Appendix

F

Free Lattices

Ralph Freese

In this appendix, we survey some of the major developments in the theory of free lattices and certain related topics. J. Ježek, J. B. Nation, and I have completed a monograph on the subject [F15], which contains the proofs of most the results here. The first chapter of this monograph is suitable as an introduction to the theory of free lattices. It also contains more details of the interesting history of the subject, as well as a chapter on algorithms for finite and free lattices.

1. Whitman's Solutions; Basic Results

The classical papers Ph. M. Whitman [1941] and [1942] solved the word problem for free lattices: Whitman gave an algorithm for determining if two lattice terms (polynomials) were equal in all lattices. He showed that each element of the free lattice has a shortest term representing it, known as the *canonical form*, and gave an algorithm to put an arbitrary term into canonical form (see Section VI.2). This canonical form is closely connected with the arithmetic of the free lattice and Whitman exploited this connection to obtain important results about free lattices.

The word problem for free lattices, in fact for finitely presented lattices, was first solved by T. Skolem in 1920 [F28, F29]. (This went unnoticed until it was recently discovered by S. Burris [F2].) What was interesting about Skolem's solution is that it is polynomial time, unlike some of the later solutions [F24, F8, F7]. The computational complexity of Whitman's algorithm is discussed in [F10] and Chapter XI of [F15]. Even though there is such a nice canonical form in free lattices and an easy algorithm for obtaining it, there is no term rewrite system for lattice theory, see [F14]. This is also proved in Chapter XII of [F15] along with some further results in this area.

The key ingredient of Whitman's solution is the following condition known as *Whitman's condition* (see Section VI.1):

(W) If $v = v_1 \wedge \cdots \wedge v_r \leq u_1 \vee \cdots \vee u_s = u$, then either $v_i \leq u$, for some i, or $v \leq u_j$, for some j.

Let $w \in FL(X)$ be join-reducible and let us suppose that $t = t_1 \vee \cdots \vee t_n$ (with $n > 1$) is the canonical form of w. Let w_i be the function associated with T_i in $FL(X)$. Then $\{w_1, \ldots, w_n\}$ are called the *canonical joinands* of w. We also say $w = w_1 \vee \cdots \vee w_n$ *canonically* and that $w_1 \vee \cdots \vee w_n$ is the *canonical join-representation* (or *canonical join-expression*) of w. If w is join-irreducible, we define the canonical joinands of w to be the set $\{w\}$. Of course the *canonical meet-representation* and *canonical meetands* of an element in a free lattice are defined dually.

The aforementioned connection between the canonical form and the arithmetic of free lattices is summarized in the following theorem which shows that the canonical form corresponds to the best way to express an element of a free lattice as a join or meet. For any lattice L and finite subsets A and B of L, we say that A *join-refines* B, and we write $A \ll B$, if for each $a \in A$, there is a $b \in B$ with $a \leq b$.

The dual notion is called *meet-refinement* and is denoted $A \gg B$.

Theorem 1 Let $w = w_1 \vee \cdots \vee w_n$ canonically in $FL(X)$. If also $w = u_1 \vee \cdots \vee u_m$, then

$$\{w_1, \ldots, w_n\} \ll \{u_1, \ldots, u_m\}.$$

Thus $w = w_1 \vee \cdots \vee w_n$ is the unique minimal join-representation of w.

B. Jónsson observed that there is a close connection between canonical form and a weak form of distributivity known as semidistributivity. A lattice is *join-semidistributive* if it satisfies the following condition (see Definition VI.1.5):

(SD$_\vee$) $a \vee b = a \vee c \quad \text{implies} \quad a \vee b = a \vee (b \wedge c)$.

Of course *meet-semidistributivity* is defined dually and a lattice is *semidistributive* if it satisfies both conditions. B. Jónsson and J. E. Kiefer [1962] showed that lattices with canonical forms, in particular free lattices, are semidistributive; see Theorem 1.21 and Lemma 2.22 of [F15]. Thus all sublattices of free lattices are semidistributive.

2. Classical Results

The early deep work on free lattices centered on sublattices. Ph. Whitman showed that $FL(\aleph_0)$ can be embedded into $FL(3)$ (see Theorem VI.2.8). B. Jónsson and J. E. Kiefer proved the following result.

Theorem 2 Let L be a lattice satisfying (W). Suppose that the elements a_1, a_2, a_3, and $v \in L$ satisfy

1. $a_i \not\leq a_j \vee a_k \vee v$, whenever $\{i, j, k\} = \{1, 2, 3\}$;
2. $v \not\leq a_i$ for $i = 1, 2, 3$;
3. v is meet-irreducible.

