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A decomposition theorem for modular lattices containing an n-diamond

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Dedicated to the memory of András Huhn

In the 1930's Von Neumann developed his concept of an n-frame in order to study the coordinatization of complemented modular lattices. In the late 1960's and early 1970's A. P. Huhn revived a variant of this concept, which he called n-diamonds, and used it in his work on modular lattices which were not necessarily complemented. He developed the basic theorems for this concept including the result that n-diamonds (and n-frames) are a projective configuration for the class of modular lattices, [12]. This means that if $f\colon L \twoheadrightarrow M$ is a surjection of modular lattices and M contains an n-diamond then this n-diamond can be pulled back through f to an n-diamond in L.

One of the main themes of modern lattice theory has been the study of lattice varieties. By Birkhoff's theorem, in order to study varieties one needs to understand the operators **H**, **S** and **P** (the closure of classes of algebras under homomorphisms, subalgebras, and direct products, respectively). In the post Jónsson's theorem era of the 1970's, the major unsolved problems on varieties of lattices centered on **H**. It is here that Huhn's result is so useful. Von Neumann showed that associated with each n-frame (and hence each n-diamond) in a modular lattice is a ring. This fact, together with Huhn's projectivity result, has played a crucial role in many of the most important results on modular varieties, certainly in the author's best work.

In this paper we prove the following decomposition theorem, analogous to Fitting's lemma, for finite dimensional modular lattices containing an n-frame. The definitions will be given below.

Theorem 1. *Let L be a finite dimensional modular lattice containing a spanning n-frame, $n \geqq 4$. Then L is a finite direct product of lattices L_i where the ring,*

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$R(L_i)$, associated with (the frame in) L_i has prime power characteristic or the field of rational numbers $\mathbf{Q}$ *is a subring of* $R(L_i)$.

One of the deepest and most important results on modular lattices containing an n-frame is Christian Herrmann's characterization of all subdirectly irreducible modular lattices generated by an n-frame, $n \geqq 4$ [9]. Herrmann's result builds on Huhn's idea of representing automorphisms of frames [13] and the author's result [5] which proves Herrmann's result in the case that the ring associated with the frame has prime characteristic. With the aid of his theorem Herrmann was able to prove the following very powerful result on varieties of modular lattices. Let $\mathcal{M}_0$ denote the variety generated by all subspace lattices of vector spaces over $\mathbf{Q}$.

Theorem 2. (Herrmann [9]). *Every variety of modular lattices which contains* $\mathcal{M}_0$ *either is not generated by its finite dimensional members or does not have a finite equational basis.*

The following corollary illustrates the power of this theorem. Let $\mathcal{M}$ denote the variety of all modular lattices and let $\mathcal{M}_{\mathrm{f}}$ and $\mathcal{M}_{\mathrm{fd}}$ denote the variety generated by all finite (respectively finite dimensional) modular lattices. Let $\mathcal{A}$ be the variety of all arguesian lattices. (The arguesian law, which is due to Jónsson [14], is stronger than the modular law and related to Desargues law in projective geometry.)

Corollary 3. $\mathcal{A}$ *is not generated by its finite dimensional members. Neither* $\mathcal{M}_{\mathrm{f}}$ *nor* $\mathcal{M}_{\mathrm{fd}}$ *is finitely based. Moreover,* $\mathcal{M}_{\mathrm{f}} < \mathcal{M}_{\mathrm{fd}} < \mathcal{M}$.

Since Herrmann's proof of Theorem 2 uses his characterization described above, his proof is quite lengthy. We use the decomposition theorem to give a short proof of his theorem.

The first two sections of this paper give the basic definitions and some results about these concepts. Theorem 1 is proved in the third section. The fourth section proves Herrmann's result, Theorem 2. The fifth section uses a new result of the author to show that the lattices Herrmann used to prove his theorem have a representation by permuting equivalence relations, i.e., a type I representation. The sixth section examines the case of 3-frames. In this case the "ring" associated with the frame may not really be a ring. Nevertheless, an analogue of Theorem 1 can be proved.

1. Preliminaries. We use $+$ and $\cdot$ or juxtaposition to indicate lattice join and meet. An *n-frame* in a lattice L is a subset $\{a_i, c_{ij}\colon i \neq j \text{ and } 1 \leqq i \leqq n\}$ of L such that

$$a_i \cdot \bigvee_{j \neq i} a_j = \bigwedge_{k=1}^{n} a_k, \tag{1}$$

(2) $$a_i \cdot c_{ij} = a_j \cdot c_{ij} = a_i \cdot a_j = \bigwedge_k a_k,$$

(3) $$a_i + c_{ij} = a_j + c_{ij} = a_i + a_j,$$

(4) $$c_{ij} = c_{ji},$$

(5) $$(c_{ij} + c_{jk})(a_i + a_k) = c_{ik}$$

for all distinct i, j, k between 1 and n. A set $\{a_1, \dots, a_n\}$ is called *independent over* $\bigwedge_k a_k$ if (1) holds. An n-frame in L is called a *spanning n-frame* if $\bigwedge_k a_k = 0_L$ and $\bigvee_k a_k = 1_L$. Let $\{a_i, c_{ij}\}$ be an n-frame, $n \geqq 4$, in a modular lattice L. The ring associated with this frame is, (where we use $\oplus$ and $\otimes$ to denote the addition and multiplication to avoid confusion with the lattice operations)

(6) $$R = \{x \in L\colon\ x + a_2 = a_1 + a_2,\ x \cdot a_2 = a_1 \cdot a_2\},$$

and for $x, y \in R$,

(7) $$x \oplus y = [(x + c_{13})(a_2 + a_3) + (y + a_3)(a_2 + c_{13})](a_1 + a_2),$$

(8) $$x \otimes y = [(x + c_{23})(a_1 + a_3) + (y + c_{13})(a_2 + a_3)](a_1 + a_2).$$

By Theorem 8.4 and Lemma 6.1 of [15] R is a ring with zero a_1 and unit c_{12}. From now on L will denote a lattice containing a fixed spanning n-frame, $n \geqq 4$, and $R(L)$ will denote the ring associate with this n-frame. At the end of the paper the case $n = 3$ will be discussed.

Lemma 1.1. *An element $x \in R(L)$ is invertible if and only if $x + a_1 = a_1 + a_2$ and $x \cdot a_1 = 0$.*

Proof. An elementary proof is given in [6].

