

FINITELY PRESENTED LATTICES: CONTINUITY AND SEMIDISTRIBUTIVITY

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In [3] we investigated finitely presented lattices and the closely related subject of lattices generated by a finite partial lattice. We described a canonical form for the elements of such a lattice and used this to study the covering relation. We showed that there is an effective procedure for finding the covers of any element of a finitely presented lattice. We gave an example of a finitely presented lattice which has no cover at all.

In the present paper we use some of the results of [3] to prove two new theorems about finitely presented lattices. We show that such lattices are both upper and lower continuous, generalizing the corresponding result for free lattices which is due to Whitman [7]. We also characterize those finitely presented lattices which are semidistributive.[1]

An overview of [3]**.** A **partial lattice P** is a partially ordered set $\langle P, \leq\rangle$ together with two partial functions $\bigvee$ and $\bigwedge$ from the set of subsets of P into P such that if $p = \bigvee S$ then p is the least upper bound of S in $\langle P, \leq\rangle$, and the dual condition holds. $\mathbf{FL}(\mathbf{P})$ denotes the lattice freely generated by $\mathbf{P}$. More formally, $\mathbf{FL}(\mathbf{P})$ is defined by the property that it is generated by P, satisfies the order, join and meet relations of $\mathbf{P}$, and any map f from P to a lattice $\mathbf{L}$, which preserves the order of $\mathbf{P}$ and the joins and meets defined in $\mathbf{P}$, can be extended to a lattice homomorphism of $\mathbf{FL}(\mathbf{P})$ to $\mathbf{L}$.

There is a close connection between finitely presented lattices and lattices freely generated by finite partial lattices. This is explained in [3].

The canonical form of an element of a free lattice is the term of shortest length which represents it. Whitman [6] showed that this term is unique up to commutativity and associativity. We define the **canonical form** of an element of $\mathbf{FL}(\mathbf{P})$ to be the shortest *adequate* term representing it. A term t with variables from P is called **adequate** if it is an element of P or if $t = t_1 \vee \cdots \vee t_n$ is a formal join, each t_i is adequate, and if $p \leq t$ for $p \in P$ then $p \leq t_i$ for some i. If t is formally a meet the dual condition must hold. Adequate terms behave very nicely and the canonical form of an element of $\mathbf{FL}(\mathbf{P})$ shares the important properties of the canonical form of elements of free lattices.

If S and T are subsets of a lattice, we say that S **refines** T if for all $s \in S$ there is a $t \in T$ with $s \leq t$. We denote this by $S \ll T$.

This research was partially supported by NSF grant no. DMS-8521710

[1] The problem of characterizing those finitely presented lattices which are semidistributive was suggested to the author by Philip Dwinger.

A set $S \subseteq \mathrm{FL}(\mathbf{P})$ is said to form a **join representation** of $w \in \mathrm{FL}(\mathbf{P})$ if $w = \bigvee S$ and S is said to form a **join cover** of w if $w \leq \bigvee S$. A join representation (join cover) S is called **nonrefinable** if $w = \bigvee T$ and $T \ll S$ imply $S \subseteq T$. If S is any subset of a lattice, we let $S^\vee$ denote the closure of S under joins. The next theorem summarizes the results we will need from [3].

Theorem 1. *Each element $w \in \mathbf{FL}(\mathbf{P})$ has only finitely many nonrefinable join representations and every join representation of w can be refined to one of these. If w is in P then all of the elements of every nonrefinable join representation (and of every join cover) of w lie in P. If the canonical form of w is*

$$w = w_1 \vee \cdots \vee w_n \vee x_1 \vee \cdots \vee x_k$$

where $x_i \in P$ and $w_i \notin P$, then the nonrefinable join representations of w all have the form

$$w = w_1 \vee \cdots \vee w_n \vee y_1 \vee \cdots \vee y_r$$

where $y_j \in P$ for $j = 1, \ldots, r$ and $\{y_1, \ldots, y_r\} \ll \{x_1, \ldots, x_k\}$.

Semidistributivity. A lattice is **join semidistributive** if it satisfies the implication

$$(\mathrm{SD}_\vee) \qquad d = a \vee b = a \vee c \quad \text{implies} \quad d = a \vee (b \wedge c).$$

Meet semidistributivity is defined by the dual implication and denoted by $(\mathrm{SD}_\wedge)$. A lattice is said to be **semidistributive** if it satisfies both of these implications. The next lemma is due to Jónsson and Kiefer [5].

Lemma 2. *A lattice satisfies* $(\mathrm{SD}_\vee)$ *if and only if*

$$c = \bigvee a_i = \bigvee b_j \quad \textit{implies} \quad c = \bigvee (a_i \wedge b_j)$$

Notice that this lemma implies that any two join representations in a lattice satisfying $(\mathrm{SD}_\vee)$ have a common refinement.

Theorem 3. *The following are equivalent for* $\mathbf{FL}(\mathbf{P})$.

(1) $\mathbf{FL}(\mathbf{P})$ *is satisfies* $(\mathrm{SD}_\vee)$.
(2) *Every element of* $\mathrm{FL}(\mathbf{P})$ *has a unique nonrefinable join representation.*
(3) *Every element of $P^\vee$ has a unique nonrefinable join representation in* $\mathbf{FL}(\mathbf{P})$.
(4) *The lattice $P^\vee \cup \{0\}$ satisfies* $(\mathrm{SD}_\vee)$.