Then L contains a sublattice isomorphic to $FL(3)$.

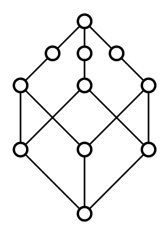

Figure 1: The bottom of $FL(3)$

They used this result to prove some interesting corollaries. If $x \in X$, then $\underline{x} = \bigwedge(X - \{x\})$ is an atom of $FL(X)$ and every atom has this form. Let u be the join of the atoms. When $|X| = 3$, then the interval $u/0$ has 11 elements; see Figure 1. When $|X| = 4$, this interval is infinite but the sublattice generated by the atoms is finite; see Figure 1.1 and 3.5 of [F15]. But when $|X| \geq 5$, we have the following:

Corollary 3. *If $n \geq 5$, the sublattice of $FL(n)$ generated by the atoms is infinite.*

Even though every nontrivial lattice equation (identity) fails in a finite lattice, finite sublattices of free lattices do satisfy an equation.

Corollary 4. *If L is a finite lattice satisfying* (W)*, then the breadth of L is at most* 4*. The variety generated by finite lattices that satisfy* (W) *is not the variety of all lattices. In particular, finite sublattices of a free lattice have breadth at most four and satisfy a nontrivial lattice equation.*

Theorem VI.2.11 and Corollary VI.2.12 give some applications of free lattice techniques to lattice theory due to R. Dean, R. P. Dilworth, and Ju. I. Sorkin.

Jónsson's conjecture.

For about 20 years, work on free lattices was dominated by a problem posed by Jónsson. He had observed that every sublattice of a free lattice satisfies (W) and is semidistributive and asked if every finite lattice satisfying these properties is a sublattice of a free lattice. This problem was finally solved by J. B. Nation in [F26].

Theorem 5 A finite lattice is isomorphic to a sublattice of a free lattice if and only if it is semidistributive and satisfies (W).

Dean's Problem.

In a recent communication, R. A. Dean informed me of an old problem in free lattice theory: can a free lattice have an ascending chain of sublattices all isomorphic to $FL(3)$? This problem did not appear in print and I was unaware of it. See [F13] for some partial results on this interesting problem.

3. Covers in Free Lattices

In his original papers, Whitman observed that even though $FL(X)$ is infinite if $|X| \geq 3$, it did have some elements which covered others. Additional covers were discovered in R. A. Dean [1961a]; see Theorem VI.2.4.

R. N. McKenzie's studies of lattice varieties naturally lead him to covers in free lattices. A lattice of the form $L = FL(X)/\Psi$, where $u \succ v$ and Ψ is the unique largest congruence separating u and v, is called a *splitting lattice*. Such lattices are finite and subdirectly irreducible and satisfy a strong form of Jónsson's Theorem (Theorem V.1.9): if a splitting lattice lies in an arbitrary join of lattice varieties, it is in one of them. Splitting lattices naturally lead to a splitting of the lattice of all lattice varieties into a principal ideal and a principal filter. This motivated A. Day [F4] to prove the following theorem which is one of the most important results on covers in free lattices. It is so important that it is proved twice in [F15].

Theorem 6 Finitely generated free lattices are weakly atomic, that is, every nontrivial interval contains a cover.

A systematic theory of covers in free lattices was developed in [F19] and is covered more thoroughly in Chapter III of [F15]. Here we give a brief overview of the main ideas of the theory.

A join-irreducible element of a lattice is *completely join-irreducible* when it has a lower cover, which of course must be unique. When w is completely join-irreducible, its lower cover is denoted w_*. The unique upper cover of a completely meet-irreducible element q is denoted q^*.

Theorem 7 Let w be a completely join-irreducible element of $FL(X)$. Then there is a unique canonical meetand $\kappa(w)$ of w_* which is not above w. Every element of $FL(X)$ which is above w_* is either above w or below $\kappa(w)$. Dually, if v is completely meet-irreducible in $FL(X)$, then $\kappa^d(v)$ is the unique canonical joinand of v^* which is not below v and every element below v^* is either below v or above $\kappa^d(v)$.

The element $\kappa(w)$ is always completely meet-irreducible and the association $w \longleftrightarrow \kappa(w)$ is a bijection from the completely join-irreducible elements to the completely meet-irreducible elements; see Theorem 3.3 of [F15].

