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IDEAL LATTICES OF LATTICES

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This paper shows that any compactly generated lattice is a subdirect product of subdirectly irreducible lattices which are complete and upper continuous. An example of a compactly generated lattice which cannot be subdirectly decomposed into subdirectly irreducible compactly generated lattices is given. In the case of an ideal lattice of a lattice L, the decomposition into subdirectly irreducible complete lattices is tied, via a special completion process, to the finitely subdirectly irreducible homomorphic of images L. It is also shown that any finite lattice satisfying the Whitman condition is a retract of the ideal lattice of the dual ideal lattice of a free lattice.

Let L be a lattice and $\mathscr{I}(L)$ the lattice of all ideals of L, and $\mathscr{F}(L)$ the lattice of all dual ideals of L ordered by reverse set inclusion. This paper is concerned with the subdirect decompositions of $\mathscr{I}(L)$ and how they relate to L. First it is shown that $\mathscr{I}(L)$, and indeed any compactly generated lattice, is a subdirect product of subdirectly irreducible lattices which are complete and upper continuous. A special class $\mathscr{K}$ of lattices is defined which contains the class of all subdirectly irreducible lattices and is contained in the class of all finitely subdirectly irreducible lattices (a lattice is finitely subdirectly irreducible if it has no nontrivial subdirect decomposition into finitely many lattices). Each lattice L of $\mathscr{K}$ has a *special completion* which is subdirectly irreducible, upper continuous and satisfies all the identities of L. The main result of the paper is that $\mathscr{I}(L)$ is a subdirect product of the special completions of those members of $\mathscr{K}$ which are homomorphic images of L.

The paper also shows that any lattice satisfying both chain conditions and the Whitman condition free of the generators is a retract of the lattice of all ideals of the lattice of all dual ideals of a free lattice. This is used to show that if L is a modular lattice then every finite dimensional homomorphic image of L must be used in every subdirect decomposition of $\mathscr{I}(\mathscr{F}(L))$ into subdirectly irreducibles. Finally, an application to the notion of transferability of lattices is given.

The terminology of the paper is taken from [2]. In particular, a lattice is weakly atomic if every quotient sublattice contains a prime quotient. If $x \succ y$ (x covers y) in a lattice, then we let $\psi(x, y)$ denote the unique maximal congruence not containing (x, y).

2. Completeness. The lattice $\mathscr{I}(L)$ is always compactly generated and every compactly generated lattice is weakly atomic. If $a \succ b$ in a lattice K then there is a unique maximal congruence $\psi(a, b)$ on K such that $(a, b) \notin \psi(a, b)$ and $\psi(a, b)$ is completely meet irreducible. It follows that if K is weakly atomic then it is a subdirect product of the subdirectly irreducible lattices $K/\psi(a, b)$, $a \succ b$ in K.

In general, homomorphic images of complete lattices need not be complete even if the lattice is compactly generated. However, we have the following.

THEOREM 2.1. *If K is a compactly generated lattice and $a \succ b$ in K, then $K/\psi(a, b)$ is a complete and upper continuous lattice. Moreover, the natural homomorphism from K to $K/\psi(a, b)$ preserves arbitrary supremums.*

Proof. We first prove the following lemma. Let K, a, b be as in the theorem and suppose a/b is weakly projective into x/y in K. *Then there exists a compact element c such that $c \vee y \leqq x$ and a/b is weakly projective into $c \vee y/y$.* Since a/b is weakly projective into x/y there exist quotients $x_0/y_0 = x/y, x_1/y_1, \cdots, x_n/y_n = a/b$ such that alternately either $x_i \vee y_{i+1} = x_{i+1}$ and $y_i \leqq y_{i+1}$ or $y_i \wedge x_{i+1} = y_{i+1}$ and $x_i \geqq x_{i+1}$.

