $b_{i+1}$, $c_{i+1}$, and $d_{i+1}$ defined similarly. We claim that if $u(a, b, c, d) \in \mathrm{FL}(a, b, c, d)$ and $u^{L_2}(a, b, c, d) \geq a$

$L_1 =$




$L_2 =$

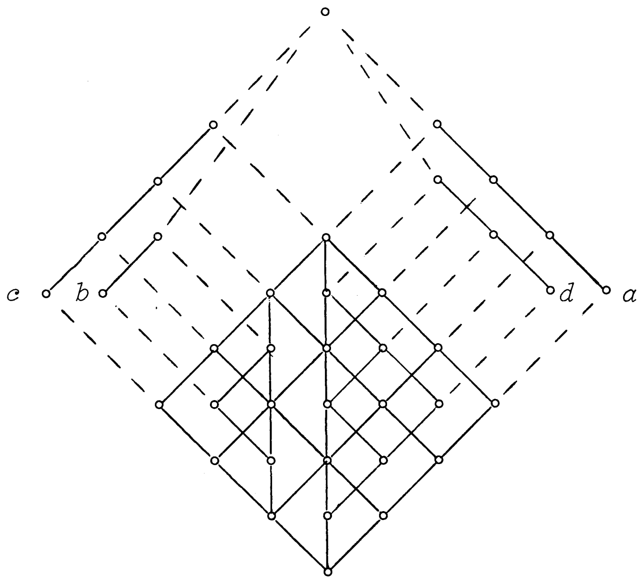



FIGURE 1

holds in $L_2$ then $u(a, b, c, d) \geq a_i$ holds in FL$(a, b, c, d)$ for some $i$ and that similar statements hold for $b$, $c$, and $d$. We prove this lemma by induction on the length of $u$. Suppose $u^{L_2} \geq a$ and suppose $u = u_1 \vee u_2$ where $u_1$ and $u_2$ have shorter length than $u$. If either $u_1^{L_2} \geq a$ or $u_2^{L_2} \geq a$ holds, it is easy to complete the proof. So we assume neither is above $a$. By examining $L_2$ we see that this implies that one of them contains $b$ and the other $c$, or one of them contains $b$ and the other $d$, or one of them contains $c$ and the other $d$. Assume the former. Then by induction $u_1 \geq b_i$ and $u_2 \geq c_j$ for some $i$ and $j$. Letting $k = \max\{i,j\}$ we have $u_1 \geq b_k$ and $u_2 \geq c_k$ and hence, $u = u_1 \vee u_2 \geq b_k \vee c_k \geq a_{k+1}$. The cases where $u$ is a generator or a meet are easier and left to the reader.

Now if $w^{L_2} = a$ in $L_2$ then $w \geq a_i$ for some $i$. But then $w^{L_1} \geq a_i^{L_1}$. It is easy to see that $a_i^{L_1} > 0$ for all $i$. Hence $w^{L_1} \neq 0$, showing that $(0, a) \notin L$.

It is possible to draw $L$. However it has a strange twist which makes its diagram awkward.

In summary, F does not contain all lattices but does contain all complete lattices and all locally finite lattices. A common generalization of these last two classes is the class of *locally complete lattices*; i.e., those lattices such that every finitely generated sublattice is a complete lattice. Clearly this class is contained in F. *Does F contain any other lattices?*

## §5 MODULAR LATTICES

Problems in the theory of modular lattices are usually more difficult than the corresponding problems for the class of all lattices or of distributive lattices. Free modular lattices are much less understood than free lattices. In this section we discuss some of the results and problems concerning free modular lattices which are related to order.

For free lattices we have Whitman's algorithm for deciding if two lattice words are equal. We now know that *there is no algorithm to decide if two lattice words are equal in the free modular lattice*. It follows that *the variety of modular lattices is not generated by its finite members*. These results depend on the "projectivity" of certain partial modular lattices. We will briefly outline the main ideas of the proofs.

