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# CONGRUENCE LATTICES OF ALGEBRAS OF FIXED SIMILARITY TYPE, I

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**We prove that if $V$ is any infinite-dimensional vector space over any uncountable field $F$, then the congruence lattice (=subspace lattice) of $V$ cannot be represented as a congruence lattice (of any algebra) without using at least $|F|$ operations. This refutes a long-standing conjecture—that one binary operation would always suffice.**

Our result implies that the natural representation of Sub $V$ as a congruence lattice is the best one, i.e., uses the minimum number of operations. This result is easy to obtain—see § 1.

In the remainder of the paper we find further necessary conditions for the representability of an algebraic lattice $L$ with $<\kappa$ operations. Necessary and sufficient conditions seem impossible at this stage of knowledge. Part II of this paper (by W. A. Lampe) gives some interesting sufficient conditions for representability of $L$ with one binary operation, for instance: the unit element of $L$ is compact.

The conjecture which we have refuted dates back at least to 1959 and the theorem of Grätzer and Schmidt ([6], [3], [14]) that every algebraic lattice can be represented as the congruence lattice Con $A$ of some unary algebra $A$; the hope was to use say a single binary operation $+$ to code the unary operations $f(x)$ as $x + a$ (with $a \in A$ depending on $f$). (See e.g., [9], [8, p. 209].)

By some results of § 3 and Part II, every algebraic lattice can be embedded as a principal ideal in an algebraic lattice which is representable with one binary operation, and also in one requiring $\kappa$ operations. This may partly explain why the conjecture resisted settlement for so long.

§ 1 contains the main result. §§ 2 and 3 contain refinements and variations on the ideas of § 1, and § 4 contains some open problems.

Some of these results were announced in [12], [13], and [18].

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**1. The main result.** All our results depend on the following lattice-theoretic property:

(∗) For every compact $\Psi$, there exist compact $\Theta, \Phi$ such that $\Theta \vee \Phi \geqq \Psi$ and $\Theta \wedge \Psi = \Phi \wedge \Psi = 0$.

It is easy to see that our subspace lattice obeys (∗) (see the proof of Theorem 1); in fact all examples in this article will obey (∗).

LEMMA 1. *If* Con $A$ *satisfies* (∗), *then* $A$ *satisfies*

$$\exists x(t(x, p) = t(x, q)) \longrightarrow \forall x(t(x, p) = t(x, q))$$

(*for* $x$ *denoting* $(x_1, \cdots, x_n)$, $p, q \in A^m$ *and* $t$ *any* $(n+m)$-*ary term, i.e., polynomial*).

REMARKS. (∗) is somewhat elusive—every algebraic lattice can be embedded as a principal ideal in an algebraic lattice obeying (∗), and also in one where (∗) fails. And so perhaps it is not surprising that Lemma 1 was overlooked for so long. It arose from a study of the lattices $L_{\lambda,\phi}$ of § 2.

*Proof.* We will assume that $t(a, p) = t(a, q)$ for some $a \in A^n$, and prove that $t(b, p) = t(b, q)$ for any $b \in A^n$. To see this, we will apply (∗) to the compact congruence $\Psi = \Theta(a, b) \vee \Theta(p, q)$ (i.e., the congruence generated by $\{(a_1, b_1), \cdots, (a_n, b_n), (p_1, q_1), \cdots, (p_m, q_m)\}$). Taking $\Phi$ and $\theta$ as supplied by (∗), we have

$$\Theta(a, b) \subseteqq \Psi \subseteqq \Theta \vee \Phi ,$$

and so

$$a = s_0 \Theta s_1 \Phi s_2 \Theta s_3 \Phi \cdots s_k = b .$$

Now since $s_0 \Theta s_1$, we have

$$t(s_1, p) \Theta t(s_0, p) = t(s_0, q) \Theta t(s_1, q) ,$$

and thus $t(s_1, p) \Theta t(s_1, q)$. Since $p \Psi q$, we also have $t(s_1, p) \Psi t(s_1, q)$. Finally, since $\Theta \cap \Psi = 0$, we have $t(s_1, p) = t(s_1, q)$. A similar argument (with $\Phi$ in place of $\Theta$) yields $t(s_2, p) = t(s_2, q)$. Proceeding by induction, we obtain $t(b, p) = t(s_k, p) = t(s_k, q) = t(b, q)$.

DEFINITION. $\# A$ denotes the number of nonnullary operations of $A$.