An element b in L is called *homogeneous* (*with respect to the frame* $\{a_i, c_{ij}\}$) if $b_i = a_i \cdot b$, $i = 1, \dots, n$, satisfy $b = \bigvee_i b_i$ and

(9) $$b_j = a_j(b_i + c_{ij}).$$

Whenever b is homogeneous, we shall use the notation $b_i = a_i \cdot b$. The next lemma can be proved with easy calculations.

Lemma 1.2. (i) *Let $k \leqq n$ and suppose that we have an element $b_k \in L$ such that $0 \leqq b_k \leqq a_k$. Let $b_i = a_i(b_k + c_{ik})$, $i \neq k$, and $b = \bigvee_i b_i$. Then b is homogeneous.*

(ii) *If b is homogeneous then $\{a_i + b,\ c_{ij} + b\}$ is an n-frame which spans the interval $1/b$, and $\{a_i \cdot b,\ c_{ij} \cdot b\}$ is an n-frame spanning $b/0$.*

We denote the rings associated with these frames by $R(1/b)$ and $R(b/0)$. More generally, if $b \leqq e$ are both homogeneous, then $\{a_i \cdot e + b,\ c_{ij} \cdot e + b\}$ is a frame which spans e/b. Its ring is denoted $R(e/b)$.

2. Stabilizers. One of the difficulties of these concepts is that if $x \in R(L)$ it does not follow that $x+b \in R(1/b)$. If b is homogeneous we define the *stabilizer of* b, denoted R_b, by

$$R_b = \{x \in R(L):\ b_1 \leqq x + b_2\}, \tag{10}$$

where, of course, $b_i = a_i \cdot b$. We say that $x \in R(L)$ is *stable* if it is in R_b for every homogeneous b. (This differs slightly from Herrmann's use of this term in [8].) The next lemmas collect the basic information on stable elements.

Lemma 2.1. *Let L be a modular lattice containing a spanning n-frame and let b be a homogeneous element. Let $x \in R(L)$. Then the following are equivalent:*

(i) $x \in R_b$,
(ii) $x+b \in R(1/b)$,
(iii) $x \cdot b \in R(b/0)$.

Proof. Suppose that x satisfies (i). To show that $x+b \in R(1/b)$ we need to prove that $(x+b)(a_2+b)=b$. Using (i) and the independence of the a_i's we calculate

$$\begin{aligned}(x+b)(a_2+b) &= a_2(x+b)+b = a_2(a_1+a_2)(x+b)+b = \\ &= a_2\big(x+(a_1+a_2)b\big)+b = a_2(x+b_1+b_2)+b = \\ &= a_2(x+b_2)+b = a_2 \cdot x + b_2 + b = b.\end{aligned}$$

Thus (i)→(ii). To see that (i)→(iii) we need to show that $x \cdot b + a_2 \cdot b = b_1 + b_2$.

$$\begin{aligned}x \cdot b + a_2 \cdot b &= x \cdot b + b_2 = (x+b_2) \cdot b = (x+b_1+b_2) \cdot b = x \cdot b + b_1 + b_2 = \\ &= x \cdot (a_1+a_2) \cdot b + b_1 + b_2 = x \cdot (b_1+b_2) + b_1 + b_2 = b_1 + b_2.\end{aligned}$$

Now if $x \cdot b \in R(b/0)$ then $x \cdot b + b_2 = b_1 + b_2$. Hence $b_1 \leqq x + b_2$. Hence (iii)→(i). Similarly (ii)→(i).

Lemma 2.2. *R_b is a subring of $R(L)$ closed under taking inverses when they exist. The maps $x \mapsto x+b$ and $x \mapsto x \cdot b$ are ring homomorphisms form R_b to $R(1/b)$ and $R(b/0)$, respectively.*

Proof. By (9) both a_1 and c_{12} (the zero and one of $R(L)$) are in R_b. If $x, y \in R_b$ then using (8) and (9)

$$\begin{aligned}x \otimes y + b_2 &= [(x+c_{23})(a_1+a_3)+(y+c_{13}+b_2)(a_2+a_3)](a_1+a_2) = \\ &= [(x+c_{23})(a_1+a_3)+(y+c_{13}+b_1+b_2)(a_2+a_3)](a_1+a_2) = \\ &= [(x+c_{23})(a_1+a_3)+(y+c_{13}+b_1+b_2+b_3)(a_2+a_3)](a_1+a_2) = \\ &= [(x+c_{23})(a_1+a_3)+b_3+(y+c_{13}+b_1+b_2)(a_2+a_3)](a_1+a_2) \geqq \\ &\geqq (x+c_{23}+b_3)(a_1+a_3)(a_1+a_2) = (x+c_{23}+b_3+b_2)(a_1+a_3)(a_1+a_2) \geqq b_1.\end{aligned}$$

Thus $x \otimes y \in R_b$. Similarly $x \oplus y \in R_b$. If x is invertible in $R(L)$ then a formula for x^{-1} is given in [6]. Using this, one can show that $x^{-1} \in R_b$.

We let $\otimes^b$ denote the multiplication for $R(1/b)$ and $\otimes_b$ for $R(b/0)$. Let $x, y \in R_b$. Then $b_1 \leqq b_2 + y$ and since $b_i + c_{ij} = b_j + c_{ij}$ by (9), we have,

$$(x+b)\otimes^b(y+b) = [(x+c_{23}+b)(a_1+a_3+b)+(y+c_{13}+b)(a_2+a_3+b)](a_1+a_2+b) =$$

$$= [(x+c_{23}+b_1+b_3)(a_1+a_3)+(y+c_{13}+b)(a_2+a_3)+b](a_1+a_2)+b =$$

$$= [(x+c_{23})(a_1+a_3)+(y+c_{13}+b)(a_2+a_3)+b](a_1+a_2)+b =$$

$$= [(x+c_{23})(a_1+a_3)+(y+c_{13}+b_2)(a_2+a_3)+b_1+b_2](a_1+a_2)+b =$$

$$= [(x+c_{23})(a_1+a_3)+(y+c_{13})(a_2+a_3)](a_1+a_2)+b = x\otimes y+b.$$

These and similar calculations show that $x \mapsto x+b$ and $x \mapsto x \cdot b$ are ring homomorphisms from R_b into $R(1/b)$ and $R(b/0)$.