The lower covers of arbitrary elements in $FL(X)$ are determined in a straightforward manner from the lower covers of their canonical joinands; see Theorem 3.5 of [F15]. Thus we can concentrate on join-irreducible elements.

We associate with each join-irreducible element $w \in FL(X)$ a finite set $\mathrm{J}(w)$ of join-irreducible elements. If $w \in X$ then $\mathrm{J}(w) = \{w\}$. Otherwise, let

$$w = \bigwedge_i \bigvee_j w_{ij} \wedge \bigwedge_k x_k$$

be the canonical representation of w, where $x_k \in X$. Then

$$\mathrm{J}(w) = \{w\} \cup \bigcup_{i,j} \mathrm{J}(w_{ij}).$$

Essentially, $\mathrm{J}(w)$ consists of the join-irreducible subterms of w. If we take the join-closure (including the empty join) of $\mathrm{J}(w)$ in $FL(X)$, we get a finite lattice $L(w)$ and w is completely join-irreducible if and only if $L(w)$ is semidistributive; see Theorem 3.26 of [F15]. This gives a nice visual way of testing if w is completely join-irreducible; see Figure 3.3 and Table 3.1 of [F15] for several examples. However, because we have to take the join-closure, this method can be exponential. The next theorem gives a syntactic algorithm which is polynomial time. Let

$$w_\dagger = \bigvee \{u \in \mathrm{J}(w) \mid u < w\},$$
$$\mathrm{K}(w) = \{v \in \mathrm{J}(w) \mid w_\dagger \vee v \not\geq w\}.$$

Theorem 8 Let w be a join-irreducible element of $FL(X)$, for a finite set X. Then w is completely join-irreducible in $FL(X)$ if and only if the following two conditions are satisfied:

1. every $u \in \mathrm{J}(w) - \{w\}$ is completely join-irreducible;
2. $w \not\leq \bigvee \mathrm{K}(w)$.

What is even more interesting is that there is a simple recursive formula for $\kappa(w)$ and hence for w_*.

Theorem 9 Let w be a completely join-irreducible element of $FL(X)$. Then

$$\kappa(w) = \bigvee\{x \in X \mid w_\dagger \vee x \not\geq w\} \\ \vee \bigvee\{k^\dagger \wedge \kappa(v) : v \in \mathrm{J}(w) - \{w\},\ w \not\leq \kappa(v)\},$$

where

$$k^\dagger = \bigwedge \left\{\kappa(v) \mid v \in \mathrm{J}(w) - \{w\},\ \kappa(v) \geq \bigvee \mathrm{K}(w)\right\}.$$

Chains of covers.

Using these results, we can answer basic question about free lattices such as the existence of chains of covers, finite intervals, and the connected components of the covering relation. We start with a chain of length two. Since by (W) every element is either join- or meet-irreducible, we may assume that the middle element w of this chain is join-irreducible and hence completely join-irreducible. Let u be the top of this chain so that $w_* \prec w \prec u$. Let q be the canonical meetand of w not above u. The uniqueness of q follows from the dual of Theorem 1. The same theorem also shows that $q \vee u \succ q$; that is, q is completely meet-irreducible. By Theorem 8, each of the canonical joinands of q is completely join-irreducible, and this implies (by Theorem 3.5 of [F15]) that q is *lower atomic*, that is, every element properly below q is below some lower cover of q. Of course, q is upper atomic since it is completely meet-irreducible. Elements which are both lower and upper atomic are called *totally atomic*. Thus associated with a three-element chain of two covers $w_* \prec w \prec u$ is a totally atomic element.

Now it turns out that there are a limited number of totally atomic elements in free lattices and their form can be completely characterized; see Theorem 6.10 and Corollary 6.11 of [F15] for the details. A more detailed analysis of totally atomic elements allows us to greatly restrict the length of a chain of covers in free lattices. For an element a in a lattice, define the *connected component* of a to be the set of those b's such that there is a sequence $a = a_0, a_1, \dots, a_n = b$ where, for each i, either $a_i \prec a_{i+1}$ or $a_i \succ a_{i+1}$. The connected component of 0 in $FL(3)$ is the interval diagrammed in Figure 1. This interval has chains of four covers. The connected component of 0 in $FL(n)$, $n \geq 4$, has chains of covers of length 3 and no longer.

Theorem 10 A chain of covers in a free lattice can have length at most 4. Chains of covers of length 3 and 4 in $FL(n)$ occur only in the connected component of 0 or of 1. On the other hand, $FL(n)$ has infinitely many chains of covers of length 2, for all $n \geq 3$.