LEMMA 2. *If* Con $A$ *obeys* (∗), *then every block of every compact congruence of* $A$ *has power* $\leqq \# A + \aleph_0$.

*Proof.* Let $S$ be a block of a compact congruence $\Theta$; thus $\Theta = \Theta(a, b)$ for some $a, b \in A^n$. Fixing $c \in S$ and considering arbitrary $d \in S$, we have $(c, d) \in \Theta(a, b)$, and hence, for some terms $t_1, \cdots, t_n$ and $p_1, \cdots, p_n \in A^m$, we have

$$\begin{aligned} c &= t_1(a, b, p_1) \\ t_1(b, a, p_1) &= t_2(a, b, p_2) \\ &\vdots \\ t_{n-1}(b, a, p_{n-1}) &= t_n(a, b, p_n) = d \end{aligned}$$

(as follows from well known and easy descriptions of compact congruences—see e.g., [3, Theorem 10.4]).

To see our upper bound on $|S|$, it is clearly enough (by elementary cardinal arithmetic) to see that $d$ depends only on the sequence $(t_1, \cdots, t_n)$, and *not* on $(p_1, \cdots, p_n)$. To see this we suppose we also have

$$\begin{aligned} c &= t_1(a, b, p_1') \\ t_1(b, a, p_1') &= t_2(a, b, p_2') \\ &\vdots \\ t_{n-1}(b, a, p_{n-1}') &= t_n(a, b, p_n') = d' . \end{aligned}$$

The proof will be complete when we have shown $d = d'$. From $t_1(a, b, p_1) = c = t_1(a, b, p_1')$ we obtain

$$t_2(a, b, p_2) = t_1(b, a, p_1) = t_1(b, a, p_1') = t_2(a, b, p_2') ,$$

(using Lemma 1 for the middle equality). Thus we have obtained $t_2(a, b, p_2) = t_2(a, b, p_2')$; proceeding by induction yields $d = d'$.

The above lemma tells us that blocks cannot be too big. The next lemma says that sometimes there must be a large block.

LEMMA 3. *Suppose that* $\Psi \in \operatorname{Con} A$, $X \subseteq \operatorname{Con} A$ *is infinite, and for distinct* $\Theta, \Phi \in X$, $\Theta \wedge \Phi = 0$ *and* $\Theta \vee \Phi = \Psi$. *Then each nontrivial block of* $\Psi$ *has power* $\geq |X|$.

*Proof.* Let $\Pi$ be any partition of $X$ into two-element sets, and take $\langle c, e\rangle \in \Psi$ with $c \neq e$. For each $B \in \Pi$, $\vee B = \Psi$, and so we have

$$c\Theta_1 d\Theta_2 \cdots \Theta_k e \qquad (\text{each } \theta_i \in B) .$$

Of course we may assume that $d \neq c$. It will be enough to see that distinct $B$'s yield distinct $d$'s (for then the set of all $d$'s has power $\geq |\Pi| = |X|$, and certainly $c\Psi d$ holds for each $d$). But this is easy, since if we also had $c\Theta_1' d$ with $\Theta_1' \in B' \neq B$, then we would have $c = d$, since $\Theta_1 \cap \Theta_1' = 0$.

THEOREM 1. *If $V$ is an infinite-dimensional vector space over an uncountable field $F$, then* $\mathrm{Con}\, A \cong \mathrm{Con}\, V$ *implies that $A$ has at least $|F|$ operations.*

*Proof.* We will see that $\mathrm{Con}\, V$ (hence $\mathrm{Con}\, A$) satisfies the hypotheses of Lemma 3, with $\Psi$ actually compact. Fix one 2-dimensional subspace $W$ and $|F|$ one-dimensional subspaces $U_\alpha \subseteq W$ $(\alpha < |F|)$. With the usual identification of subspaces with congruences, we take $\Psi = W$ and $X = \{U_\alpha : \alpha < |F|\}$. Clearly $U_\alpha \vee U_\beta = W$ for $\alpha \neq \beta$, and so Lemma 3 yields a compact congruence with a block of power $|F|$.

To see that $\mathrm{Con}\, V$ $(= \mathrm{Sub}\, V)$ obeys $(*)$, let $U$ be any finite-dimensional subspace of $V$ with basis $\{v_1, \cdots, v_n\}$. Extend this to a linearly independent subset $\{v_1, \cdots, v_{2n}\}$ of $V$. Now take $U'$ to be spanned by $\{v_1 + v_{n+1}, \cdots, v_n + v_{2n}\}$ and $U''$ to be spanned by $\{v_{n+1}, \cdots, v_{2n}\}$. Then certainly $U \subseteq U' + U''$ and $U \cap U' = U \cap U'' = \{0\}$.