Suppose $y_{n-1} \wedge a = b$ and $a \leqq x_{n-1}$. Choose c a compact element below a but not below b. Then $c \vee y_{n-1} \geqq c \vee b = a$ and thus a/b is weakly projective into $y_{n-1} \vee c/y_{n-1}$. Let $x_{n-2} = \bigvee S$ where S is the set of compact elements below x_{n-2}. Then $c \leqq y_{n-1} \vee x_{n-2} = \bigvee_{s \in S} s \vee y_{n-1}$. Since c is compact there exists an $s \in S$ such that $c \leqq s \vee y_{n-1}$. Now it is easy to verify that a/b is weakly projective into $s \vee y_{n-2}/y_{n-2}$ and it is also weakly projective into $s \vee y_{n-3}/y_{n-3}$. Continuing in this way we get the desired conclusion. The case where $x_{n-1} \vee b = a$ and $b \geqq y_{n-1}$ can be handled similarly.

Let f denote the natural homomorphism from K to $K/\psi(a, b)$, where a covers b. We shall show that f preserves arbitrary supremums. The rest of the theorem follows immediately from this. Let Y be an arbitrary subset of $K/\psi(a, b)$. Since f is onto, there is a subset X of K such that $Y = \{f(x): x \in X\}$. Let $x_0 = \bigvee X$. Since f preserves order, $f(x_0) \geqq f(x)$ for all $x \in X$. If $f(x_0)$ is not the least upper bound for Y then we can find $z \in K$ such that $z < x_0$, $f(z) < f(x_0)$ and $f(z) \geqq f(x)$ for all $x \in X$. Now it is easy to see that a quotient x/y is collapsed by $\psi(a, b)$ if and only if a/b is not weakly projective into x/y. Since $f(z) < f(x_0)$, a/b is weakly projective into x_0/z. By the lemma, we can find a compact element c such that $x_0 \geqq c \vee z$ and a/b is weakly projective into $c \vee z/z$. Since $\bigvee X = x_0 \geqq c$, there exists $x_1, \cdots, x_n \in X$ such that $x_1 \vee \cdots \vee x_n \geqq c$. Since $f(z) \geqq$

$f(x_i)$, $i = 1, \cdots, n$, $f(z) = f(z \vee x_1 \vee \cdots \vee x_n)$. Hence a/b is not weakly projective into $z \vee x_1 \vee \cdots \vee x_n/z$. Since $c \vee z/z \subseteqq z \vee x_1 \vee \cdots \vee x_n/z$, this is a contradiction. The proof is complete.

It is interesting to note that the lattice $K/\psi(a, b)$ in the theorem need not be compactly generated. To see this let S be a countable set and let A_i be a group isomorphic to the group Z_{2^∞} for each $i \in S$. Let G be the direct product of the A_i's. For each subset T of S we let G_T be the subgroup of G consisting of all elements of G whose ith-projection is 0 for all i in T. Let L denote the collection of subgroups H of G such that either H contains G_T for some finite subset T of S, or H is the zero subgroup. Then L is a complete, upper continuous, modular, subdirectly irreducible lattice which is not compactly generated. Let $K = \mathscr{I}(L)$ and let a and b be principal ideals corresponding to a prime quotient in L. Then it follows from Lemma 5.1 below that $K/\psi(a, b) \cong L$. Moreover, Lemma 6.2 below shows that L must be used in every subdirect decomposition of $\mathscr{I}(L)$ into subdirectly or even finitely subdirectly irreducible lattices.

3. Projectivities in $\mathscr{I}(L)$.

LEMMA 3.1. *Suppose that $(a)/J$ is weakly projective into $(c)/J_0$ in $\mathscr{I}(L)$ and let $d \in J_0$. Then there is a $b \in J$ such that a/b is weakly projective into c/d in L.*

Proof. Since $(a)/J$ is weakly projective into $(c)/J_0$ there exist quotients $I_0/J_0 = (c)/J_0, I_1/J_1, \cdots, I_n/J_n = (a)/J$ such that alternately either $I_i \vee J_{i+1} = I_{i+1}$ and $J_i \leqq J_{i+1}$ or $J_i \wedge I_{i+1} = J_{i+1}$ and $I_i \geqq I_{i+1}$. Let us assume that $J_{n-1} \wedge (a) = J$ and $(a) \leqq I_{n-1}$ and $I_{n-2} \vee J_{n-1} = I_{n-1}$ and $J_{n-2} \leqq J_{n-1}$. Since $(a) \leqq I_{n-1} = J_{n-1} \vee I_{n-2} = J_{n-1} \vee \bigvee_{u \in I_{n-2}} (u)$, there is $a_{n-2} \in I_{n-2}$ such that $(a_{n-2}) \vee J_{n-1} \geqq (a)$. It follows from induction now that there is a b_{n-2} in $(a_{n-2}) \wedge J_{n-2}$ such that a_{n-2}/b_{n-2} is weakly projective into c/d in L.

