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Whaley's Theorem for Finite Lattices

Dedicated to the memory of Ivan Rival

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Abstract. Whaley's Theorem on the existence of large proper sublattices of infinite lattices is extended to ordered sets and finite lattices. As a corollary it is shown that every finite lattice $\mathbf{L}$ with $|\mathbf{L}| \geqslant 3$ contains a proper sublattice $\mathbf{S}$ with $|\mathbf{S}| \geqslant |\mathbf{L}|^{1/3}$. It is also shown that that every finite modular lattice $\mathbf{L}$ with $|\mathbf{L}| \geqslant 3$ contains a proper sublattice $\mathbf{S}$ with $|\mathbf{S}| \geqslant |\mathbf{L}|^{1/2}$, and every finite distributive lattice $\mathbf{L}$ with $|\mathbf{L}| \geqslant 4$ contains a proper sublattice $\mathbf{S}$ with $|\mathbf{S}| \geqslant \frac{3}{4}|\mathbf{L}|$.

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In [5], Tom Whaley proved the following classic result about sublattices of lattices.

THEOREM 1. *If* $\mathbf{L}$ *is a lattice with* $\kappa = |\mathbf{L}|$ *infinite and regular, then either*

(1) *there is a proper principal ideal of* $\mathbf{L}$ *of size* κ, *or*
(2) *there is a proper principal filter of* $\mathbf{L}$ *of size* κ, *or*
(3) $\mathbf{M}_\kappa$, *the modular lattice of height* 2 *and size* $\kappa + 2$, *is a* 0, 1*-sublattice of* $\mathbf{L}$.

COROLLARY 2. *If* $|\mathbf{L}|$ *is infinite and regular, then* $\mathbf{L}$ *has a proper sublattice of cardinality* $|\mathbf{L}|$.

Whether or not the corollary is true for singular cardinals is an interesting question; the stronger statement of the theorem does not hold for singular cardinals [5].

Although Whaley's Theorem is about sublattices, there is an interesting generalization for ordered sets which we present before giving our finitary version of his theorem. The proof is actually somewhat easier than Whaley's original.

Let $\mathbf{P}$ be an ordered set. We say that a subset A of $\mathbf{P}$ is a *super-antichain* if no pair of distinct elements of A has a common upper bound or a common lower bound.

THEOREM 3. *If* **P** *is an ordered set with* $\kappa = |\mathbf{P}|$ *infinite and regular, then either*

(1) *there is a principal ideal of* **P** *of size* κ, *or*
(2) *there is a principal filter of* **P** *of size* κ, *or*
(3) **P** *contains a super-antichain of size* κ.

Proof. Suppose that (1) and (2) fail. We will construct a super-antichain by transfinite induction. For every $a \in \mathbf{P}$ set

$$N_a = \bigcup_{x \geqslant a} \downarrow x \cup \bigcup_{y \leqslant a} \uparrow y$$

and note that, since κ is regular, $|N_a| < \kappa$. Pick $a_0 \in \mathbf{P}$ arbitrarily. Let $\delta < \kappa$ be an ordinal, and suppose we have a_γ for all $\gamma < \delta$ forming a super-antichain. Then $|\bigcup_{\gamma<\delta} N_{a_\gamma}| < \kappa$, whence $\mathbf{P} - \bigcup_{\gamma<\delta} N_{a_\gamma} \neq \emptyset$. Choose a_δ in this latter set. Then $\{a_\gamma : \gamma < \kappa\}$ is a super-antichain. □

Whaley's Theorem follows by applying Theorem 3 to the ordered set $\mathbf{P} = \mathbf{L} - \{0, 1\}$ obtained by removing the least and greatest elements of **L** (if they exist). Assume κ is a regular cardinal. If **P** has a super-antichain of size κ, then 0 is the pairwise meet and 1 is the pairwise join of these elements in **L**. Thus $\mathbf{M}_\kappa$ is a 0, 1-sublattice of **L**. If **P** has a principal filter $\uparrow x$ of size κ, it is possible that this is all of **P**. Regardless, the corresponding filter $\uparrow x$ of **L** is proper because x cannot be the least element of **L**.

Now we prove an analogue of Theorem 3 for finite ordered sets.

THEOREM 4. *Let* **P** *be a finite ordered set with* $|\mathbf{P}| = n$. *Let* $\gamma = \lceil n^{1/3} \rceil$. *Then either*

(1) *there is a principal ideal* I *of* **P** *with* $|I| \geqslant \gamma$, *or*
(2) *there is a principal filter* F *of* **P** *with* $|F| \geqslant \gamma$, *or*
(3) **P** *contains a super-antichain* A *with* $|A| \geqslant \gamma$.