Proof. Using Theorem 1, Lemma 2 and the remark after it, we see that (1) implies (2). It is a standard result, which is easy to prove, that (2) implies (1). Obviously (2) implies (3).

To see that (3) implies (2), suppose (3) holds and let $v = \bigvee \{p \in P : p \leq w\}$. Suppose that

$$w = w_1 \vee \cdots \vee w_n \vee y_1 \vee \cdots \vee y_r \tag{1}$$

is a nonrefinable join representation of w. If $p \leq w$ with $p \in P$, then (1) represents a join cover of p. By Theorem 1 any such join cover can be refined to one in whose elements all lie in P. It follows that

$$v = \bigvee\{p \in P : p \leq w_1 \vee \cdots \vee w_n\} \vee y_1 \cdots \vee y_r. \tag{2}$$

Moreover, if one replaces any y_j in (2) with $\bigvee\{p \in P : p < y_j\}$ the resulting element is strictly below v; otherwise we would violate the nonrefinability of (1). Thus $\{y_1, \ldots, y_r\}$ is contained in a nonrefinable join representation of v and the remaining elements of such a representation all lie below $w_1 \vee \cdots \vee w_n$. From this fact and Theorem 1 and our assumption that v has a unique nonrefinable join representation, it follows that (1) is the unique nonrefinable join representation of w. Hence (3) implies (2).

Thus (1), (2), and (3) are equivalent. Now by §4 of [3] the map which sends $w \in \mathrm{FL}(\mathbf{P})$ to $\bigvee\{p \in P : p \leq w\}$ defines a homomorphism from $\mathbf{FL}(\mathbf{P})$ onto the lattice $\mathbf{P}^\vee \cup \{0\}$. Obviously each element of this lattice has a least preimage under this map (namely itself) and from this it is easy to establish that (1) implies (4). Finally it is not hard to show that (4) implies (3), completing the proof. □

Continuity. We call a lattice **lower continuous** if whenever $a = \bigwedge a_i$, where the a_i's form a chain, then

$$a \vee b = \bigwedge (a_i \vee b)$$

for all b in the lattice. (This differs slightly from the notation of [1] where it is assumed that the lattice is complete.) Upper continuity is defined by duality.

Theorem 4. *Finitely presented lattices are both upper and lower continuous.*

Proof. It suffices to prove this for $\mathbf{FL}(\mathbf{P})$ with P finite (since every finitely generated lattice is isomorphic to a lattice of this form). Suppose $a = \bigwedge_{i=0}^{\infty} a_i$, where the a_i's are decreasing, and that $a \vee b \neq \bigwedge(a_i \vee b)$. Clearly $a \vee b$ is a lower bound for the set $\{a_i \vee b\}$. Since it is not the greatest lower bound, there is another lower bound c with

$$c \nleq a \vee b. \tag{3}$$

Choose such a c of least rank (for the purposes of this paper we can let the rank of c be the length of the shortest term representing c). We show that there is no such c by induction on the rank of c. For $w \in \mathbf{FL}(\mathbf{P})$ let $P_w = \{p \in P : p \leq w\}$. Clearly $\bigcap P_{a_i} = P_a$. Since P is finite

$$P_{a_k} = \bigcap P_{a_i} = P_a. \tag{4}$$

for all sufficiently large k. By removing finitely many a_k's, we may assume (4) holds for all k.

Suppose $c \in P$. Now by Lemma 2 of [3], $c \leq a_i \vee b$ implies that $c \leq \bigvee(P_{a_i} \cup P_b)$, for all i. Hence, by (4)

$$c \leq \bigvee(P_{a_k} \cup P_b) = \bigvee(P_a \cup P_b) \leq a \vee b.$$

This is a contradiction.

Since the a_i's form a chain, $c \leq a_i$ can hold for only finitely many i's or else $c \leq a$, contradicting (3). By removing finitely many a_i's, we can assume that $c \nleq a_i$ for all i. If $c = c_1 \wedge c_2$ then by Dean's solution to the word problem (see [2] or [3]) we have for each i that one of the following must hold.

$$c \leq a_i, \quad c \leq b,$$
$$c \leq p \leq a_i \vee b \quad \text{for some } p \in P,$$
$$c_1 \leq a_i \vee b, \quad c_2 \leq a_i \vee b$$

We have already ruled out the first possibility. The second violates (3) and the argument above rules out the third. Thus either the fourth or the fifth case must hold. Since there are infinitely many a_i's, one of the c_j's, say c_1, must satisfy $c_1 \leq a_i \vee b$ for infinitely many i's. But since the a_i's form a descending chain, this implies $c_1 \leq a_i \vee b$ for all i. Since c_1 has lower rank than c, we must have $c_1 \leq a \vee b$. But then $c \leq c_1 \leq a \vee b$, a contradiction.

The case when c is a join is easy. □

It is an interesting historical fact that Whitman did not know if his theorem (that free lattices are continuous) had any content: it was conceivable that there were no proper meets of infinite descending chains in free lattices. R. A. Dean was the first to establish that there were such, see [4]. He showed that $x \vee (y \wedge z)$ had no cover in $\mathbf{FL}(x, y, z)$ and thus this element is the meet of a properly descending chain.

References

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