Since $(a) \leqq (a_{n-2}) \vee J_{n-1} = \bigvee_{J_{n-1}} (a_{n-2} \vee u)$, there is an element b_{n-1} of J_{n-1} such that $a_{n-2} \vee b_{n-1} \geqq a$. Clearly b_{n-1} may be chosen so that $b_{n-1} \geqq b_{n-2}$. Let $a_{n-1} = a_{n-2} \vee b_{n-1}$ and $b = a \wedge b_{n-1}$. Then it is clear that a/b is weakly projective into a_{n-2}/b_{n-2} in L which is weakly projective into c/d in L. The case when $(a) = I_{n-1} \vee J$ and $J \geqq J_{n-1}$ is treated similarly.

The following corollary shows that any congruence of L can be extended to a congruence of $\mathscr{I}(L)$ in such a way that the restriction to L of the extended congruence is the original one.

COROLLARY 3.2. *Suppose $(a)/(b)$ is weakly projective into $(c)/(d)$ in $\mathscr{I}(L)$. Then a/b is weakly projective into c/d in L.*

If L is a modular lattice then the analogous results may be proved with weakly projective replaced by projective.

4. **Subdirect decompositions of $\mathscr{I}(L)$.** Let $\mathscr{K}$ be the class of all lattices K such that for some $a \in K$ and $J \in \mathscr{I}(K)$, $(a) \succ J$ and any nontrivial congruence of K collapses a and b for some $b \in J$. Notice that $\mathscr{K}$ lies properly between the class of subdirectly irreducible lattices and the class of finitely subdirectly irreducible lattices. If K is in $\mathscr{K}$ and a and J are as above then we define $K' = \mathscr{I}(K)/\psi((a), J)$ to be the *special completion* of K with respect to a and J. Let $\mathscr{K}'$ be the subclass of $\mathscr{K}$ which consists of those members of $\mathscr{K}$ which are complete, upper continuous, subdirectly irreducible with a prime quotient which is always collapsed. The next theorem is an easy consequence of Theorem 2.1, Lemma 3.1, and Lemma 5.1 below.

THEOREM 4.1. *Let $K \in \mathscr{K}$ then $K' \in \mathscr{K}'$ and K is a sublattice of K' and K' is in the variety (equational class) generated by K. Moreover, if $K \in \mathscr{K}'$ then $K' = K$. In particular, $K'' = K'$ and if K satisfies the ascending chain condition then $K' = K$.*

LEMMA 4.2. *Let L be a lattice, $a \in L$ and $J \in \mathscr{I}(L)$ such that $(a) \succ J$. Let K be the image of L in $\mathscr{I}(L)/\psi((a), J)$. Then $K \in \mathscr{K}$ and $\mathscr{I}(L)/\psi((a), J) \cong K'$.*

Proof. By Lemma 3.1 $K \in \mathscr{K}$. Let f be the natural map from $\mathscr{I}(L)$ onto $\mathscr{I}(K)$, and let φ be the kernel of f. Since $f(a) \neq f(b)$ for all b in J, $((a), J) \notin \varphi$. Consequently $\varphi \leqq \psi((a), J)$ and thus $\mathscr{I}(L)/\psi((a), J)$ is a homomorphic image of $\mathscr{I}(K)$. The kernel of this map is clearly $\psi((f(a)), f(J))$. Hence $\mathscr{I}(L)/\psi$ $(a), J) \cong \mathscr{I}(K)/\psi(f(a), f(J)) = K'$.