However, nothing like Day's Theorem holds for chains of covers of length 2: there are infinite intervals in $FL(3)$ that do not contain any chain of covers of length 2. (The reader might try to prove this using the connection of totally atomic elements to chains of two covers described above and the fact that there are only finitely many totally atomic elements in $FL(n)$.)

Finite intervals.

With a little extra work, it is possible to find all finite intervals. The bottom of $FL(3)$, diagrammed in Figure 1, is an interval and so every subinterval of it is also an interval. It turns out these and their duals are the only intervals in free lattices. So the lattices of Figure 2 and their duals are the only nontrivial finite intervals in free lattices.

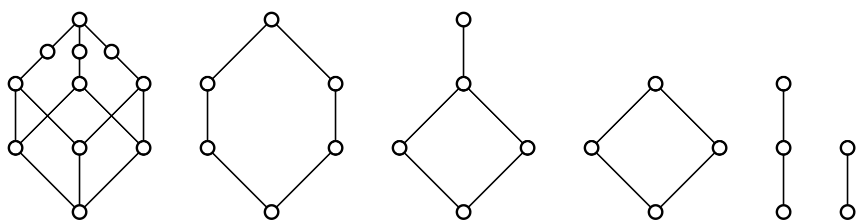

Figure 2: Finite intervals of free lattices

Actually, most of these are rare. The first two (and the dual of the first) only occur in $FL(3)$ and only at the bottom or top. The next two occur in every $FL(n)$ but only in the connected component of 0 or 1. But the three-element interval occurs infinitely often. This answers Problem VI.17.

Connected components.

Using similar ideas, it is possible to characterize the connected components of the covering relation in free lattice. This was first done in [F9] and is presented in Chapter VII of [F15]. The connected component of 0 is easy to describe. For $FL(3)$ it is diagrammed in Figure 1 and for $FL(n)$, $n > 3$, it is described in Example 3.45 and Figure 3.5 of [F15]. The fact that outside these components chains of covers can have length at most 2 severely restricts the possible connected components. Nevertheless, connected components that do occur are interesting. There are two types. The first consists of any number of copies of

$\mathfrak{N}_5$ with a common least element and one common atom. Figure 3 shows one with three copies of $\mathfrak{N}_5$. The dotted lines indicate a noncover while the solid lines indicate a cover.

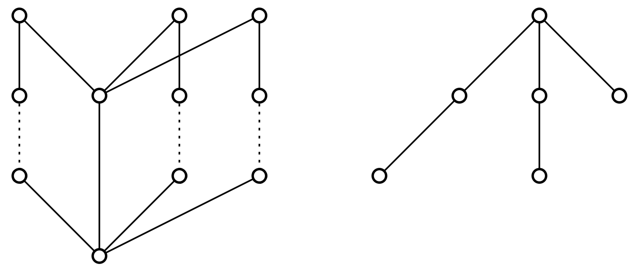

Figure 3: Two connected components

The only other type of connected component consists of a collection of chains each of length one or two with a common top element but otherwise disjoint. One example is given in Figure 3. It is interesting that all possibilities occur *except* the three-element chain and two three-element chains with a common top.

4. Semisingular Elements and Tschantz's Theorem

Suppose w is a completely join-irreducible element. Then $w_* = w \wedge \kappa(w)$ and, by its definition, $\kappa(w)$ is a canonical meetand of w_*. If $w = \bigwedge w_i$ canonically, then

$$w_* = \kappa(w) \wedge w = \kappa(w) \wedge \bigwedge w_i.$$

Is this the canonical form of w_*? Surprisingly, it turns out that it usually is.

Theorem 11 Let w be a completely join-irreducible element of $FL(X)$ with $w = w_1 \wedge \cdots \wedge w_m$ canonically. Then

$$\{\kappa(w)\} \cup \{w_i : w_i \not\geq \kappa(w)\}$$

is the set of canonical meetands of w_*.

We originally proved this in [F19] to simplify making a table of examples. But it turns out this result plays an important role in the proofs of several of the deeper theorems, including the characterization of chains of covers and of finite intervals given above. To explore some of these, we define a completely join-irreducible element of a free lattice to be *semisingular*, if $\kappa(w) \leq w_i$, for at least one of the canonical meetands w_i of w. If every $\kappa(w) \leq w_i$, for *every* i, we call w *singular*. In this case, $w_* = \kappa(w)$. Singular elements are characterized in [F16]. They are very rare: if w is singular, then either w is the meet of two coatoms or w_* is the join of two atoms; see Theorem 8.6 of [F15].