Notice that this lemma implies that if x is in the prime subring of $R(L)$ (the subring generated by 1) or is the inverse of an element in the prime subring then x is stable.

Notation and motivation. If S is a ring and M is a unitary left S-module then the lattice of submodules, $L(M^n)$, of the module M^n, contains a natural spanning n-frame, namely,

$$a_i = \{(0, \ldots, \overset{i\text{th}}{x}, \ldots, 0) \colon x \in M\},$$

$$c_{ij} = \{(0, \ldots, \overset{i\text{th}}{x}, \ldots, \overset{j\text{th}}{-x}, \ldots, 0) \colon x \in M\}.$$

Linear algebraic calculations show that the ring associated with this frame, $R(L(M^n))$, is the endomorphism ring of M. A homogeneous element has the form $\{(x_1, \ldots, x_n) \colon x_i \in B\}$ for some submodule B of M. The stabilizer of this homogeneous element is the subring of those endomorphisms of M which map B into itself. Simple calculations also show that if $r \in R(L(M^n))$ then $a_1 \cdot r = \{(x, 0, \ldots, 0) \colon xr = 0\}$, i.e., $a_1 \cdot r$ is the kernel of r (in the first coordinate). Similarly, $a_2(a_1+r)$ is the range of r (in the second coordinate).

For a general modular lattice containing an n-frame, and $x \in R(L)$, there are, by Lemma 1.2, homogeneous elements $b(x)$ and $d(x)$ such that $b(x)_1 = a_1 \cdot x$ and $d(x)_2 = a_2(a_1+x)$. Thus $b(x)$ corresponds to the kernel of x and $d(x)$ to the image.

Lemma 2.3. *Let $x \in R(L)$ and let $b = b(x)$ and $d = d(x)$. Then $x \in R_b$ and $x \in R_d$ and*

(i) *$x+d$ is the zero element of $R(1/d)$, and*

(ii) *$x \cdot b$ is the zero element of $R(b/0)$.*

Proof. Since $x+a_2=a_1+a_2$,

$$x+d_2 = x+a_2(x+a_1) = (x+a_2)(x+a_1) = x+a_1(x+a_2) = x+a_1 \geqq a_1.$$

Thus $x \in R_d$. Trivially $x \in R_b$. The above calculation shows that $x+d_2=x+a_1$. Hence,

$$a_1+d = a_1+d_2+d = (a_1+a_2)(x+a_1)+d = x+a_1+d = x+d_2+d = x+d.$$

Since a_1+d is the zero element of $R(1/d)$, this proves (i). Again (ii) is trivial; $x \cdot b=b_1$, which is the zero of $R(b/0)$.

3. Proof of Theorem 1.

Lemma 3.1. *If* $u \leqq a_1$ *then*

$$(u+c_{12})a_2 = \big((u+c_{13})a_3+c_{23}\big)a_2.$$

Proof. Let w be the right side of the above equation. Then

$$w+c_{23}+c_{13} = \big((u+c_{13})a_3+c_{23}\big)(a_2+a_3)+c_{13} =$$

$$= (u+c_{13})a_3+c_{23}+c_{13} = u(a_1+a_3)+c_{23}+c_{13} = u+c_{23}+c_{13}.$$

Meeting both sides with a_1+a_2 gives $w+c_{12}=u+c_{12}$. Thus since $w \leqq a_2$, $w= =(w+c_{12})a_2=(u+c_{12})a_2$, as desired.

For $x \in R(L)$ we let x^2 denote $x \otimes x$.

Lemma 3.2. *Let* $x \in R(L)$ *and suppose that* $a_1 \cdot x=a_1 \cdot x^2$. *Let* $b=b(x)$. *Then*

$$(a_1+b)(x+b) = b.$$

Proof. Since $x \leqq a_1+a_2$, $a_1(x+b)=a_1(a_1+a_2)(x+\bigvee b_i)=a_1(x+b_1+b_2)= =b_1+a_1(x+b_2)$. (In the future we shall omit the details of these independence type arguments.) Thus

$$(a_1+b)(x+b) = b+a_1(x+b) = b+a_1(x+b_2) = b+a_1\big(x+a_2(c_{12}+a_1 \cdot x)\big).$$

Now we calculate, using the last lemma

$$a_1 \cdot x^2 = a_1[(x+c_{23})(a_1+a_3)+(x+c_{13})(a_2+a_3)] =$$

$$= a_1(a_1+a_3)[(x+c_{23})(a_1+a_3)+(x+c_{13})(a_2+a_3)] =$$

$$= a_1[(x+c_{23})(a_1+a_3)+a_3(x+c_{13})] = a_1(a_1+a_3)\big(x+c_{23}+a_3(x+c_{13})\big) =$$

$$= a_1\big(x+c_{23}+a_3(a_1+a_3)(x+c_{13})\big) = a_1\big(x+c_{23}+a_3(a_1 \cdot x+c_{12})\big) \geqq$$

$$\geqq a_1\big(x+a_2(c_{23}+a_3(a_1 \cdot x+c_{12}))\big) = a_1\big(x+a_2(c_{12}+a_1 \cdot x)\big).$$

Thus $(a_1+b)(x+b) \leqq b+a_1 \cdot x^2=b+a_1 \cdot x=b+b_1=b$.

Lemma 3.3. *If L is finite dimensional and $x\in R(L)$ satisfies $a_1\cdot x=0$, then $a_1+x=a_1+a_2$ and thus x is invertible.*

Proof. Suppose that $a_1\cdot x=0$. It is easy to see that we have the following transpositions

$$(a_1+x)/a_1 \searrow x/0 \nearrow (a_1+a_2)/a_2 \searrow c_{12}/0 \nearrow (a_1+a_2)/a_1.$$

Thus the dimension of $(a_1+x)/a_1$ equals that of $(a_1+a_2)/a_1$. Since $a_1+x\leqq a_1+a_2$, this forces equality and the result follows.