It is not hard to show that if K is a homomorphic image of L lying in $\mathscr{K}$, then the special completion of K is a subdirectly irreducible homomorphic image of $\mathscr{I}(L)$. These homomorphic images are sufficient to form a subdirect product. Indeed, since $\mathscr{I}(L)$ is weakly atomic the set of congruences on $\mathscr{I}(L)$ of the form $\psi(I', J')$ where $I' \succ J'$ is adequate to form a subdirect representation of $\mathscr{I}(L)$ into subdirectly irreducibles. If we choose $c \in I'$, $c \notin J'$ and using Zorn's lemma, choose $J \in \mathscr{I}(L)$ such that $(c) \wedge J' \subseteqq J \prec (c)$, then $\psi((c), J) \leqq \psi(I', J')$. Consequently, congruences of the form $\psi((c), J)$ where $(c) \succ J$ are adequate to form a subdirect representation of $\mathscr{I}(L)$. Combining the previous theorems we have the following description of the subdirect decomposition of $\mathscr{I}(L)$ in terms of L.

THEOREM 4.3. *Let L be a lattice. Then $\mathscr{I}(L)$ is a subdirect* *product of the special completions of the homomorphic images of L which lie in $\mathscr{K}$.*

5. Retracts of ideal lattices of dual ideal lattices of free lattices. We let (W) denote the Whitman condition which is free of the generators; that is

$$\text{(W)} \qquad a \wedge b \leqq c \vee d \quad \text{implies} \quad a \leqq c \vee d \quad \text{or} \quad b \leqq c \vee d \quad \text{or} \quad a \wedge b \leqq c \quad \text{or} \quad a \wedge b \leqq d \, .$$

In this section we show that any lattice which satisfies both chain conditions and (W) is a retract of the lattice of ideals of the lattice of dual ideals of a free lattice. Some of the ideas of the proof are borrowed from R. McKenzie and A. Kostinsky [6]. The principal ideal generated by x is denoted (x) and the principal dual ideal $[x]$. Dual ideals are ordered by reverse set inclusion.

LEMMA 5.1. *Let f be a homomorphism of L into K, where K is a complete and upper continuous lattice. Then f can be extended to a homomorphism, f', mapping $\mathscr{I}(L)$ into K by $f'(I) = \bigvee_{u \in I} f(u)$. Furthermore, f' preserves arbitrary supremums in $\mathscr{I}(L)$.*

Proof. It is easy to see that f' preserves arbitrary joins. To see that f preserves finite meets, let $I_1, I_2 \in \mathscr{I}(L)$. Then since $\{f(u) \mid u \in I_i\}$ is a directed set, $i = 1, 2$, we have by upper continuity,

$$\begin{aligned} f'(I_1) \wedge f'(I_2) &= \Big(\bigvee_{u_1 \in I_1} f(u_1)\Big) \wedge \Big(\bigvee_{u_2 \in I_2} f(u_2)\Big) = \bigvee_{u_1 \in I_1} \bigvee_{u_2 \in I_2} f(u_1) \wedge f(u_2) \\ &= \bigvee_{u_1 \in I_1} \bigvee_{u_2 \in I_2} f(u_1 \wedge u_2) = \bigvee_{u \in I_1 \wedge I_2} f(u) = f'(I_1 \wedge I_2) \, . \end{aligned}$$

THEOREM 5.2. *Let K be a lattice satisfying both chain conditions and* (W). *Then K is a retract of $\mathscr{I}\mathscr{F}(FL(X))$, where $|X| = |K|$.*

Proof. Let $L = FL(X)$ and let f be a homomorphism from L onto K. By the dual of Lemma 5.1 f can be extended to a homomorphism of $\mathscr{F}(L)$ onto K, which in turn can be extended to a homomorphism of $\mathscr{I}\mathscr{F}(L)$ onto K. We let f denote both of these extensions. Let β be the one-to-one map of K into $\mathscr{I}\mathscr{F}(L)$ which maps $b \in K$ to the principal ideal generated by the dual ideal $\{u \in L: f(u) \geqq b\}$. Let α mapping K one-to-one into $\mathscr{I}\mathscr{F}(L)$ be given by $\alpha(b) = \{D \in \mathscr{F}(L): f(D) \leqq b\}$. Clearly $\alpha(b)$ is the greatest element in $\mathscr{I}\mathscr{F}(L)$ whose image under f is b, for all b in K. We claim that $\beta(b)$ is the least element of $\mathscr{I}\mathscr{F}(L)$ mapping to b under f. Indeed, suppose $f(I) = b$ for some $I \in \mathscr{I}\mathscr{F}(L)$. Then by Lemma 5.1 $b = f(I) = \bigvee_{D \in I} f(D)$. The ascending chain condition in K yields that $b = f(D)$