Semisingular elements play an important role in free lattices, for example, in finding maximal chains without covers. The following theorem gives a very nice characterization of semisingular elements.

Theorem 12 A completely join-irreducible element $w \in FL(X)$ is semisingular if and only if $w_* = \kappa(w)$ (that is, w is singular) or w is the middle element of a three-element interval.

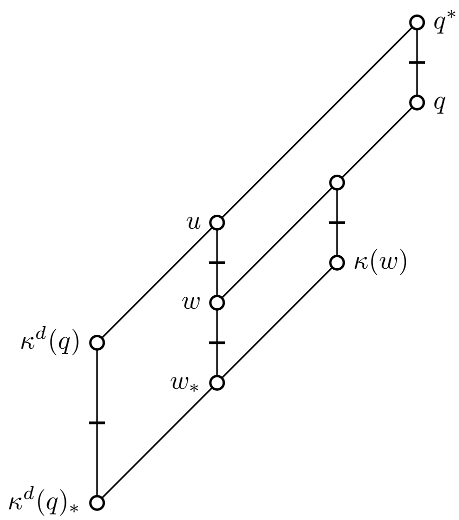

Figure 4: The form of a three-element interval u/w_*

Theorems 7.9 and 7.10 of [F15] give strong characterizations of three-element intervals. Figure 4 illustrates some of the key information in these theorems. (The crosshatches indicate coverings.) It is an important fact that the canonical form of the element w of that figure is $w = u \wedge q$. This has the following useful corollary.

Corollary 13. *Let w be a completely join-irreducible element of $FL(X)$ with $\kappa(w) \neq w_*$. If u is an element such that $w_* = u \wedge \kappa(w)$, then either*

$$u = w_* \quad or \quad u = w \quad or \quad u \succ w.$$

Moreover, the last case only occurs if w is semisingular.

Tschantz's Theorem.

Nation and I felt that it would be relatively easy to show that every infinite interval of a free lattice would contain $FL(\omega)$ as a sublattice. We were surprised when we were not even able to rule out the possibility of an infinite interval which is a chain. (See the introduction and Chapter IX of [F15] for more of the interesting history of this problem.) This problem was solved by Tschantz [F30] with the following deep theorem.

Theorem 14 Every infinite interval of a free lattice contains a sublattice isomorphic to $FL(\omega)$.

The major part of the proof is showing that, in fact, there is no infinite interval which is a chain. This can be derived easily from Corollary 13: Suppose that the interval a/b is an infinite chain. By Day's Theorem and the fact that there are no long chains of covers, this interval contains a completely join-irreducible element w with $b < w_* \prec w < a$ and both a/w and w_*/b are infinite (or the dual situation holds). But this clearly cannot happen by Corollary 13.

Tschantz raised the following problem: can a free lattice have elements $a > c > b$ such that both a/c and c/b are infinite and every element of a/b is comparable with c. It is shown that no such element exists in Theorem 10.1 of [F15], using rather different techniques.

Maximal chains.

The characterization of semisingular elements allows us to answer some important questions about maximal chains in free lattices. Indded, this was our original motivation for characterizing them. The problem is this: for which pairs $a > b$ in $FL(X)$ is there a maximal chain in a/b without any covers? If a/b is atomic, then, since an element of a free lattice can have at most finitely many covers, there is no such maximal chain. It turns out (Theorem 9.15 of [F15]) that in all other cases there is such a dense maximal chain.

Theorem 15 If $a > b$ in $FL(X)$ and a/b is neither atomic nor dually atomic, then there is a maximal chain from b to a without covers.

Actually, Theorem 9.15 proves more: we can find such a chain each of whose elements, except possibly a and b, is coverless, that is, has no upper and no lower cover in $FL(X)$.

5. Applications and Related Areas

Free lattices and the techniques used to study them have applications in several areas of lattice theory and even in universal algebra. This section explores these areas. We begin by introducing some basic concepts.

Bounded homomorphisms.

Theorem 8 gives a very easy (and very efficient) way to test if an element w is completely join-irreducible in a free lattice and Theorem 9 shows how to find $\kappa(w)$ and therefore w_*. While these theorems are very nice, they obscure some of the details behind them and since these methods are applicable in several other areas, we describe them now.