The classical von Stadt coordinatization of projective geometries of dimension at least three shows how to define a ring from the geometry. Von Neumann showed how to extend these ideas to obtain a ring from any modular lattice which contains a certain partial lattice known as a 4-frame (here we shall use the term frame). A frame in a modular lattice is a set of seven elements $\{a_1, a_2, a_3, a_4, c_{12}, c_{13}, c_{14}\}$ such that $a_1 \wedge (a_2 \vee a_3 \vee a_4) = a_1 \wedge a_2 \wedge a_3 \wedge a_4$ and symmetrically $a_1 \vee c_{1j} = a_1 \vee a_j = a_j \vee c_{1j}$ and $a_1 \wedge c_{1j} = a_1 \wedge a_j = a_j \wedge c_{1j}$ for $j = 2, 3, 4$. This says $\{a_1, a_2, a_3, a_4\}$ generate a 16 element Boolean algebra with $a_1, a_2, a_3, a_4$ as atoms and $a_1, a_j, c_{1j}$ generate $M_3$. In a modular lattice containing a frame we let

$$R_L = \{x \in L: \ x \vee a_2 = a_1 \vee a_2, \ x \wedge a_2 = a_1 \wedge a_2\}$$

and define for $x, y \in R_L$

$$x+y = ([(x \vee c_{13}) \wedge (a_2 \vee a_3)] \vee [(y \vee a_3) \wedge (a_2 \vee c_{13})]) \wedge (a_1 \vee a_2)$$

(*)

$$xy = ([(x \vee c_{23}) \wedge (a_1 \vee a_3)] \vee [(y \vee c_{13}) \wedge (a_2 \vee a_3)]) \wedge (a_1 \vee a_2)$$

where

$$c_{23} = (c_{12} \vee c_{13}) \wedge (a_2 \vee a_3).$$

Von Neumann shows that with these operations $R_L$ is a ring with 1. To get an idea of how these definitions arise let $R$ be a ring with 1 and let $L$ be the lattice of submodules of $R^4$ as a left $R$ module. Let $a_1 = \{(x, 0, 0, 0): \ x \in R\}, \ldots, a_4 = \{(0, 0, 0, x): \ x \in R\}$, $c_{12} = \{(x, -x, 0, 0): x \in R\}, \ldots, c_{14} = \{(x, 0, 0, -x): \ x \in R\}$. The reader can check that $R(-1, r, 0, 0) \in R_L$ and that $R(-1, r, 0, 0) + R(-1, s, 0, 0) = R(-1, r+s, 0, 0)$ where the first "+" is defined by (*).

Frames are projective in the class of modular lattices. This means that if we map a free modular lattice, FM($X$), onto the lattice $L$ of submodules described above, we can find a frame $\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{c}_{12}, \bar{c}_{13}, \bar{c}_{14}$ in FM($X$) which maps to $a_1, a_2, a_3, a_4, c_{12}, c_{13}, c_{14}$ in $L$. In particular we get a ring in FM($X$).

Unfortunately it need not be $R$. In fact it is probably $\mathbb{Z}$ in most cases.

Now let $K_p$ and $K_q$ be countable fields of distinct finite characteristics $p$ and $q$. The lattice $L_p$ of subspaces of a four dimensional vector space over $K_p$ has a frame $a_1, \ldots, c_{14}$ similar to the one defined above for $L$. Also $L_q$, the lattice of subspaces of a four dimensional vector space over $K_q$, has a frame $a_1'$, $a_2'$, $a_3'$, $a_4'$, $c_{12}'$, $c_{13}'$, $c_{14}'$. In $L_q$ the lattice of all subspaces of $a_1' \vee a_2'$ is just the lattice of subspaces of a two dimensional vector space over $K_q$. It is easy to see that this lattice is $M_\omega$, the lattice with 0, 1, and countably many atoms. Similarly the quotient sublattice $1/a_3 \vee a_4$ of $L_p$ is $M_\omega$. Form the lattice $L$ from $L_p$ and $L_q$ by identifying these two copies of $M_\omega$. This is a modular lattice by the results of R.P. Dilworth and M. Hall [15], [28].

Consider a homomorphism of FM($X$) onto $L$. We can find frames $\overline{a_1}, \ldots, \overline{c_{14}}$ and $\overline{a_1'}, \ldots, \overline{c_{14}'}$ in FM($X$) which map to the corresponding frames in $L$. These frames determine two rings $R_p$ and $R_q$, and by choosing $\overline{a_1}, \ldots, \overline{c_{14}}$ properly, $R_p$ has characteristic $p$ and $R_q$ has characteristic $q$. Furthermore the two frames can be lined up so that there is a natural bijection from $R_p$ onto $R_q$.