for some $D \in I$. Now $b = f(D) = \bigwedge_{u \in D} f(u)$. Hence $f(u) \geqq b$ for all $u \in D$. Thus $D \subseteqq \{u \in L: f(u) \geqq b\}$. Consequently $D \geqq \{u \in L: f(u) \geqq b\}$ in $\mathscr{F}(L)$ and thus $(D) \geqq \beta(b)$. Since $I \geqq (D)$ we have $I \geqq \beta(b)$.

The map α preserves meets in K. For joins we have the following formula.

$$(*) \qquad \alpha(b_0 \vee b_1) = \alpha(b_1) \vee \alpha(b_1) \vee \bigvee \{([x]): x \in X \text{ and } f(x) \leqq b_0 \vee b_1\} .$$

Let I_0 be the right-hand side of the above equation. It is clear that $f(I_0) = b_0 \vee b_1$, so that $I_0 \leqq \alpha(b_0 \vee b_1)$. Suppose that for some I, $f(I) \leqq b_0 \vee b_1$. If $D \in I$ then $f(D) \leqq b_0 \vee b_1$. Suppose that D is principal; $D = [u]$. We shall show that $D \in I_0$ by induction on the length of the word u. Let $T = \{u \in L: f(u) \leqq b_0 \vee b_1$ implies $[u] \in I_0\}$. Clearly $X \subseteqq T$ and T is closed under joins. Let $u = u_0 \wedge u_1$ with $u_0, u_1 \in T$ and $b_0 \vee b_1 \geqq f(u) = f(u_0) \wedge f(u_1)$. It follows from (W) that either $f(u_0)$ or $f(u_1) \leqq b_0 \vee b_1$ or $f(u_0) \wedge f(u_1) \leqq b_0$ or b_1. If $f(u_0) \leqq b_0 \vee b_1$ then $[u_0] \in I_0$ and thus $[u] = [u_0 \wedge u_1] \in I_0$. If $f(u_0) \wedge f(u_1) \leqq b_0$ then $([u_0 \wedge u_1]) \leqq \alpha(b_0) \leqq I_0$. Now let D be any filter in I. Then $\bigwedge_{u \in D} f(u) = f(D) \leqq b_0 \vee b_1$. Since K satisfies the descending chain condition and $\{f(u): u \in D\}$ is closed under finite meets, there exists $u_0 \in D$ such that $f(u_0) = f(D)$. Hence $f(u_0) \leqq b_0 \vee b_1$ and thus $[u_0] \in I_0$. Now since $D \leqq [u_0]$, $D \in I_0$, proving (*).