Theorem 3.4. *Let L be finite dimensional, $x\in R(L)$ and let $b=b(x)$ and $d=d(x)$ be the elements defined in Section 2. Suppose that x satisfies $a_1\cdot x=a_1\cdot x^2$, then $x+b$ is invertible in $R(1/b)$ and $a_1=b_1+d_1$.*

Proof. That $x+b$ is invertible in $R(1/b)$ follows from Lemmas 1.1, 3.2, and 3.3. Now let $e_1=b_1+d_1$ and let $e=b+d$ be the homogeneous element associated with e_1 (cf. Lemma 1.2). By Lemma 2.3 $x\in R_b$ and R_d. From this it follows that $x\in R_e$. Now since $e\geqq d$, e is a homogeneous element for the frame $\{a_i+d, c_{ij}+d\}$ and by Lemma 2.1 $x+d\in R(1/d)_e$ since $x+d+e=x+e$. By Lemma 2.2 we have three ring homomorphisms, $f\colon R_d\to R(1/d)$, $g\colon R(1/d)_e\to R(1/e)$, and $h\colon R_e\to$ $\to R(1/e)$. Clearly, $g(f(x))=h(x)$. Since $f(x)=x+d$ is the zero element of $R(1/d)$ by Lemma 2.3, $h(x)=x+e$ is the zero element of $R(1/e)$. However, $x+b$ is invertible in $R(1/b)$. By Lemma 2.3 there is a homomorphism of $R(1/b)_e$ into $R(1/e)$ and the image of $x+b$ is $x+b+e=x+e$. Thus $x+e$ is an invertible element of $R(1/e)$. Checking the definition of the ring of a frame one sees that the only way an element of the ring can be both zero and invertible is if the frame is trivial. Thus $e=1$ and thus $a_1\cdot e=a_1=b_1+d_1$, as desired.

Theorem 3.5. *Let L be finite dimensional, $x\in R(L)$ and let $b=b(x)$ and $d=d(x)$ be the elements defined in Section 2. Suppose that x* satisfies *$a_1+x=a_1+x^2$, then $x\cdot d$ is invertible in $R(d/0)$ and $0=b_1\cdot d_1$.*

Proof. Since $x\leqq a_1+a_2$, we have using (9)

$$\begin{aligned} d_2 &= a_2(a_1+x) = a_2(a_1+x^2) = \\ &= a_2[a_1+(x+c_{13})(a_2+a_3)+(x+c_{23})(a_1+a_3)] = \\ &= a_2[(x+c_{13})(a_2+a_3)+(x+a_1+c_{23})(a_1+a_3)] = \\ &= a_2[(x+c_{13})(a_2+a_3)+a_1+a_3(x+a_1+c_{23})] = \\ &= a_2[(x+c_{13})(a_2+a_3)+a_3(x+a_1+c_{23})] = a_2\big(x+c_{13}+a_3(x+a_1+c_{23})\big) = \\ &= a_2\big[x+c_{13}+a_3\big(c_{23}+(x+a_1)(a_2+a_3)\big)\big] = a_2\big[x+c_{13}+a_3\big(c_{23}+a_2(x+a_1)\big)\big] = \\ &= a_2\big[x+(a_1+a_2)\big(c_{13}+a_3(c_{23}+a_2(x+a_1))\big)\big] = a_2\big[x+a_1\big(c_{13}+a_3(c_{23}+a_2(x+a_1))\big)\big] = \\ &= a_2\big[x+a_1\big(c_{13}+a_3(c_{23}+d_2)\big)\big] = a_2[x+a_1(c_{13}+d_3)] = a_2(x+d_1). \end{aligned}$$

Thus $d_2 \leqq x+d_1$ and so $x \cdot d+d_1=d(x+d_1) \geqq d \cdot d_2=d_2$. Hence $x \cdot d+d_1=d_1+d_2$. By an argument similar to the proof to Lemma 3.3, this in turn implies $x \cdot d \cdot d_1= =x \cdot d_1=0$, and thus $x \cdot d$ is invertible in $R(d/0)$ by Lemma 1.1. If we let $e=b \cdot d$ then by the argument of the last theorem, $x \cdot e$ is both the zero and invertible in $R(e/0)$, showing that $e=0$. Hence, $b_1 \cdot d_1=0$, as desired.

It is easy to see that for $x \in R(L)$, $a_1 \cdot x \leqq a_1 \cdot x^2 \leqq a_1 \cdot x^3 \leqq \ldots$, and $a_1+x \geqq \geqq a_1+x^2 \geqq a_1+x^3 \geqq \ldots$. For example, to see the former let $y \in R(L)$. Then

$$a_1 \cdot (x \otimes y) = a_1[(x+c_{23})(a_1+a_3)+(y+c_{13})(a_2+a_3)] =$$

$$= a_1[(x+c_{23})(a_1+a_3)+a_3(y+c_{13})] = a_1(x+c_{23}+a_3(y+c_{13})) \geqq a_1 \cdot x$$

from which $a_1 \cdot x \leqq a_1 \cdot x^2 \leqq \ldots$ follows.

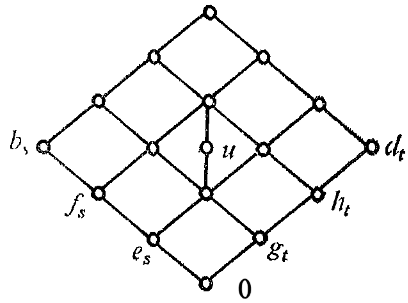
Now if L is finite dimensional there is a k such that $a_1 \cdot x^{2k}=a_1 \cdot x^k$ and $a_1+x^{2k}= =a_1+x^k$. Thus we have the following corollary.

Corollary 3.6. *Let L be a finite dimensional modular lattice containing a spanning n-frame, $n \geqq 4$, and let $x \in R(L)$. Then there are homogeneous elements b and d such that b and d are complements, $x \in R_b \cap R_d$, for some k, x^k is the zero of $R(b/0)$ and x is invertible in $R(d/0)$.*

Proof. As above we choose k such that $a_1 \cdot x^{2k}=a_1 \cdot x^k$ and $a_1+x^{2k}=a_1+x^k$. Let $b=b(x^k)$ and $d=d(x^k)$. Now the result will follow from the previous results once it is shown that $x \in R_b \cap R_d$. To see that $x \in R_b$ we need to show that $b_1 \leqq x+b_2$, i.e., $a_1 \cdot x^k \leqq x+a_2(c_{12}+a_1 \cdot x^k)$. We will actually show $a_1 \cdot x^k \leqq x+a_2(c_{12}+a_1 \cdot x^{k-1})$, which is stronger by the above remarks. We argue by induction on k. It is clear when $k=1$. Now $a_1 \cdot x^k=a_1[(x+c_{23})(a_1+a_3)+(x^{k-1}+c_{13})(a_2+a_3)]$. By the same argument as given in the displayed calculations in the proof of Lemma 3.2, this is equal to $a_1(x+c_{23}+a_3(a_1 \cdot x^{k-1}+c_{13}))$. This equals $a_1(x+a_2(c_{23}+a_3(a_1 \cdot x^{k-1}+c_{13})))$, since $x \leqq a_1+a_2$. By (9) the latter equals $a_1(x+a_2(a_1 \cdot x^{k-1}+c_{12}))$. Thus $a_1 \cdot x^k= =a_1(x+a_2(a_1 \cdot x^{k-1}+c_{12})) \leqq x+a_2(c_{12}+a_1 \cdot x^{k-1})$, as desired. The proof that $x \in R_d$ is similar.