Proof. Let λ be the largest size of a principal ideal or filter of **P**, so that $|\downarrow x| \leqslant \lambda$ for all $x \in \mathbf{P}$ and $|\uparrow y| \leqslant \lambda$ for all $y \in \mathbf{P}$.

For $a \in \mathbf{P}$, as in the previous proof, define $N_a = \bigcup_{x \geqslant a} \downarrow x \cup \bigcup_{y \leqslant a} \uparrow y$. We claim that $|N_a| \leqslant \lambda^2$. Indeed, if $u = |\downarrow a - \{a\}|$ and $v = |\uparrow a - \{a\}|$, then

$$N_a = \bigcup_{x > a} (\downarrow x - \downarrow a) \cup \bigcup_{y < a} (\uparrow y - \uparrow a) \cup \{a\}$$

and thus, since $u,\ v \leqslant \lambda - 1$,

$$\begin{aligned} |N_a| &\leqslant v(\lambda - u - 1) + u(\lambda - v - 1) + 1 \\ &= (\lambda - 1)(u + v) - 2uv + 1 \\ &= (\lambda - 1)^2 + 1 - (\lambda - 1 - u)(\lambda - 1 - v) - uv \\ &\leqslant (\lambda - 1)^2 + 1 \\ &\leqslant \lambda^2. \end{aligned}$$

We form a super-antichain A as follows. Choose $a_1 \in \mathbf{P}$ arbitrarily. Suppose $a_1, \dots, a_k$ have been chosen and that they form a super-antichain. Choose $a_{k+1} \in \mathbf{P} - \bigcup_{1 \leqslant i \leqslant k} N_{a_i}$ as long as this last set is nonempty. Thus we obtain a sequence $a_1, \dots, a_\beta$ where $\{a_1, \dots, a_\beta\}$ is a super-antichain and $\mathbf{P} = \bigcup_{1 \leqslant i \leqslant \beta} N_{a_i}$ so $n \leqslant \beta((\lambda - 1)^2 + 1)$. Since $\beta\lambda^2 \geqslant n$, either $\lambda \geqslant n^{1/3}$ or $\beta \geqslant n^{1/3}$. □

As before, we translate this result to lattices by taking $\mathbf{P} = \mathbf{L} - \{0, 1\}$. Note that this makes $|\mathbf{L}| = n + 2$ and $\lambda + 1$ the largest size of a principal ideal or filter of $\mathbf{L}$.

THEOREM 5. *Let* $\mathbf{L}$ *be a finite lattice with* $|\mathbf{L}| = n + 2$. *Then either*

(1) *there exists* $x < 1$ *with* $|{\downarrow}x| \geqslant n^{1/3}$, *or*
(2) *there exists* $y > 0$ *with* $|{\uparrow}y| \geqslant n^{1/3}$, *or*
(3) $\mathbf{M}_\beta$ *is a* 0, 1*-sublattice of* $\mathbf{L}$, *where* $\beta = \lceil n^{1/3} \rceil$.

In the construction of the super-antichain in Theorem 4 we started with an arbitrary element a_1, and in fact proved the following stronger theorem.

THEOREM 6. *Let* $\lambda + 1$ *be the largest size of a proper ideal or filter of* $\mathbf{L}$, *and let* $|\mathbf{L}| = n + 2$. *Then every element of* $\mathbf{L}$ *is contained in a* 0, 1*-sublattice* $\mathbf{M}_\beta$ *of* $\mathbf{L}$ *with* $\beta((\lambda-1)^2+1) \geqslant n$. *In particular, if* $n > (\lambda-1)^2+1$, *then* $\mathbf{L}$ *is complemented.*

We can see that this result is essentially the best possible. Let $\mathbf{P}$ be the lattice of subspaces of a projective plane of order q, and let $\mathbf{K}$ be the lattice formed by taking k copies of $\mathbf{P}$ and identifying their 0's and 1's. A 0, 1-$\mathbf{M}_\beta$ can contain at most one point and one line from each copy of $\mathbf{P}$. Thus the parameters for $\mathbf{K}$ are

$$\begin{aligned} n &= 2k(q^2 + q + 1), \\ \lambda &= q + 2, \\ \beta &= 2k. \end{aligned}$$

In particular, if we take $k = \lceil q/2 \rceil$, the largest proper ideals and proper filters and 0, 1-$\mathbf{M}_\beta$'s in the resulting lattice have size approximately $|\mathbf{K}|^{1/3}$. Of course $\mathbf{K}$ has larger sublattices, but not of the form produced by Whaley's theorem.