Let $h\colon K \to L$ be a lattice epimorphism. Then we say that h is a *lower bounded homomorphism* if $h^{-1}(a)$ has a least element for each $a \in L$. An *upper bounded homomorphism* is defined dually and a homomorphism is *bounded* if it is both lower and upper bounded. When h is a lower bounded homomorphism, we let $\beta(a)$ denote the least preimage of a. $\alpha(a)$ is the dual concept. This concept was introduced in R. N. McKenzie [1972]. In this paper, McKenzie makes a thorough study of lattice varieties and free lattices play an important role. Lattices which are bounded homomorphic images of free lattices (*bounded lattices* for short) are closely connected with covers in free lattices. In fact, a join-irreducible element $w \in FL(X)$ is completely join-irreducible if and only if the finite lattice $L(w)$ defined in the previous section is bounded and this fact is used in the proofs of Theorems 8 and 9.

B. Jónsson, in his studies of projective lattices and sublattices of free lattices, defined a dependency relation D on a lattice L:

$$a \, D \, b \quad \text{if } a \neq b,\ b \text{ is join-irreducible, and there is a } \\ p \in L \text{ with } a \leq b \vee p \text{ and } a \nleq c \vee p \text{ for } c < b.$$

We say that *a depends on b*. Of course, (the transitive closure of) D defines a quasiorder on L. We view D as 'less than or equal to,' that is, if $a\,D\,b$ then a is less than or equal to b in this quasiorder.

Although this dependency relation and the notion of bounded homomorphisms seem quite different, they are actually closely related as the following theorem (Corollary 2.39 of [F15]) shows.

Theorem 16 A finite lattice L is lower bounded if and only if it contains no D–cycle.

Having no D–cycle means that the D–relation determines a partial order. When L is finite, this means the elements can be ranked by their depth in this ordered set. The elements of rank 0 are the maximal elements of this order; these elements are the join-prime elements. Using this ranking, one can inductively calculate $\beta(a)$ (for any homomorphism h onto L). Even when there is a D–cycle one can calculate a sequence $\beta_n(a)$, $n \geq 0$, which is coinitial in $h^{-1}(a)$.

These concepts arose in connection with free lattices and play an important role there but they also play an important role in several other areas.

Congruence lattices.

There is a strong connection between the D-relation and congruences on L, especially when L is finite. Indeed, an easy calculation shows that if $a\ D\ b$ for join-irreducibles a and b in a finite lattice, then $\Theta(a, a_*) \leq \Theta(b, b_*)$.

Let L be a finite lattice. Even if the D-relation contains a cycle, it still determines a quasiorder on the set of nonzero join-irreducible elements. Of course a quasiorder induces an equivalence relation ($a \sim b$ if and only if $a \leq b$ and $b \leq a$) and if we factor by this equivalence relation, we get an ordered set. By Theorem 2.35 of [F15], this ordered set is isomorphic to the ordered set of join-irreducible congruences on L. This is the basis of a very efficient method of finding the ordered set of join-irreducible congruences; see [F12] and Section 5 of Chapter XI of [F15].

Theorem 17 Let L be a lattice with n elements. Then each of the following can be determined in time $O(n^2)$:

1. If L is simple.
2. If L is subdirectly irreducible.
3. If L is directly indecomposable.
4. If L is a bounded homomorphic image of a free lattice.
5. If L is semidistributive.
6. If L is a splitting lattice.

Most of the above are based on an $O(n^2)$ algorithm which finds the ordered set of join-irreducible congruences. This algorithm finds the underlying set and a relation whose transitive closure is the order relation and this is good enough for the above facts. But for others things we need to find the full $\leq$ relation. In [F12], I give an $O(n^2 \log n)$ algorithm for doing this. This is based on the fact that given an ordered set P having e covers, a lattice whose join-irreducible congruences is isomorphic to P cannot be too small. Specifically, we have:

Theorem 18 If the ordered set of join-irreducible congruences on a finite lattice L has $e_\prec$ covers with $e_\prec > 2$, then

$$|L| \geq \frac{e_\prec}{2 \log_2 e_\prec}.$$

Investigations into the relation of $|L|$ to the number of join-irreducible congruences were done by G. Grätzer, H. Lakser, and E. T. Schmidt [F20] and G. Grätzer, I. Rival, and N. Zaguia in [F22]; see Section C.1.

Sublattices of free lattices.