Theorem 3.7. *Let L be a finite dimensional modular lattice containing a spanning n-frame, $n \geqq 4$, and let p be a prime. Then L can be decomposed as $L \cong L_1 \times \times L_2$ in such a way that the characteristic of $R(L_1)$ is a power of p and p is invertible in $R(L_2)$.*

Proof. We view p as an element of $R(L)$. As in the last corollary there is a k such that $a_1+p^{2k}=a_1+p^k$ and $a_1 \cdot p^{2k}=a_1 \cdot p^k$. We again let $b=b(p^k)$ and $d=d(p^k)$. Let $L_1=b/0$ and $L_2=d/0$. By the last result p^k is zero in $R(b/0)=R(L_1)$ and is invertible in $R(d/0)=R(L_2)$. Hence the characteristic of $R(L_1)$ is p^s for some $s \leqq k$, and p is invertible in $R(L_2)$. Also b and d are complements. Since L is modular, this implies that $L_1 \times L_2 \subseteqq L$ (this is a "folklore" theorem of lattice theory, see [1] p. 73). In order to show that $L_1 \times L_2 = L$ we need to show that b and d are a *distributive pair*, i.e., for any $u \in L$, b, d, and u generate a distributive sublattice of L (see Theorem 5.2, p. 33 of [15] or 15.9 of [2]). Now we use the following easy result (see Lemma 5.1, p. 36 [15]): if both (b', d) and (b'', d) are distributive pairs and if $b' \cdot d = 0 = b'' \cdot d$, then $(b' + b'', d)$ is a distributive pair and $d(b' + b'') = 0$. Now if (b, d) is not a distributive pair then by repeatedly applying this result there are indices s and t such that (b_s, d_t) is not a distributive pair, i.e., there is a $u \in L$ such that the sublattice generated by b_s, d_t, and u, $\langle b_s, d_t, u\rangle$, is not distributive. Then $\langle b_s, d_t, u(b_s + d_t)\rangle$ is also nondistributive. Hence we may assume that $u \leqq b_s + d_t$. Thus the sublattice generated by u, b_s, and d_t will be a (nondistributive) homomorphic image of the following:



Note that

$$b_s(u+d_t)/b_s \cdot u \nearrow (u+b_s)(u+d_t)/u \searrow d_t(u+b_s)/d_t \cdot u.$$

Since L is finite dimensional we may assume that $b_s \cdot u \prec b_s(u+d_t)$. Let $e_s = u \cdot b_s$, $f_s = b_s(u+d_t)$, $g_t = u \cdot d_t$, and $h_t = d_t(u+b_s)$. We let e be the homogeneous element associated with e_s using Lemma 1.2. We define homogeneous elements f, g, and h in a similar way. Now since $f_i + e \succ e$, f is the join of the atoms above e. This implies that f/e is complemented, see 4.1 of [2]. A complemented modular lattice containing an n-frame, $n \geqq 4$, is isomorphic to the lattices of subspaces, $L(V)$, of an n-dimensional vector space, V, over a skew field F, see 13.4 and 13.5 of [2]. Since $e \leqq b$, and p^k is a stable element of $R(b/0)$, and p^k is zero in $R(b/0)$, the characteristic of F is p. By a similar argument h/g is isomorphic to the lattice of subspaces, $L(U)$, of a vector space, U, over a skew field K in which p^k, and hence p, is invertible.

Since $b \cdot d = 0$ we have that $f/e \nearrow f+g/e+g$ and $h/g \nearrow h+e/e+g$ and $(f+g)(h+e) = e+g$. Thus both $L(V)$ and $L(U)$ can be embedded into $f+h/g+e$. Moreover, since the atoms of $f+h/e+g$ join to $f+h$, $f+h/e+g$ is a complemented modular lattice of length $2n$. Now

$$f_s + e + g/e + g \nearrow f_s + u + e + g/u + e + g = h_t + u + e + g/u + e + g \searrow h_t + e + g/e + g.$$

Since the f_i+e are part of an n-frame, $f_i+e+g/e+g$ is projective to $f_j+e+g/e+g$ for any i and j. Similarly, $h_i+e+g/e+g$ and $h_j+e+g/e+g$ are projective. Hence $f+g$ is the join of pairwise perspective atoms in $f+h/e+g$. Consequently, $f+h/e+g$ is a simple, complemented modular lattices and thus isomorphic the lattice of subspaces of a vector space. But this vector space lattice contains subspace lattices of different characteristics, an impossibility. This contradiction proves the theorem.

To prove Theorem 1 is now easy. Let L be a finite dimensional modular lattice containing an n-frame, $n \geqq 4$. If every prime is invertible in $R(L)$, then $\mathbf{Q}$ is embedded in $R(L)$. If p is not invertible in $R(L)$, then $L \cong L_1 \times L_2$ with p invertible in $R(L_2)$ and the characteristic of $R(L_1)$ a power of p by the last theorem. Now we apply the same procedure to L_2. Since L is finite dimensional, this must stop after finitely many steps and we arrive at the conclusion of the theorem.