And so Lemma 2 tells us that $|F| \leqq \sharp A + \aleph_0 = \sharp A$.

2. Refinements. In this section we present further means of finding large congruence blocks and thus seeing (*via* Lemma 2) that certain algebraic lattices obeying $(*)$ require many operations in their congruence representations. Lemma 4 below may be proved in a manner similar to Lemma 3. Lemma 5 is far more general than Lemma 4, but also more complicated. Lemma 6 below produces large congruence blocks in an entirely different manner.

LEMMA 4. *Suppose that* $\Psi \in \mathrm{Con}\, A$, $X \subseteq \mathrm{Con}\, A$ *and a partition $\Pi$ of $X$ obey these conditions*:

(1) *for each* $B \in \Pi$, $\Psi \leqq \vee B$;

(2) *for distinct* $B, B' \in \Pi$, $\Theta \in B$ *and* $\Phi \in B'$, $\Theta \wedge \Phi = 0$.

*Then every nontrivial block of $\vee X$ has power* $\geqq |\Pi|$.

In what follows, $\kappa$ is an infinite cardinal. We say that a poset $P$ is *$\kappa$-directed* if and only if each $S \subseteq P$ with $|S| < \kappa$ has an upper bound in $P$.

THEOREM 2. *If* $\mathrm{Con}\, A$ *satisfies* $(*)$ *and the compact elements of* $\mathrm{Con}\, A$ *are $\kappa^+$-directed, then* $\kappa \leqq \sharp A + \aleph_0$.

*Proof.* By Lemma 2, it will be enough to produce a compact congruence with one block of power $\geqq \kappa$. For this, we use $(*)$ and $\kappa^+$-directedness. For $\alpha \leqq \kappa$ we recursively define compact congruences $\Psi_\alpha$, $\Theta_\alpha$, $\Phi_\alpha$ as follows:

$$\begin{aligned}&\Psi_\alpha \geqq \Psi_\beta, \Theta_\beta, \Phi_\beta \qquad (\alpha > \beta)\\ &\Theta_\alpha \vee \Phi_\alpha \geqq \Psi_\alpha\\ &\Theta_\alpha \wedge \Psi_\alpha = \Phi_\alpha \wedge \Psi_\alpha = 0\end{aligned}$$

(the first condition is possible by $\kappa^+$-directedness, the last two by (*)). It is clear that if we take $X = \{\theta_\alpha, \Phi_\alpha: \alpha < \kappa\}$ and $\Pi = \{\{\Theta_\alpha, \Phi_\alpha\}: \alpha < \kappa\}$, then the hypotheses of Lemma 4 are fulfilled, and moreover $\vee X \subseteqq \Psi_\kappa$. Thus $\Psi_\kappa$ is a compact congruence with a block of power $\geqq \kappa$.

To illustrate Theorem 2, we suppose $V$ is a vector space of dimension $\kappa^+$ over some field. We let $S$ be the lattice of all subspaces of $V$ of dimension $\leqq \kappa$, and let $L$ be the lattice of all ideals of $S$. Theorem 2 says that $\# A \geqq \kappa$ whenever $L \cong \operatorname{Con} A$.

LEMMA 5. *Suppose that* $\Psi$, $Z \in \operatorname{Con} A$ *and a family* $P$ *of finite subsets of* $\operatorname{Con} A$ *obey these conditions*:
(1) $|\cup P|$ *is regular and infinite*;
(2) $\Psi$ *is compact and* $\Psi \not\leqq Z$;
(3) $\forall B \in P$, $\Psi \leqq (\vee B) \vee Z$;
(4) *each* $B \in P$ *is minimal for condition* (3);
(5) $\forall T \subseteqq \cup P$, *if* $|T| = |\cup P|$, *then* $\wedge T \leqq Z$.
*Then* $\vee(\cup P) \vee Z$ *has a block of power* $\geqq |\cup P|$.

REMARKS. Of course the finiteness of the sets $B \in P$ follows from (2) and (4). This lemma is more versatile than Lemma 4 since, firstly, $P$ need not be a partition, secondly, the disjointness condition (5) has been weakened considerably, and thirdly, 0 has been replace by an arbitrary $Z \not\geqq \Psi$. Obviously the new assumption (1) makes no difference for our applications in conjunction with Lemma 2, because each singular cardinal is the supremum of all smaller regular cardinals.