In proving his theorem characterizing finite sublattices of free lattices, Theorem 5, Nation used two relations, A and B, originating with B. Jónsson, which refine the D–relation. If Nation's Theorem failed, there would be a finite semidistributive lattice containing a D–cycle. Such a lattice would contain a cycle of join-irreducible elements such that consecutive elements were related by either A or B. Nation's proof begins with a beautiful duality result [F25] (Theorem 2.63 of [F15]):

$$\begin{aligned} a\ A\ b &\iff \kappa(a)\ B^{\mathrm{d}}\ \kappa(b), \\ a\ B\ b &\iff \kappa(a)\ A^{\mathrm{d}}\ \kappa(b). \end{aligned} \tag{1}$$

Now $a\ A\ b$ implies that $a < b$, so clearly there can be no cycle of all A's; the duality result immediately implies there is no cycle of all B's, and this plays a key role in the proof of Nation's Theorem. It also immediately gives an important result of A. Day [F5]:

Theorem 19 A finite lattice which is a lower bounded homomorphic image of a free lattice is bounded if and only if it is semidistributive.

This result is used in the fast algorithm to determine if an element w of a free lattice has a lower cover given in Theorem 8.

Projective lattices.

Projective lattices, that is, retracts of free lattices, and projective configurations play and important role in the study of lattice varieties. Of course projective lattices are sublattices of free lattices. B. Jónsson, A. Kostinsky, and R. N. McKenzie (see R. N. McKenzie [1972] and A. Kostinsky [1972]) proved the converse for finitely generated lattices.

Theorem 20 A finitely generated lattice is projective if and only if it is isomorphic to a sublattice of a free lattice. These conditions hold if and only if the lattice satisfies (W) and is a bounded homomorphic image of a free lattice.

Thus by Nation's Theorem a finite lattice is projective if and only if it is semidistributive and satisfies (W).

As mentioned above, a finite lattice is a lower bounded homomorphic image of a free lattice if and only if the D-relation is acyclic. In this case, the D-relation defines a partial order. For a finitely generated lattice to be lower bounded we need that all chains above each element of this order are finite. The converse also holds.

Arbitrary projective lattices were characterized in [F18]. The most difficult and perhaps most surprising part of the proof consists of showing that the above condition on the D-relation holds in all projective lattices.

Theorem 21 A lattice L is projective if and only if it satisfies the following conditions:

1. Whitman's condition (W);
2. The D relation is acyclic and every D–chain above each element is finite;
3. The D^{d} relation is acyclic and every D^{d}–chain above each element is finite;
4. L has the minimal join-cover refinement property and the dual property;
5. L is finitely separable.

A finite subset S of L is a *join-cover* of $a \in L$ if $a \leq \bigvee S$. We say L has the *minimal join-cover refinement property* if for each $a \in L$ there is a finite set, $\mathcal{M}(a)$, of join-covers of a such that if T is a join-cover of a there is an $S \in \mathcal{M}(a)$ with $S \ll T$. The lattice L is *finitely separable* if for each $a \in L$ there are two finite sets $\mathrm{A}(a)$ and $\mathrm{B}(a)$ such that $\mathrm{A}(a) \subseteq \{c \in L : c \geq a\}$, $\mathrm{B}(a) \subseteq \{c \in L : c \leq a\}$, and if $a \leq b$, then $\mathrm{A}(a) \cap \mathrm{B}(b)$ is nonempty.

Sharply transferable lattices.

Transferable and sharply transferable lattices are discussed in the text and in Appendix A. As pointed out there, for finite lattices these concepts coincide and, in fact, are precisely the finite sublattices of free lattices. So Nation's Theorem gives a strong characterization of these lattices. Although there is no good characterization of general transferable lattices, there is a very nice characterization of sharply transferable lattices due to G. Grätzer and C. R. Platt [F21]. (Note the similarity to arbitrary sublattices of free lattice for which there is no good characterization and projective lattices for which there is.)

Theorem 22 A lattice L is sharply transferable if and only if it satisfies the following four conditions:

1. Whitman's condition (W);
2. The D relation is acyclic and every D–chain *below* each element is finite;
3. The D^{d} relation is acyclic and every D^{d}–chain above each element is finite;
4. For each $x \in L$, the set $\{y \mid y \ngeq x\}$ is finite.

Notice how similar Theorems 21 and 22 are. The last condition is a finiteness condition as are the two last conditions of Theorem 21, which are, in fact, implied by the last condition of Theorem 22. The first and third conditions of the two theorems are the same but note the curious difference in the second conditions.