4. Herrmann's Theorem. In this section we use Theorem 1 to prove Herrmann's result. Let p be a prime and let $R = \check{\mathbf{Z}}_p$ be the ring of p-adic integers. Recall that the only nonzero ideals of R are $p^k R$, $k=0, 1, \ldots$. Thus the lattice of submodules of R as a left R-module, $L({}_R R)$, is a descending chain with 0, i.e., the dual of $\omega+1$. Hence $L_1 = L({}_R R^n)$ also has the ascending chain condition. If we let a_i be the submodule of R^n generated by $(0, \ldots, 1, \ldots, 0)$, 1 in the i^{th} place, c_{ij} the submodule generated by $(0, \ldots, 1, \cdots, -1, \ldots, 0)$, where the 1 and -1 are in the i^{th} and j^{th} position, then $\{a_i, c_{ij}\}$ is an n-frame in L_1. Now in a modular lattice the relation which identifies a and b if $a+b/a \cdot b$ is finite dimensional is a congruence which we denote here by Θ. Note that $\{a_i/\Theta,\ c_{ij}/\Theta\}$ is a spanning n-frame of L_1/Θ and that a_i/Θ covers 0 in L_1/Θ. As in the last section this implies that $L_1/\Theta \cong L({}_F F^n)$ for some skew field F. Since the operations of $R(L_1)$ are defined from the lattice operations, the homomorphism $L_1 \to L_1/\Theta$ induces a ring homomorphism of $R(L_1)$ into F. It is not hard to see that each member of the frame is the greatest member of its Θ-class. Consequently the only element of $R(L_1)$ which is Θ-equivalent to a_1 is a_1, i.e., the ring homomorphism $R \to F$ is a monomorphism. Hence R is a subring of F. Thus the field of fractions, $\hat{\mathbf{Q}}_p$ of $R = \hat{\mathbf{Z}}_p$, is a subfield of F. (Actually it is not hard to see that $F = \hat{\mathbf{Q}}_p$ and that the homomorphism of L_1 onto L_1/Θ is given by the tensor product: $U \mapsto U \otimes_R \hat{\mathbf{Q}}_p$. This follows from the flatness of $\hat{\mathbf{Q}}_p$ as $\hat{\mathbf{Z}}_p$-module, see 3.32 of [16].) In particular F is uncountable and has characteristic 0.

Since L_1 satisfies the ascending chain condition, each element x of L_1/Θ has a largest inverse image, denoted αx. Thus α is a meet preserving map from L_1/Θ into L_1 mapping the frame in L_1/Θ to the frame of L_1.

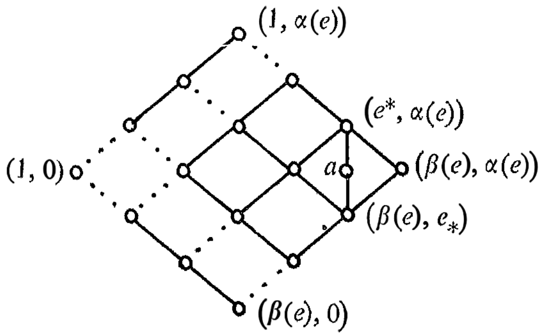
Now let S be the nonmodular lattice obtained from $L({}_{\mathbf{Q}}\mathbf{Q}^n)$ by adjoining an extra element e which is between 0 and 1 and a complement of all other elements.

Since $\mathbf{Q}$ is embedded in F, $L({}_{\mathbf{Q}}\mathbf{Q}^n)$ is a sublattice of $L({}_F F^n)$ in a natural way. (This map sends a subspace U to $U \otimes_{\mathbf{Q}} F$, which is just the F-subspace generated by U.) Extend this map to S by mapping e to a point which is on no rational hyperplane. For example, e can be sent to a one dimensional subspace spanned by a vector in F^n whose coordinates are linearly independent over $\mathbf{Q}$. Combining this map with α we obtain a meet embedding of S into L_1, which is canonical on the frames. We also use α to denote this map. Thus α maps e to a rank 1 free submodule, and hence $\alpha(e)/0$ is dually isomorphic to the ordinal $\omega+1$.

Let L_0 be the lattice which is dual to L_1, except we use the prime q in place of p. (L_0 may be taken to be the lattice of subgroups of the direct product of n copies of the Prüfer group $\mathbf{Z}_{q^\infty}$.) Then there is a join embedding β of S into L_0 such that $1/\beta(e)$ is isomorphic to $\omega+1$. Let

$$A_{pq} = \{(u, v) \in L_0 \times L_1 \colon \exists x \in S,\ \beta(x) \leqq u,\ v \leqq \alpha(x)\}.$$

It is easy to check that this is a sublattice which contains the spanning frame $\{(a_i, a_i), (c_{ij}, c_{ij})\}$. By the above remarks the interval $(1, \alpha(e))/(\beta(e), 0)$ is isomorphic to $(\omega+1)\times(\omega+1)^d$. We let $e^* \in L_0$ denote the upper cover of $\beta(e)$ and $e_* \in L_1$ the lower cover of $\alpha(e)$. We let L_{pq} be the lattice obtained from A_{pq} by adjoining a new element a so that $(e^*, \alpha(e))/(\beta(e), e_*)$ is isomorphic to M_3. Since $(\beta(e), \alpha(e))$ is both join and meet irreducible in A_{pq}, it is easy to see that L_{pq} is a modular lattice. The interval $(1, \alpha(e))/(\beta(e), 0)$ of L_{pq} is drawn below where the solid lines indicate coverings.



Now $R(L_1) \cong \hat{\mathbf{Z}}_p$ and simple linear algebraic calculations show that this isomorphism is given by $r \mapsto \{(-x, rx, 0, \ldots, 0) \colon x \in \hat{\mathbf{Z}}_p\}$. Below we identify r and this submodule. Again by linear algebraic calculations we have that $a_1 \cdot r = \{(y, 0, \ldots, 0) \colon ry = 0\}$. In our case $R(L_1) \cong \hat{\mathbf{Z}}_p$ is an integral domain. Hence we have $a_1 \cdot r = 0$ for each $r \in \hat{\mathbf{Z}}_p$ except $r = 0$. Similarly, we have that $(a_1 + p)a_2 = \{(0, px, 0, \ldots, 0) \colon x \in \hat{\mathbf{Z}}_p\}$. Since $p\hat{\mathbf{Z}}_p$ is the unique maximal ideal of $\hat{\mathbf{Z}}_p$, $(a_1+p)a_2$ is the unique lower cover of a_2 in L_1.