If there was a lattice homomorphism $g$ from FM($X$) to a finite lattice such that $g(\overline{a_1}) \neq g(\overline{a_1} \wedge \overline{a_2})$, then $g$ is one-to-one on the frames and these frames would determine two nontrivial finite rings $S_p$ and $S_q$ of characteristics $p$ and $q$ with $|S_p| = |S_q|$. But this is clearly impossible since a finite ring (with 1) of characteristic $p$ must have order $p^n$. This implies that the variety of modular lattices is not generated by its finite members. By using some ingenious skew-field constructions of A. Macintyre and P.M. Cohn, it is possible to extend these ideas to show that the word problem for the free modular lattice on five or more generators is unsolvable.

The free modular lattice on three generators is finite [14]; however, FM(4) is infinite and it is an open problem to determine whether it has a solvable word problem. In the remaining part of this section we review some of the important problems for free modular lattices and their connection with the word problem for FM(4).

As we pointed out earlier, covers in free modular lattices are particularly important in structure theory. If $u \succ v$ in $\mathrm{FM}(n)$, then by Dilworth's characterization of lattice congruences [17] there is a unique largest congruence $\psi(u, v)$ not containing $(u, v)$. Thus associated with every cover, $u \succ v$, of $\mathrm{FM}(n)$ is a subdirectly irreducible modular lattice, namely $M = \mathrm{FM}(n)/\psi(u, v)$. If $x_1, \ldots, x_n$ are the generators of $\mathrm{FM}(n)$ and $a_1, \ldots, a_n$ are their images in $M$, then since $(u, v) \notin \psi(u, v)$, $u(a_1,\ldots,a_n) \neq v(a_1,\ldots,a_n)$. Thus the relation $u = v$ fails in $M$ for $a_1, \ldots, a_n$. Now the modularity of $\mathrm{FM}(n)$ implies that $\theta(u, v)$, the smallest congruence identifying $u$ and $v$, is an atom of the congruence lattice of $\mathrm{FM}(n)$ ([7] 10.3 and 10.5). Hence $\theta(u, v) \wedge \psi(u, v) = 0$ in this congruence lattice. This says that $\mathrm{FM}(n)$ is a subdirect product of $M$ and $\mathrm{FM}(n)/\theta(u, v)$. Of course $\mathrm{FM}(n)/\theta(u, v)$ satisfies the relation $u = v$. In fact it is not hard to show that if $L$ is any modular lattice generated by $n$ elements then either $L$ satisfies the relation $u = v$ for its generators or $L$ is a subdirect product of $M$ and a lattice $L_1$ which satisfies the relation $u = v$.

There are several important open questions about these ideas.

(1) *If* $u \succ v$ *in* $\mathrm{FM}(n)$ *is* $\mathrm{FM}(n)/\psi(u, v)$ *finite?*

(2) *Is* $\mathrm{FM}(n)$ *weakly atomic?*

In the case of free lattices both of the above questions can be answered yes [37], [9]. For modular lattices we only know that at least one of them has a negative answer. As pointed out above, the elements $\overline{a_1} > \overline{a_1} \wedge \overline{a_2}$ in FM(5) become identified under any homomorphism of FM(5) into a finite lattice. If FM(5) were weakly atomic there would be elements $u$, $v$ with $\overline{a_1} \geq u \succ v \geq \overline{a_1} \wedge \overline{a_2}$. But then the natural map from FM(5) to $M = \mathrm{FM}(5)/\psi(u, v)$ does not identify $u$ and $v$ and thus does not identify $\overline{a_1}$ and $\overline{a_1} \wedge \overline{a_2}$. Hence $M$ must be infinite. That is *either weak atomicity fails for* FM(5) *or there is an infinite lattice of the form* $\mathrm{FM}(5)/\psi(u, v)$, $u \succ v$.

A subdirectly irreducible modular lattice $L$ is called a *splitting modular lattice* if there is a lattice equation $\varepsilon$ which fails in $L$ such that every variety of modular lattices either satisfies $\varepsilon$ or contains $L$. For example, $M_3$ is a splitting modular lattice since we can take $\varepsilon$ to be the distributive law. Also every lattice of the form $\mathrm{FM}(n)/\psi(u, v)$ for $u \succ v$ is a splitting modular lattice with $\varepsilon$: $u = v$.

(3) *Is every splitting modular lattice of this form?*

Although these concepts are not well understood, we do have several examples. In [19] infinitely many lattices of the form $FM(4)/\psi(u, v)$, for $u \succ v$, are given. In [21] it is shown that the lattice of subspaces of an n-dimensional vector space over a finite prime field is a splitting lattice if $4 \leq n < \infty$. On the other hand, $M_4$, the six element lattice of length two, is not a splitting lattice [10].