Nation used Theorem 22 and his duality theorem given in formula (1) to show that strongly transferable lattices are projective [F25]. Although his duality theorem depended on L being finite, he showed that for a lattice satisfying the last condition of Theorem 22 that at least the forward implications of (1) hold. Now the crux of proving that sharply transferable lattices are projective is showing that they satisfy the second condition of Theorem 21. But the third condition of Theorem 22 together with the forward direction of his duality theorem show this:

Theorem 23 Every sharply transferable lattice is projective.

Nation also gives an example of a lattice which is projective and even satisfies the last condition of Theorem 22 but is not transferable.

Varieties of lattices and congruence varieties.

Both of these topics are covered in Appendix G so we will just make a few comments. Jónsson's Theorem (B. Jónsson [1967]) motivated McKenzie to make a thorough study of lattice varieties in R. N. McKenzie [1972]. The concept of a bounded homomorphism began with this paper. A finite, subdirectly irreducible lattice L which is a bounded homomorphic image of a free lattice is called a *splitting lattice*. By Jónsson's Theorem, if a subdirectly irreducible lattice is in the join of finitely many varieties, it is in one of them. For a splitting lattice this is true for an arbitrary join of lattice varieties and this leads to a 'splitting' of the lattice of all lattice varieties. So if L is a splitting lattice, there is a conjugate equation ε such that every variety either contains L or satisfies ε, but not both.

As we pointed out above, splitting lattices are closely associated with covers in free lattices. If w is a join-irreducible element of a free lattice $FL(X)$ with a lower cover w_*, then $FL(X)/\Psi(w, w_*)$ is a splitting lattice with conjugate equation $w \approx w_*$, where $\Psi(w, w_*)$ is the unique largest congruence separating w and w_*. Moreover, every splitting lattice has this form. Day's Theorem, Theorem 6, shows that there are many splitting lattices; in fact, they generate the variety of all lattices.

A *congruence variety* is a variety of lattices generated by all the congruence lattices of the members of some variety of algebras. Bounded homomorphisms and splitting lattices played an essential role in this area; see [F23] for a survey of the results up to about 1979 and [F11] for subsequent developments. One of the major questions of this area was the existence of nonmodular congruence variety other than the variety of all lattices. S. V. Polin [F27] answered this question by constructing a variety, $\mathcal{P}$, whose associated congruence variety was nonmodular but did satisfy a nontrivial lattice equation. A thorough analysis of Polin's congruence variety is carried out in [F6]. It is there shown that the congruence lattice of finitely generated free algebra in $\mathcal{P}$ is a splitting lattice and the associated splitting equations are given. Using these splitting equations, I proved with A. Day that Polin's congruence variety is the unique minimal nonmodular congruence variety. With G. Czédli [F3], I gave an effective characterization of lattice equations which imply congruence modularity (and distributivity).

Theorem 24 Polin's congruence variety is the unique minimal nonmodular congruence variety. There is an effective procedure to determine if a lattice equation implies congruence modularity.

As a second example, consider the variety of all (meet) semilattices. The two-element semilattice, $\mathfrak{C}_2$, is the only subdirectly irreducible (even finitely subdirectly irreducible), hence the meet-irreducible congruences of a semilattice are all coatoms. If a is a nonzero element of a semilattice S, then let Ψ_a be the congruence with two equivalence classes, $\{x \mid x \geq a\}$ and its complement. This is always a coatom in $\operatorname{Con} S$ and when S is finite, these are the only coatoms. Now it is not hard to show that if $\Psi_a \; D^{\mathrm{d}} \; \Psi_b$, then $a > b$ in S; see Lemma 2.86 of [F15]. Thus D^{d} is acyclic. In particular, we get the theorem of K. V. Adaricheva [F1]:

Theorem 25 If S is a finite semilattice, then $\operatorname{Con} S$ is an upper bounded homomorphic image of a free lattice.

D. Papert [1964] had shown that the congruence lattice of a semilattice satisfies $(\mathrm{SD}_\wedge)$ but being upper bounded is much stronger. In fact this relation between $(\mathrm{SD}_\wedge)$ and upper boundedness holds much more generally; see [F17].

Theorem 26 If $\mathcal{V}$ is a variety of algebras such that the congruence lattices of the members of $\mathcal{V}$ satisfy $(\mathrm{SD}_\wedge)$, then the congruence lattices of the finite algebras in $\mathcal{V}$ are upper bounded homomorphic images of a free lattice. If the congruence lattices of the algebras in a variety $\mathcal{W}$ satisfy $(\mathrm{SD}_\vee)$, then the congruence lattice of the finite algebras in $\mathcal{W}$ are both upper and lower bounded homomorphic images of a free lattice.

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