Recall that L_0 is isomorphic to the lattice of subgroups of the direct product of n copies of $\mathbf{Z}_{q^\infty}$. In general the ring associated with the direct product of n copies of a module is the endomorphism ring of the module. Thus in our case $R(L_0)\cong \cong \operatorname{End}(\mathbf{Z}_{q^\infty})\cong \hat{\mathbf{Z}}_q$. With the aid of these facts, it is not hard to verify that in L_0 $a_1+r=a_1+a_2$ for all $r\in R(L_0)$ except $r=0_R=a_1$, and that $a_1\cdot q$ is the unique atom below a_1. It follows that in L_{pq} $(a_1,a_1)\cdot(q,q)=(a_1\cdot q,0)\succ 0$, and $[(a_1,a_1)+(p,p)]\cdot(a_2,a_2)=(a_2,(a_1+p)a_2)\prec(a_2,a_2)$. Now in L_0 $a_1\cdot\beta(e)=0$ so that $a_1\cdot q+\beta(e)\succ\beta(e)$. Hence in L_{pq}

$$(a_1\cdot q,0)/(0,0)\nearrow(e^*,\alpha(e))/(\beta(e),\alpha(e)).$$

Similarly,

$$(\beta(e),\alpha(e))/(\beta(e),e_*)\nearrow(1,1)/(1,a_1+p+a_3+\ldots+a_n)\searrow(a_2,a_2)/(a_2,(a_2(a_1+p))).$$

Thus in L_{pq} $(a_2,a_2)/(a_2,a_2(a_1+p))$ and $(a_1\cdot q,0)/(0,0)$ are projective prime quotients.

We will show that L_{pq} is not in $\mathcal{M}_{\mathrm{fd}}$. Suppose that $L_{pq}\in\mathcal{M}_{\mathrm{fd}}$. Then L_{pq} is a homomorphic image of a lattice M which is residually finite dimensional. By Huhn's theorem, [12], M has a frame $\{a_i,c_{ij}\}$, which we may assume spans M, mapping onto the frame $\{(a_i,a_i),(c_{ij},c_{ij})\}$ in L_{pq}. By an easy application of Dedekind's transposition principle, we have, in M, that $a_1q/0$ and $a_2/a_2(a_1+p)$ have nontrivial subquotients which are projective. Thus there are elements $b_1,c_1,f_2,g_2\in M$ such that $0\leqq c_1<b_1\leqq a_1\cdot q$ and $a_2(a_1+p)\leqq g_2<f_2\leqq a_2$ and b_1/c_1 and f_2/g_2 are projective. Let $b,c,f,g\in M$ be the homogeneous elements associated with b_1,c_1,f_2,g_2, see Lemma 1.2. Since M is residually finite dimensional, there is a homomorphism $\psi\colon M\twoheadrightarrow K$ with K finite dimensional such that $\psi(b_1)\neq\psi(c_1)$. By Theorem 3.7 $K\cong K_1\times K_2$ where $R(K_1)$ has characteristic p^i for some i, and p is invertible in $R(K_2)$. Let $\pi_i\colon K\twoheadrightarrow K_i$, $i=1,2$, be the projection homomorphisms.

Since q is in the subring of $R(M)$ generated by 1, it is stable. Thus by Lemma 2.2, q in $R(b/0)$ is the element $q\cdot b$. But since $b_1\leqq a_1\cdot q\leqq q$, $q\cdot b=q(b_1+b_2)=b_1+q\cdot b_2= =b_1$. Thus $R(b/0)$, and hence $R(b/c)$, has characteristic q. Since $a_2(a_1+p)\leqq g_2$, we have, by joining a_1 to both sides, $p\leqq a_1+p\leqq(a_1+a_2)(a_1+p)\leqq a_1+g$. Hence $p+g\leqq a_1+g$, which implies that $p=0$ in $R(1/g)$. Thus $p=0$ in $R(f/g)$, again by Lemma 2.2.

It follows that in K_1, $R(\pi_1\psi(b)/\pi_1\psi(c))$ satisfies $q=0$. But $R(K_1)$ satisfies $p^i=0$ and thus $R(\pi_1\psi(b)/\pi_1\psi(c))$ also satisfies $p^i=0$ by Lemma 2.2. Since p and q are relatively prime, this ring must satisfy $1=0$, i.e., $\pi_1\psi(b)=\pi_1\psi(c)$, so that $(\psi(b),\psi(c))\in\ker\pi_1$. Hence $(\psi(b_1),\psi(c_1))\in\ker\pi_1$. Similarly $(\psi(f_2),\psi(g_2))\in\ker\pi_2$. But $\psi(b_1)/\psi(c_1)$ projects to $\psi(f_2)/\psi(g_2)$. Thus $(\psi(b_1),\psi(c_1))\in\ker\pi_1\cap\ker\pi_2=0$. Hence $\psi(b_1)=\psi(c_1)$, a contradiction. Hence $L_{pq}\notin\mathcal{M}_{\mathrm{fd}}$, as claimed.

Let p^+ be the first prime after p. The next step in the proof is to show that any nonprincipal ultraproduct of $\{L_{pp^+}\colon p \text{ a prime}\}$ lies in $\mathcal{M}_0$. This is a fairly standard argument and we shall only sketch it. HERRMANN's original proof [9] contains more details.

Let $L=(\Pi_p L_{pp^+})/\mathcal{U}$ be a nonprincipal ultraproduct of $\{L_{pp^+}\}$. The corresponding ultraproduct of rings $R=(\Pi_p \hat{\mathbf{Z}}_p)/\mathcal{U}$ has characteristic 0 and every prime is invertible, since these facts hold in $\hat{\mathbf{Z}}_p$ for almost all p. Hence $\mathbf{Q}$ is a subring of R. Now the ultraproduct $(\Pi_p L(\hat{\mathbf{Z}}_p^n))/\mathcal{U}$ is a lattice of submodules of a module over R. Since $\mathbf{Q}\subseteq R$, this may be viewed as a module over $\mathbf{Q}$. Hence this lattice can be embedded into the lattice of subspaces of a vector space over $\mathbf{Q}$. Let $A=(\Pi_p A_{pp^+})/\mathcal{U}$. Then A can be embedded into the direct product $L(V_0)\times L(V_1)$ of vector space lattices over $\mathbf{Q}$. Now A is just L with the element $(\Pi a)/\mathcal{U}$ removed. Also in L and in A we have $(\Pi(\beta(e), e_*))/\mathcal{U}\prec(\Pi(\beta(e), \alpha(e)))/\mathcal{U}\prec(\Pi(e^*, a(e)))/\mathcal{U}$. By changing V_0 and V_1 we may assume that $(e^*, \alpha(e))/(\beta(e), \alpha(e))$ and $(\beta(e), \alpha(e))/(\beta(e), e_*)$ have the same dimension. Now $L(V_0)\times L(V_1)$ is a sublattice of $L(V_0\times V_1)$ and L can be embedded into this lattice by choosing a to be a common complement of (e^*, e_*) and $(\beta(e), \alpha(e))$ in $(e^*, \alpha(e))/(\beta(e), e_*)$. Thus $L\in\mathcal{M}_0$.