Although we do not know if the word problem for FM(4) is solvable, C. Herrmann has shown that the free modular lattice generated by two complemented pairs does have a solvable word problem [29]. In fact he lists all subdirectly irreducible modular lattices which can be generated by two complemented pairs. Most of these lattices are of the form $FM(4)/\psi(u, v)$ for $u \succ v$ in FM(4). In his proof he assumes $L$ is a subdirectly irreducible modular lattice generated by two complemented pairs but not on his list. Then $L$ must satisfy all the relations $u = v$. Herrmann uses these relations in an ingenious way to eventually show that $0 = 1$ in $L$; i.e., $L$ is trivial.

There is some hope this sort of technique can be applied to FM(4). Indeed there is some evidence that four generated subdirectly irreducible modular lattices are a much more restricted class than five generated ones. Representation theorists have considered endomorphisms of vector spaces which preserve four given subspaces. These lead to algebras of "tame representation type" whereas with five subspaces one gets algebras of "wild representation type" [3], [4], [26].

On the other hand the class of four generated subdirectly irreducible lattices is not too restrictive, as the next theorem shows.

THEOREM 5. *There are* $2^{\aleph_0}$ *projective planes generated by four points. Thus there are* $2^{\aleph_0}$ *simple, complemented modular lattices which are four-generated.*

In order to prove this result we need to recall Marshall Hall's free plane construction [27]. We view a projective plane $\pi$ as a set of points and set of lines and an incidence relation between them (which says whether or not a given point is on a given line.) The axioms are: (i) each pair of distinct points are on a unique line, (ii) for each pair of distinct lines there is a unique point which is on both, (iii) $\pi$ contains a quadrangle (four points, no three on a line). Projective planes may be viewed as modular lattices with the points as atoms, the lines as coatoms and $p \leq \ell$ if and only if $p$ is on $\ell$.

A collection $\pi_0$ of points, lines, and an incidence relation between them is a partial plane provided each pair of distinct points is on at most one line. It follows that each pair of lines has at most one point on both. If $\pi_o$ is a partial plane containing a quadrangle, it can be completed to a plane as follows. For each pair of distinct points $p_1$ and $p_2$ in $\pi_o$ with no line containing both, we add a new line $\ell$ and extend the incidence relation by saying $p_1$ and $p_2$ are on $\ell$. The result is a new partial plane $\pi_1$. $\pi_2$ is formed by a dual process from $\pi_1$. Thus we obtain a sequence $\pi_o \subseteq \pi_1 \subseteq \pi_2 \subseteq \ldots$ . We call $\pi_n$ the $n$-step free extension of $\pi_o$. Then it is easy to see that $\pi = \bigcup \pi_i$ is a projective plane extending $\pi_o$.

We shall construct our $2^{\aleph_o}$ planes by modifying the above construction. In each case $\pi_o$ will consist of four points and no lines. Then $\pi_1, \pi_2, \ldots$ shall be constructed so that (i) $\pi_i \subseteq \pi_j$ if $i < j$ (this implies that both incidence and non-incidence is preserved); (ii) $\pi_n$ is a partial plane generated by $\pi_o$; (iii) if $n$ is even, $\pi_{n+1} - \pi_n$ has only lines and every pair of points in $\pi_n$ lies on a line in $\pi_{n+1}$, and dually for $n$ odd. If $x \in \pi_n - \pi_{n-1}$, we say $x$ has rank $n$ and write $r(x) = n$; note lines always have odd rank and points even. Note that these conditions imply $\pi = \bigcup \pi_n$ will be a projective plane generated by four points. Also note that these conditions imply that each line $\ell$ of $\pi$ contains only finitely many points with rank less than $r(\ell)$ and that all but at most one of these points must have rank $r(\ell) - 1$.

Let $p_1, p_2, p_3, \ldots$ be a set of prime numbers, with $p_i \geq 5$. Let $\rho_i$ be a projective plane coordinatized by the field with $p_i$ elements, $i = 1, 2, 3, \ldots$ . Recall that every quadrangle in $\rho_i$ generates it. We shall construct $\pi$ with $\rho_i \subseteq \pi$, $i = 1, 2, 3, \ldots$ . Let $\pi_o$ have only four points and let $\pi_6$ be the 6-step free extension of $\pi_o$. The reader can check that $\pi_6$ has 8 points with no two joined by a line. Use 4 of these points to generate a subplane isomorphic to $\rho_1$. We shall denote this subplane $\rho_1$. This construction differs from the free plane construction in that, for example, if $p, q, r \in \rho_1$ are points in $\pi_n$ with no two on a line in $\pi_n$, in $\pi_{n+1}$ we can add one line which contains all three provided that $p, q, r$ are collinear in $\rho_1$. All other lines are added freely.