Momentarily considering β to be a map from K to $\mathscr{F}(L)$, we let h be the unique homomorphism of $L = FL(X)$ into $\mathscr{F}(L)$ extending the map $h(x) = \beta(f(x))$, $x \in X$. By the dual of Lemma 5.1 h can be extended to a homomorphism of $\mathscr{F}(L)$ into $\mathscr{F}(L)$. Since $\mathscr{F}(L)$ is naturally embedded in $\mathscr{I}\mathscr{F}(L)$, this extension may be thought as a homomorphism of $\mathscr{F}(L)$ into $\mathscr{I}\mathscr{F}(L)$. By Lemma 5.1 again, this homomorphism can be extended to a homomorphism from $\mathscr{I}\mathscr{F}(L)$ into $\mathscr{I}\mathscr{F}(L)$. We use h to denote this homomorphism. We claim that $\beta(f(I)) \leqq h(I) \leqq \alpha(f(I))$, $I \in \mathscr{I}\mathscr{F}(L)$, and consequently $f(h(I)) = f(I)$. To see this first suppose I is doubly principal, $I = ([u])$. We induct on the length of u. The inequalities hold if $u \in X$. Suppose $\beta(f(u_i)) \leqq h(u_i) \leqq \alpha(f(u_i))$, $i = 0, 1$. Then, since β preserves joins, $\beta(f(u_0 \vee u_1)) = \beta(f(u_0)) \vee \beta(f(u_1)) \leqq h(u_0) \vee h(u_1) = h(u_0 \vee u_1) \leqq \alpha(f(u_0)) \vee \alpha(f(u_1)) \leqq \alpha(f(u_0 \vee u_1))$. The calculations for $u_0 \wedge u_1$ is similar. Now suppose $I = (D)$. Then $\beta(f(u)) \leqq h(u) \leqq \alpha(f(u))$ for all $u \in D$. Hence $\bigwedge_{u \in D} \beta(f(u)) \leqq \bigwedge_{u \in D} h(u) \leqq \bigwedge_{u \in D} \alpha(f(u))$. Since K satisfies the descending chain condition, there exists $u_0 \in D$ with $f(D) = f(u_0)$. Hence $f(u) \geqq f(u_0)$ for all $u \in D$. Therefore $\beta(f(u)) \geqq \beta(f(u_0))$, for all $u \in D$. Thus $\bigwedge_D \beta(f(u)) = \beta(f(u_0)) = \beta(f(D))$. Similarly $\bigwedge_D \alpha(f(u)) = \alpha(f(D))$. Hence $\beta(f(D)) \leqq h(D) \leqq \alpha(f(D))$. Now let I be arbitrary. Then $\bigvee_I \beta(f(D)) \leqq \bigvee_I h(D) \leqq \bigvee_I \alpha(f(D))$ and as above this gives $\beta(f(I)) \leqq h(I) \leqq \alpha(f(I))$, as desired.

Now let $g: K \to \mathscr{I}\mathscr{F}(L)$ be defined by $g = h\alpha$. We wish to show that g is a homomorphism. Since both α and h preserve meets, g does also. To show that joins are preserved it suffices to show that $h(\alpha(b_0) \vee \alpha(b_1)) \geqq h(\alpha(b_0 \vee b_1))$. Now, since h preserves arbitrary joins in $\mathscr{I}\mathscr{F}(L)$, $h(\alpha(b_0 \vee b_1)) = h(\alpha(b_0)) \vee h(\alpha(b_1)) \vee \{h(([x])): x \in X$ and $f(x) \leqq b_0 \vee b_1\}$. Therefore, it suffices to show that if $x \in X$ and $f(x) \leqq b_0 \vee b_1$, then $h(\alpha(b_0) \vee \alpha(b_1)) \geqq h(x)$. For such an x, $f(\alpha(b_0) \vee \alpha(b_1)) = f(\alpha(b_0)) \vee f(\alpha(b_1)) = b_0 \vee b_1 \geqq f(x)$. Hence, $\alpha(b_0) \vee \alpha(b_1) \geqq \beta(f(x)) = h(x)$. Thus $h(\alpha(b_0) \vee \alpha(b_1)) \geqq h(h(x)) \geqq \beta(f(h(x))) = \beta(f(x)) = h(x)$, and g is a homomorphism.

It is clear that $f(g(b)) = b$ for all $b \in K$. The proof is complete.

COROLLARY 5.3. *The following are equivalent for a lattice L satisfying both chain conditions*:

(i) *L satisfies* (W).

(ii) *L is a retract of $\mathscr{I}\mathscr{F}(FL(X))$, $|X| = |L|$.*

(iii) *L is a sublattice of $\mathscr{I}\mathscr{F}(FL(X))$, $|X| = |L|$.*

The implication (iii) implies (i) follows from the fact that $\mathscr{I}\mathscr{F}(FL(X)$ satisfies (W) [1]. The rest follows from Theorem 5.2.