*Proof.* By (2) we have $a, b \in A^n$ with

$$\Psi = \Theta(a, b) = \Theta(a_1, b_1) \vee \cdots \vee \Theta(a_n, b_n) .$$

Since each $(a_i, b_i) \in \Psi$, (3) says that for each $B \in P$

$$a_i = x_0 \Theta_1 x_1 \Theta_2 x_2 \cdots x_k = b_i$$

for some $x_1, x_2, \cdots \in A$ and some $\Theta_1, \Theta_2, \cdots \in B \cup \{Z\}$. For each $i$ we form $Q_i \subseteqq A$ as follows: take one such sequence for each $B \in P$, and then let $Q_i$ be the set of all $x_1, x_2, \cdots$ occurring in all these sequences. Obviously each $Q_i$ is contained in a single $\vee(\cup P) \vee Z$ block, and so it will be enough to see that some $|Q_i| \geqq |\cup P|$.

Clearly it is enough to show that $Q = Q_1 \cup \cdots \cup Q_n$ has power $\geqq |X|$, where $X = \cup P$. To see this we define

$$\tau: (Q \times Q - Z) \longrightarrow \text{power set of } X$$

*via*

$$(p, q) \longmapsto \{\Theta \in X: p\Theta q\} .$$

We will clearly be done if we can show that $|\operatorname{Range} \tau| \geqq |X|$. Condition (5) implies that each $|\tau(p, q)| < |X|$, and so by the regularity of $|X|$ it will be enough to see that $|\cup \operatorname{Range} \tau| = |X|$. We will in fact show that $\cup \operatorname{Range} \tau = X$.

For each $\Theta \in X = \cup P$, we have $B \in P$ and

$$a_i = x_0 \Theta_1 x_1 \Theta_2 x_2 \cdots x_k = b_i \qquad (1 \leqq i \leqq n)$$

(as above) with $\Theta \in B$ and each $\Theta_j \in B \cup \{Z\}$. Now among these relations must occur $x_j \Theta x_{j+1}$ with $(x_j, x_{j+1}) \notin Z$, for otherwise we could eliminate $\Theta$ in favor of $Z$, contradicting the minimality of $B$ in (4). But clearly $\Theta \in \tau(x_j, x_{j+1})$, and hence $X \subseteqq \operatorname{Range} \tau$.

LEMMA 6. *Suppose that $C$ is the set of compact elements of* $\operatorname{Con} A$, $X \subseteqq \operatorname{Con} A$ *and $X$ is infinite. If $\vee X = 1$ and $|X| < |C|$, then some congruence in $X$ has a block of power $\geqq \kappa$, for each regular $\kappa$ satisfying $|X| < \kappa \leqq |C|$. (In particular, for $\kappa = |X|^+$.)*

*Proof.* Fixing such a regular $\kappa$, we let $Y$ be the set of finite joins of congruences from $X$. Thus $Y$ is *directed* and $|Y| < \kappa$. Fixing any $a \in A$, we have, since $A$ is the unique block of 1,

$$A = [a]_1 = \bigcup_{\theta \in Y} [a]_\Theta .$$

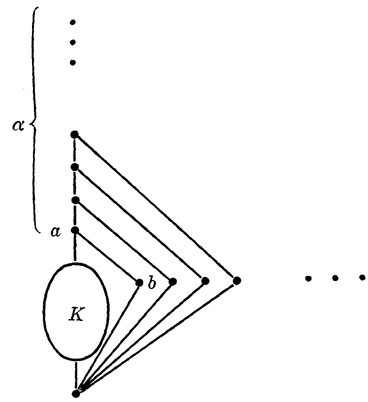
One easily checks that $|C| \leqq |A|$, and so the regularity of $\kappa$ yields $|[a]_\Theta| \geqq \kappa$ for some $\Theta \in Y$.

By definition of $Y$, we have $\Theta = \Theta_1 \vee \cdots \vee \Theta_n$ with each $\Theta_j \in X$. Let

$$Q_0 = \{a\} \quad \text{and} \quad Q_{i+1} = \{x \notin Q_i: (\exists y \in Q_i)(\exists j) x \Theta_j y\} .$$

Since $[a]_\Theta = \bigcup_{i \in \omega} Q_i$ and $\kappa$ is regular, there exists $i$ with $|Q_{i+1}| \geqq \kappa$; taking the least such $i$, we also have $|Q_i| < \kappa$. For $b \in Q_i$ and each $j$, let $R_{bj} = \{x \in Q_{i+1}: b\Theta_j x\}$. By the regularity of $\kappa$, we may choose $b$ and $j$ with $|R_{bj}| \geqq \kappa$, and clearly $|[b]_{\Theta_j}| \geqq \kappa$.