Note that the projective plane $\mathbf{P}$ of order q has $n = 2(q^2 + q + 1)$. Unless q is a square, its largest sublattices are of the form $\mathbf{S} = {\uparrow}p \cup {\downarrow}\ell$ for a point p not on a line ℓ, with $|\mathbf{S}| = 2(q + 3) \approx \sqrt{2|\mathbf{L}|}$. If $q = p^{2k}$ is a square and $\mathbf{P}$ is Desarguesian, there is also a subplane of order p^k, which is again a sublattice of size approximately $\sqrt{2|\mathbf{L}|}$. We do not know an example of any finite lattice $\mathbf{L}$ with $|\mathbf{L}| > 2$ that does not contain a sublattice of cardinality at least $\sqrt{2|\mathbf{L}|}$. This leads us to conjecture that *every finite lattice has a sublattice with* $|\mathbf{S}| > |\mathbf{L}|^{1/2}$ (perhaps with a constant thrown in). For modular lattices it is true.

THEOREM 7. *Let* $\mathbf{L}$ *be a finite modular lattice with at least* 3 *elements. Then* $\mathbf{L}$ *has a proper sublattice* $\mathbf{S}$ *with* $|\mathbf{S}| \geqslant |\mathbf{L}|^{1/2}$.

Proof. Let $|\mathbf{L}| = n+2$ and let $\lambda+1$ be the size of the largest proper ideal or filter of $\mathbf{L}$. Since $|\mathbf{L}| \geqslant 3$, $\lambda \geqslant 1$. If $(\lambda+1)^2 \geqslant |\mathbf{L}|$, then $\mathbf{L}$ has a proper filter or ideal of size at least $|\mathbf{L}|^{1/2}$. So we may assume that $(\lambda+1)^2 < n+2$, and since $\lambda \geqslant 1$ this implies $(\lambda-1)^2+1 < n$ and whence by Theorem 6, $\mathbf{L}$ is complemented. The result is true for $\mathbf{L} \cong \mathbf{M}_n$, so we may also assume that $\mathbf{L}$ has height greater than 2.

There is an atom b with $|{\uparrow}b| = \lambda+1$, or there is a coatom c such that $|{\downarrow}c| = \lambda+1$, in fact both since $\mathbf{L}$ is complemented. If we take an atom b with $|{\uparrow}b| = \lambda+1$ and let c be a complement of b, then $\mathbf{S} = {\uparrow}b \cup {\downarrow}c$ is a sublattice with $2(\lambda+1)$ elements. If this is all of $\mathbf{L}$ then $\mathbf{L}$ has a sublattice with $\frac{1}{2}|\mathbf{L}|$ elements and since $\mathbf{L}$ has at least 3 elements and is complemented, it has at least 4 elements and so $\frac{1}{2}|\mathbf{L}| \geqslant |\mathbf{L}|^{1/2}$.

Since the complements of b are coatoms, no two complements of b are complements of each other. Hence, again using Theorem 6, $\beta = 2$ and so $(\lambda-1)^2+1 \geqslant n/2$. Hence

$$|\mathbf{S}|^2 = 4(\lambda+1)^2 \geqslant 4(\lambda-1)^2+8 \geqslant 2(n+2) = 2|\mathbf{L}|$$

and so $|\mathbf{S}| \geqslant \sqrt{2|\mathbf{L}|} > |\mathbf{L}|^{1/2}$. □

Ivan Rival's Theorem, Reprise

For some other classes of lattices the conjecture that every finite lattice has a sublattice with $|\mathbf{S}| > |\mathbf{L}|^{1/2}$ is easy. For example, every finite join semidistributive lattice contains a prime ideal, and hence a sublattice of cardinality at least $|\mathbf{L}|/2$. In general the conjecture appears to be challenging, but for distributive lattices much more can be said. In [3] Ivan Rival proved every distributive lattice has a proper sublattice of size at least $\frac{2}{3}|\mathbf{L}|$. Actually his theorem is much stronger: he showed that *every* maximal sublattice has size at least $\frac{2}{3}|\mathbf{L}|$. Nothing similar holds in other natural classes of lattices: in [2] it is shown that there are arbitrarily large finite modular lattices (and an infinite modular lattice) which have a maximal sublattice with 13 elements. (That paper characterizes those finite lattices that are maximal sublattices of arbitrarily large finite lattices.)

Can more be said about the size of the largest sublattice of a distributive lattice? We will show that in fact every finite distributive lattice has a sublattice of size at least $\frac{3}{4}|\mathbf{L}|$. The proof uses some of Ivan's ideas from [4].

THEOREM 8. *Let* $\mathbf{L}$ *be a finite distributive lattice.*

(1) (Rival [3].) *If* $|\mathbf{L}| \geqslant 3$ *then every maximal sublattice of* $\mathbf{L}$ *has size at least* $\frac{2}{3}|\mathbf{L}|$.
(2) *If* $|\mathbf{L}| \geqslant 4$ *then there is a proper sublattice of* $\mathbf{L}$ *of size at least* $\frac{3}{4}|\mathbf{L}|$.