In $\pi_{12}$ the other four points will have again created 8 points no two on a line. In $\pi_{18}$ four of these points will have created 8 new points, no two on a line. Now $\rho_1 \subseteq \pi_n$ for some $n$ and it is clear that there are 8 points of rank greater than $n$ with no two on a line. Four of these points are used to generate $\rho_2$. Continuing in this way we obtain a plane $\pi$ with $\rho_i \subseteq \pi$, $i = 1, 2, \ldots$. Notice that if $x \in \rho_i$ and $y \in \rho_j$, $i \neq j$, then $r(x) \neq r(y)$. Also notice that if $q_1, \ldots, q_k$ are points on $\ell$ with $r(q_i) < r(\ell)$, $i = 1, \ldots, k$ and $k > 2$ then $\ell$ and $q_1, \ldots, q_k$ are in some $\rho_i$.

Now suppose $\rho_0$ is the plane over $GF(p_0)$ where $p_0 \geq 5$ is a prime with $p_0 \neq p_i$, $i = 1, 2, \ldots$. We claim $\rho_0$ is not a subplane of $\pi$. Suppose $\rho_0 \subseteq \pi$. Pick an element of highest rank in $\rho_0$; say it is a point $q$. In $\rho_0$ there are $p_0 + 1$ lines $\ell_1, \ldots, \ell_{p_0+1}$ containing $q$ and all but at most one of these must have rank $r(q) - 1$. By the dual of the above remarks, $q$, $\ell_1, \ldots, \ell_{p_0+1}$ must lie in some $\rho_t$. Choose one of these lines, say $\ell_1$, with $r(\ell_1) = r(q) - 1$. Then every point of $\rho_0$ on $\ell_1$ has rank less than $r(\ell_1)$ or rank equal to $r(q)$. If $\ell_1$ contains exactly two points of $\rho_0$ of lower rank, then, since $p_0 \geq 5$, $\ell_1$ contains $q_1, q_2 \in \rho_0$ with $q_1 \neq q \neq q_2$ and $r(q_1) = r(q_2) = r(q)$. Clearly $q_1, q_2 \in \rho_t$. If there is another line, say $\ell_2$ with this same situation, we obtain $q_3, q_4$ on $\ell_2$ with $q_3, q_4 \in \rho_t$. But $q_1, q_2, q_3, q_4$ form a quadrangle and so generate $\rho_0$. Hence $\rho_0 \subseteq \rho_t$, which is impossible.

Thus for all but possibly two of the lines $\ell_1, \ldots, \ell_{p_0+1}$, all points except $q$ and possibly one other must have rank less than $r(\ell_j) = r(q) - 1$. There are more than two such points and hence $\ell_j$ and all points of $\rho_0$ of lower rank on $\ell_j$ must lie in some $\rho_i$ but $\ell_j \in \rho_t$ so the points of lower rank must also be in $\rho_t$. It is now easy to see that there is a quadrangle of $\rho_0$ in $\rho_t$, which is impossible.

Since there are $2^{\aleph_0}$ subsets of the primes, we have $2^{\aleph_0}$ nonisomorphic projective planes generated by four points.

This result improves the theorem of I. Rival and B. Sands [42] that there are $2^{\aleph_0}$ four-generated simple (nonmodular) lattices. They, however, show that every countable lattice can be embedded into a four-generated simple lattice. *Is this true in the modular case?* The author conjectures it is not.

The existence of $2^{\aleph_0}$ simple four-generated modular lattices means serious modifications of Herrmann's techniques will be required in order to solve the word problem for FM(4). On the other hand these lattices do not rule out a positive solution. In fact, since every partial plane can be extended to a plane, it is easy to show that *one can algorithmically decide whether or not two lattice terms are equal in all projective planes.*

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## ADDENDUM

In §4 the proof that $(0,a) \notin L$ may be simplified as follows. It is easy to see that there are homomorphisms $\varphi_i : L_i \to M_4$, $i = 1,2$, where $M_4$ is the length two lattice with atoms $a,b,c,d$,