To illustrate how Lemmate 5 and 6 may be applied, we will present several algebraic lattices, all of the same general form, namely: $L_{\alpha,K}$ *is the ideal lattice of*




*for a limit ordinal* $\alpha$ *and a join-semilattice* $K$ *which we will specify further below.* Obviously the lattices $L_{\alpha,K}$ satisfy (*), and so we may apply Lemma 2 to them.

We first note that if $\alpha = \beta + \gamma$ with $\gamma < |\beta| + |K|$, then Lemmate 2 and 6 immediately yield $\sharp A \geqq |\beta| + |K|$ for Con $A \cong L_{\alpha,K}$. An example where neither Lemma 4 nor Lemma 5 is useful is obtained by taking $\alpha = \omega$ and $K = \omega_1$.

By way of historical comment, we should remark that Lampe's pioneering example [12] was precisely $L_{\alpha,K}$ with $\alpha = \omega_1 + \omega$, $K = \varnothing$. (He actually used Lemmate 2 and 4, which apply equally well to this example.)

The lattices $L_{\alpha,\varnothing}$ arose quite naturally during Lampe's attempts to represent each algebraic lattice as the congruence lattice of a groupoid. It is possible to view $L_{\omega,\varnothing}$ as a partial lattice in such a way that a lattice $L$ obeys (*) if and only if for every compact $a$ there is a zero-preserving embedding $\sigma$ of $L_{\omega,\varnothing}$ into $L$ with $a$ in the range of $\sigma$. Each of Lampe's attempts foundered on a lattice which had some $L_{\alpha,\varnothing}$ as a *partial* sublattice.

Next we give three examples using various features of Lemma 5, where neither Lemma 4 nor Lemma 6 can be applied. These examples have $\alpha = \omega_1$ and $|K| = \omega_1$. In each case we will see that $\sharp A \geqq \omega_1$ for $L_{\alpha,K} \cong$ Con $A$ (and of course $\sharp A = \omega_1$ is possible, by Grätzer-Schmidt).

For our first example we take $K = \omega_1^*$ ($\omega_1$ upside down), and we apply Lemma 5 with $Z = 0$, $\Psi = a$ and $P = \{\{b, \alpha\}: \alpha \in K\}$. In this example $P$ is not a partition, and only a weak disjointness assumption (5) holds.

For our next example we take $K = K_0 \cup \{p, c\}$, with $K_0 =$ the lattice of all finite or co-countable subsets of $\omega_1$, and $p$ and $c$ two new points. The natural order of $K_0$ is extended to $K$ by letting $x \leqq c$ for all $x \in K$, and letting $x \leqq p$ if and only if $x$ is a finite subset of $\omega_1$. Let $\Psi = c$, $P = \{\{p, x\}: x$ a coatom of $K_0\}$ and $Z =$ the join in

$L_{\omega_1,K}$ of all the atoms in $K_0$. Then one may easily check that $\Psi$, $P$, $Z$ satisfy the hypotheses of Lemma 5 (whenever $L_{\omega_1,K}$ is identified with Con $A$). We could not have $Z$ any smaller and still satisfy (5), for if $Y < Z$, then $Y \not\geqq$ some atom $e \in K_0$, but clearly $e = \wedge T$ for a suitable set $T$ of coatoms.

For our third example we let $X = \{x_\alpha: \alpha < \omega_1\}$ and let $FL(X)$ be the free lattice generated by $X$, and then we take $K$ to be the lattice of dual ideals of $FL(X)$, ordered by reverse set inclusion. Let $\Psi \in K$ be the dual ideal of $FL(X)$ generated by the set $\{x_0 \vee x_\alpha: 0 < \alpha < \omega_1\}$. We will think of $FL(X)$ as a sublattice of $K$ (i.e., $K$ is a special kind of completion of $FL(X)$). Straightforward calculation shows that the hypotheses of Lemma 5 hold for $\Psi$, $P = \{x_0, x_\alpha\}: 0 < \alpha < \omega_1\}$ and $Z =$ the ideal of $K$ generated by $\{\wedge T: T \subseteqq X, |T| = \omega_1\}$ (whenever $L_{\omega_1,K}$ is identified with Con $A$). Note that in this example $\Psi$ and $Z$ are not comparable.