Both results are the best possible in that there are infinitely many finite distributive lattices where equality obtains.

Proof. Let $\mathbf{L}$ be a finite distributive lattice. For $x \geqslant y$ we let $x/y = \{u \in \mathbf{L} : y \leqslant u \leqslant x\}$ denote the interval sublattice between x and y. Recall that in a distributive lattice, join irreducible elements are join prime and meet irreducible elements are meet prime. Also *if* $x, y \in \mathbf{L}$ *then the sublattice of* $\mathbf{L}$ *generated by* $x/x \wedge y$ *and* $y/x \wedge y$ *is isomorphic to* $(x/x \wedge y) \times (y/x \wedge y)$. This result, which holds for modular lattices as well, is quite old. Dilworth said that when he entered lattice theory in the 1930's this result was already considered to be folklore. A proof can be found on page 4 of [1].

Following [4] we let $Q(\mathbf{L})$ denote all intervals b/a where a is join irreducible and b is meet irreducible. In this proof we consider 0 to be join irreducible and 1 to be meet irreducible. For such an interval $\mathbf{L} - b/a$ is a sublattice of $\mathbf{L}$, as follows immediately from the fact that a is join prime and b is meet prime. Of course if a sublattice of this form is to be maximal we want b/a to be minimal in $Q(\mathbf{L})$ with respect to set containment. Conversely, if $\mathbf{M}$ is a maximal sublattice of $\mathbf{L}$ then we shall see that $\mathbf{M}$ has this form. Indeed, let $\mathbf{M}$ be a maximal sublattice of $\mathbf{L}$. Then there is a join irreducible a not in $\mathbf{M}$. The element a is the meet of the meet irreducibles above it. If for every meet irreducible $b \geqslant a$ we had $b/a \cap \mathbf{M} \neq \emptyset$, then $a \in \mathbf{M}$. Hence there is a meet irreducible b above a with $b/a \cap \mathbf{M} = \emptyset$ and thus $\mathbf{M} = \mathbf{L} - b/a$.

The restrictions on the minimum size of $\mathbf{L}$ guarantee sublattices of the form $\mathbf{L} - \{x\}$ have at least $2/3$ (or $3/4$ in the second statement) of the elements. So we may assume that $\mathbf{L}$ has no elements which are both join and meet irreducible.

Now suppose $b/a \in Q(\mathbf{L})$ is minimal with respect to set inclusion. Then by our assumptions $a < b$ and a is a proper meet $a = \bigwedge x_i$ of meet irreducible elements. Clearly there is an i such that $x_i \ngeqslant b$. By the minimality of b/a we have $x_i \nleqslant b$. It follows that there is an element $a_1 \succ a$ such that $a_1 \wedge b = a$. A dual argument yields an element $b_0 \prec b$ such that $a \vee b_0 = b$. Let $a_0 = a \wedge b_0$. Note $a_1 \wedge b_0 = a_1 \wedge b \wedge b_0 = a \wedge b_0 = a_0$. By the folklore theorem, $a_1/a_0 \times b_0/a_0$ is embedded in $\mathbf{L}$. Since $a_0 \prec a \prec a_1$, a_1/a_0 is either $\mathbf{3}$ or $\mathbf{2} \times \mathbf{2}$. In either case $\mathbf{3} \times b_0/a_0$ is embedded in $\mathbf{L}$ and so $|b/a| = |b_0/a_0| \leqslant \frac{1}{3}|\mathbf{L}|$. Thus every maximal sublattice has size at least $\frac{2}{3}|\mathbf{L}|$.

To prove the second statement of the theorem, let a be a maximal join irreducible element. By our assumption, $a \neq 1$ so there are meet irreducible elements above a. Let b be minimal among those. We construct a_1, a_0 and b_0 as above. Again $a_1/a_0 \times b_0/a_0$ is embedded in $\mathbf{L}$. Since a is a maximal join irreducible element, a_1 is join reducible. So there is an element $c < a_1$ with $c \nleqslant a$. Since a is join prime, $c \vee a_0 \neq a_1$. Hence a_1/a_0 has 4 elements and thus $|b/a| = |b_0/a_0| \leqslant \frac{1}{4}|\mathbf{L}|$. Thus $\mathbf{L} - b/a$ has size at least $\frac{3}{4}|\mathbf{L}|$, proving (2).

These inequalities are the best possible: all maximal sublattices of finite Boolean algebras with at least 4 elements have size exactly $\frac{3}{4}|\mathbf{L}|$ and $\mathbf{3} \times \mathbf{n}$ has a maximal sublattice of size $\frac{2}{3}|\mathbf{L}|$; see [3]. □

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