6. Applications. The techniques used to prove Theorem 4.3 can be used to prove an analogous result for $\mathscr{I}\mathscr{F}(L)$. Again there is a natural class $\mathscr{H}$ of finitely subdirectly irreducible lattices such that $\mathscr{H}$ includes all subdirectly irreducible lattices and a natural completion of these lattices, satisfying the same properties as the special completion of the members of $\mathscr{K}$. $\mathscr{I}\mathscr{F}(L)$ is then a subdirect product of the completions of the homomorphic images of L which lie in $\mathscr{H}$.

Since subdirect decompositions are not, in general, irredundant, the question of which of the members of $\mathscr{K}$ (resp. $\mathscr{H}$) are necessary in the decomposition of $\mathscr{I}(L)$ (resp. $\mathscr{I}\mathscr{F}(L)$) is ill-posed. However, if L is modular then, by a result of P. Crawley and R. P. Dilworth, $\mathscr{I}(L)$ and $\mathscr{I}\mathscr{F}(L)$ each have a unique irredundant decomposition into subdirectly irreducibles. Now we can ask which of the special completions of the homomorphic images of L in $\mathscr{K}$ (resp. $\mathscr{H}$) are necessary in representing $\mathscr{I}(L)$ (resp. $\mathscr{I}\mathscr{F}(L)$). The following theorem partially answers this question.

THEOREM 6.1. *Let L be a modular lattice. Then every finite dimensional subdirectly irreducible homomorphic image of L must be used in every decomposition of $\mathscr{I}\mathscr{F}(L)$ into subdirectly irreducible lattices.*

Proof. Let f be a homomorphism from L onto K, where K is a finite dimentional subdirectly irreducible lattice. Then K has a prime quotient a/b which is collapsed by all nontrivial congruences on K. By Lemma 5.1 f may be extended to a homomorphism of $\mathscr{I}\mathscr{F}(L)$ onto K. Define the maps α and β from K into $\mathscr{I}\mathscr{F}(L)$ as in the proof of Theorem 5.2. Since $\alpha(b)$ is the largest element of $\mathscr{I}\mathscr{F}(L)$ which is mapped to b under f, and since $\beta(a)$ is the smallest element mapping to a under f, it follows that $\alpha(b) \vee \beta(a) \succ \alpha(b)$ in $\mathscr{I}\mathscr{F}(L)$. Hence $\mathscr{I}\mathscr{F}(L)/\psi(\alpha(b) \vee \beta(a), \alpha(b))$ is isomorphic to K. The theorem follows from the following lemma. For any lattice A we let $\Theta(A)$ denote the lattice of congruence relations on A and 0 its least element.

LEMMA 6.2. *Let A be a modular lattice with prime quotient x/y. If S is a set of meet irreducible elements in $\Theta(A)$ such that $\bigwedge S = 0$, then $\psi(x, y) \in S$.*

Proof. Since $\bigwedge S = 0$, there must be a $\varphi \in S$ such that $(x, y) \notin \varphi$. Thus, $\varphi \leqq \psi(x, y)$. Since A is modular, the least congruence collapsing x and y, $\theta(x, y)$, covers 0 in $\Theta(A)$. Since $\Theta(A)$ is distributive, $\theta(x, y) \vee \varphi \succ \varphi$. Hence $\varphi = (\theta(x, y) \vee \varphi) \wedge \psi(x, y)$. Since φ is meet irreducible it follows that $\varphi = \psi(x, y)$, proving the lemma.

A lattice K is called transferable if, for every lattice L, whenever K is isomorphic to a sublattice of $\mathscr{I}(L)$, then K is isomorphic to a sublattice of L. The next result follows from Theorem 5.2.

COROLLARY 6.3. *If K satisfies* (W) *and both chain conditions and both K and its dual are transferable then K is isomorphic to a sublattice of a free lattice.*

It should be pointed out that B. Jónsson has shown that a sublattice of a free lattice which satisfies both chain conditions is finite [5].

Another corollary is the following: *if K is a finite transferable lattice satisfying* (W), *and K is a lower bounded homomorphic image of a free lattice* (see [7]), *then K is a sublattice of a free lattice.* Further applications of Theorem 5.2 and other results related to transferability are given by H. Gaskill, G. Grätzer, and C. R. Platt [4], and K. A. Baker and A. W. Hales [1]. For applications of ideal lattices to lattice varieties, see [3].

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