Now the proof of Theorem 2 can easily be completed. If $\mathcal{K}$ is a finitely based variety with $\mathcal{M}_0\subseteq\mathcal{K}$ then the ultraproduct $(\Pi_p L_{pp^+})/\mathcal{U}$ lies in $\mathcal{K}$. Since $\mathcal{K}$ is finitely based there must be a prime p such that $L_{pp^+}\in\mathcal{K}$. Since $L_{pp^+}\notin\mathcal{M}_{\mathrm{fd}}$, $\mathcal{K}$ is not generated by its finite dimensional members.

For Corollary 3, the fact that $\mathcal{M}_{\mathrm{f}}\neq\mathcal{M}_{\mathrm{fd}}$ is a result of FREESE [4]. The rest of the corollary follows from the fact that $\mathcal{M}_0\subseteq\mathcal{M}_{\mathrm{f}}$, which is proved by HERRMANN and HUHN in [11].

5. Type I representations. It follows from the results of FREESE, HERRMANN and HUHN [7] that if $\mathcal{V}$ is a variety of algebras all of whose congruences are modular then L_{pq} is not in the variety generated by the congruence lattices of the algebras in $\mathcal{V}$. Indeed, in the last proof we showed that $(a_2, a_2)/(a_2, a_2(a_1+p))$ and $(a_1\cdot q, 0)/(0, 0)$ are projective prime quotients in L_{pq}. Let b be the homogeneous element of L_0 with $b_1=a_1\cdot q$ and let d be the homogeneous element of L_1 with $d_2=a_2(a_1+p)$, see the notation before Lemma 2.3. It follows from Lemma 2.3 that $R(b/0)$ has characteristic q in L_0 and $R(1/d)$ has characteristic p in L_1. Thus in L_{pq} the ring of the frame in $(b, 0)/(0, 0)$ has characteristic q and the ring of the frame $(1, 1)/(1, d)$ has characteristic p. The projectivity above shows that the quotient $(b_1, 0)/(0, 0)$ in the first frame is projective to $(1, d+a_1)/(1, d)$ in the second. Now Proposition 2 of [7] shows that this situation cannot occur in a modular congruence variety. Thus L_{pq} cannot be in any modular congruence variety.

In light of the above result it is of interest to decide if L_{pq} has a representation as a lattice of permuting equivalence relations (known as a *type I representation*). The following theorem of the author shows that it does have such a representation. The proof of this theorem will appear elsewhere.

Theorem 5.1. *Let L be a modular lattice containing an element a which is both join and meet irreducible. If the sublattice $L-\{a\}$ has a type I representation then L has such a representation.*

Summarizing these results we have:

Corollary 5.2. *The lattices L_{pq} all have a type I representation. If $p\neq q$ and $\mathcal{K}$ is a variety of algebras all of whose congruences are modular, then L_{pq} is not in the lattice variety generated by the congruences lattices of the members of $\mathcal{K}$.*

6. 3-frames. Throughout the previous sections we dealt with n-frames where n was at least 4. In this section we show that an analogue of Theorem 3.7 holds for $n=3$. The definition of the ring R determined by a frame with $n=3$ given in (6) makes perfect sence. Moreover addition and multiplication, as given in (7) and (8), are well-defined. However, it is not true that $(R, \oplus, \otimes)$ satisfies the ring axioms, as the lattices associated with non-Desarguesian projective planes show. In particular neither operation is necessarily associative. We will call a term in $\oplus$ and $\otimes$ and the constant 1 and no variables a *prime term* if its evaluation in $\mathbf{Z}$ is a prime. Thus $[(1\oplus 1)\otimes(1\oplus 1)]\oplus 1$ is a prime term. By a *prime* in R we will mean the evaluation of a prime term in R. By a *power* of $x\in R$ we mean the evaluation of some term in only $\otimes$ and the variable x.

Theorem 6.1. *Let L be a finite dimensional modular lattice containing a spanning 3-frame. Let $\langle R;, \oplus, \otimes\rangle$ be the algebraic structure defined by* (6), (7) *and* (8), *and let p be a prime in R. Then L can be decomposed as $L\cong L_1\times L_2$ in such a way that in $R(L_1)$ some power of p is zero and p is invertible in $R(L_2)$.*

Proof. The proof is essentially the same as Theorem 3.7. For the most part one simply notes that the proofs work for $n=3$. There are two places where some care is necessary. Define the symmetric power, $x^{[n]}$, of $x\in R$ by $x^{[1]}=x$, and $x^{[n+1]}=x^{[n]}\otimes x^{[n]}=(x^{[n]})^2$. Now before Corollary 3.6 we showed that $a_1\cdot x\leqq a_1(x\otimes y)$, for $x, y\in R$. A similar argument shows that $a_1+(x\otimes y)\leqq a_1+y$. From this we see that $a_1\cdot x\leqq a_1\cdot x^{[2]}\leqq a_1\cdot x^{[3]}\leqq\dots$ and $a_1+x\geqq a_1+x^{[2]}\geqq a_1+x^{[3]}\geqq\dots$. Hence, by the finite dimensionality of L, it follows that some symmetric power y of x satisfies $a_1\cdot y=a_1\cdot y^2$ and $a_1+y=a_1+y^2$.

The other place that requires care is the proof, in Theorem 3.7, that (b, d) is a distributive pair. This required a vector space argument. However, the proof showed that if (b, d) failed to be a distributive pair then L contained a simple complemented sublattice of dimension $2n=6$. Since $6>4$ the classic coordinatization theorem (see 13.4 and 13.5 of [2]) shows that this sublattice is isomorphic to the lattice of subspaces of a vector space. Moreover the proof of Theorem 3.7 shows that this sublattice will have two three-dimensional sublattices. The "rings" determined by the frames of these three dimensional intervals must be real rings because they lie inside a vector space lattice. In one of the rings, a power of p is zero and in the other, it is invertible. This is of course impossible in a vector space lattice.

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