Of course, further examples could be obtained by embedding the ideals of $K$ into other lattices obeying (*), e.g., a subspace lattice. Incidentally, we are unable to decide whether $L_{\omega_1,\varnothing}$ or $L_{\omega_1,\omega_1}$ can be represented with $\aleph_0$ operations.

**3. Further speculations.** We first remark that obviously (*) can be weakened without harm to

(**) For every compact $\Psi$ there exist compact $\Theta_1, \cdots, \Theta_n$ such that $\Theta_1 \vee \cdots \vee \Theta_n \geqq \Psi$ and $\Theta_1 \wedge \Psi = \cdots = \Theta_n \wedge \Psi = 0$.

Notice also that if some principal dual ideal $D$ of $L$ requires $\kappa$ operations for a representation, then the same is true of $L$, since Con $A \cong L$ implies Con $A/\Theta \cong D$ for some $\Theta$. (And so one easily sees that *every* algebraic lattice $L$ is embeddable in one which requires $\kappa$ operations.) Thus in searching for a characterization of "$L$ is representable with $\leqq\kappa$ operations" we need to examine all principal dual ideals of $L$. Define $C(L)$ to be the set of compact elements of $L$ and then define

$$\sigma(L) = \Sigma\{|\cup P|: (\exists \Psi, Z \in L) P, \Psi \text{ and } Z \text{ are as in Lemma 5, and there exists a compact element } \geqq \vee(\cup P) \vee Z\}.$$

$$\tau(L) = \begin{cases} |C(L)| & \text{if } \exists X \subseteqq C(L) \text{ with } |X| < |C(L)| \text{ and } \vee X = 1; \\ 0 & \text{otherwise}. \end{cases}$$

$$\rho(L) = \Sigma\{\sigma(D) + \tau(D): D \text{ a principal dual ideal of } L \text{ obeying } (**)\}.$$

Clearly all our previous results combine to show that $\# A + \aleph_0 \geqq \rho(\text{Con } A)$. This last inequality summarizes all our available information on this topic. Thus, for example, we see that $\rho(L) \leqq \aleph_0$ is a necessary condition for congruence representability with $\aleph_0$ operations; but we would hardly conjecture that it is a sufficient condition.

## 4. Problems.

(1) Is the inequality $\sharp A + \aleph_0 \geqq \rho(\mathrm{Con}\, A)$ sharp, in the sense that every algebraic lattice is isomorphic to some Con $A$ with $\sharp A + \aleph_0 = \rho(\mathrm{Con}\, A) + \aleph_0$? (See § 3 just above.)

(2) If no principal dual ideal of $L$ obeys (**) of § 3, is $L$ representable with $\leqq \aleph_0$ operations? (See § 3.)

(3) Is every distributive algebraic lattice $L$ representable with $\aleph_0$ operations? ... one operation? ... as Con (some lattice)? This last property holds for $L$ finite [1, p. 83] [4, p. 95]. (See also [2], [5], [15]-[17] for further information.) A *groupoid* representation exists if $L$ is the ideal lattice of a distributive lattice (see Part II of this paper).

Notice that if 1 is true, then so is 2, and if 2 is true, then so is the first part of 3. We would be surprised if 1 is true.

(4) If the set of compact elements of $L$ is countable, then is $L$ representable with $< \aleph_0$ operations?

(5) If below each compact element of $L$ there are $\leqq \aleph_0$ compact elements, then is $L$ representable with $\aleph_0$ operations? ... $< \aleph_0$ operations? ... one operation? (W. Hanf showed [7] (see also [8, p. 106]) that the corresponding assertions hold for representations as *subalgebra lattices*.)

(6) If $A$ has countable similarity type, must there exist $\boldsymbol{B}$ of finite similarity type with Con $\boldsymbol{B} \cong$ Con $A$?

(7) If $A$ has finite similarity type, must there exist a groupoid $\boldsymbol{B}$ with Con $\boldsymbol{B} \cong$ Con $A$? (Lampe conjectures yes.) Note that Hanf [7] proved the corresponding assertion for subalgebra lattices, and McKenzie has proved a slightly weaker assertion for Con: there exists $\boldsymbol{B}$ of type (2,1) with Con $\boldsymbol{B} \cong$ Con $A$ (see Lampe's survey paper in the Proceedings of the 1977 Esztergom Colloquium).

(8) Characterize the class of lattices isomorphic to congruence lattices of groupoids. ... of algebras with $< \kappa$ operations.

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