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On some identities valid in modular congruence varieties

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Freese and Jónsson [8] showed that the congruence lattice of a (universal) algebra in a congruence modular variety is always arguesian. On the other hand Jónsson [16] constructed arguesian lattices which cannot be embedded into the normal subgroup lattice of a group. These lattices consist of two arguesian planes of different prime order glued together over a two element sublattice (cf. Dilworth and Hall [3]). In [11], Herrmann and Poguntke derived identities not valid in those lattices but valid in all lattices of normal subgroups. In the present paper we show that these (and more general) identities hold in the congruence lattice of any algebra in a congruence modular variety. This implies, in particular, that the class of arguesian lattices does not form a congruence variety in the sense of Jónsson [17]. (This result has been proved by the first author and announced in [7]). Moreover, one concludes as in [11] that a modular congruence variety cannot be defined by finitely many identities provided it contains the rational projective plane or two projective planes of distinct prime orders or a subgroup lattice of a group $C^3_{p^2}$.

1. Definitions and main results

For subgroups the verification of the lattice identities to be constructed reduces to the trivial observation that isomorphic abelian quotients have the same exponent. Consequently, we introduce "projective" lattice relations which yield for certain quotients:

I isomorphy

II "coordinate systems" allowing one to speak about "exponents".

Ad I. Projective quotients. If a and b are elements of a modular lattice such that $a \geq b$ then we write $a/b = \{x \mid a \geq x \geq b\}$. We write $a/b \nearrow c/d$ as well as

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$c/d \searrow a/b$ if $a+d=c$ and $ad=b$. Then, $\phi x = x+d$ and $\phi^{-1}y = ya$ define isomorphisms between a/b and c/d which are inverse to each other we write $a/b \nearrow_\phi c/d$. If $a/b = a_0/b_0$, $c/d = a_n/b_n$ and $a_{i-1}/b_{i-1} \nearrow_{\phi_i} a_i/b_i$ or $a_{i-1}/b_{i-1} \searrow_{\phi_i} a_i/b_i$ for $1 \leq i \leq n$ then a/b is *projective* to c/d in n steps via the a_i/b_i – and we write $a/\approx_\phi c/d$ where $\phi = \phi_n \circ \cdots \circ \phi_1$.

Ad II. v. Neumann frames. A sequence $\vec{a} = (a_0, \ldots, a_n)$ of elements of a modular lattice L is called *independent* if $a_i(a_0 + \cdots + a_{i-1}) = \Pi\vec{a}$ for $1 \leq i \leq n$. Then, $a_i = a_j$ implies $a_i = a_j = \Pi\vec{a}$ and the $a_i \neq \Pi\vec{a}$ form the set of atoms of a boolean sublattice of L with smallest element $\Pi\vec{a}$. A sequence $\vec{a} = (a_0, \ldots, a_n, a_{01}, \ldots, a_{0n})$ forms a *frame* of order $n+1$ if $(a_0, \ldots, a_n)$ is independent and if $a_0 + a_{0i} = a_i + a_{0i} = a_0 + a_i$ and $a_0 a_{0i} = a_i a_{0i} = a_0 a_i$ for $1 \leq i \leq n$. This is, in essence, the definition of v. Neumann [18], cf [10], [5]. As a general reference for lattice theory we use [1]. For arbitrary n and odd m let $J_{n,m}$ be the modular lattice freely generated by

$$\vec{a} = (a_0, \ldots, a_n, a_{01}, \ldots, a_{0n}),\ \vec{b} = (b_0, \ldots, b_n, b_{01}, \ldots, b_{0n}),\ \vec{c} = (c_1, \ldots, c_m),$$
$$\vec{d} = (d_1, \ldots, d_m)$$

subject to relations expressing that $\vec{a}$ and $\vec{b}$ are frames of order $n+1$, $\Sigma\vec{a}/(a_1 + \cdots + a_n) \nearrow c_1/d_1$, $c_i/d_i \nearrow c_{i+1}/d_{i+1}$ for even and $c_i/d_i \searrow c_{i+1}/d_{i+1}$ for odd i $(1 \leq i \leq m-1)$, and $c_m/d_m = b_0/\Pi\vec{b}$.

PROPOSITION 1. *$J_{n,m}$ is a projective modular lattice.*

Let $W_{n,m}$ and $F_{n,m}$ the word algebra and the modular lattice with $4n+2m+2$ free generators $\vec{x}$, $\vec{y}$, $\vec{u}$, $\vec{v}$ and $\phi: W_{n,m} \to F_{n,m}$ and $\pi: F_{n,m} \to J_{n,m}$ the canonical homomorphisms. By Proposition 1 there exists an embedding $\varepsilon: J_{n,m} \to F_{n,m}$ such that $\pi \circ \varepsilon$ is the identity map on $J_{n,m}$. Choose terms $\bar{a}_i$ etc. in $W_{n,m}$ such that $\phi\bar{a}_i = \varepsilon a_1$, $\phi\bar{a}_{0i} = \varepsilon a_{0i}$, $\phi\bar{b} = \varepsilon b_i$, $\phi\bar{b}_{0i} = \varepsilon b_{0i}$, $\phi\bar{c}_i = \varepsilon c_i$, $\phi\bar{d}_i = \varepsilon d_i$ (the proof of Prop. 1 indicates one particular choice). Then, for any lattice term w in $2n+1$ variables let $\gamma_{n,m}(w)$ denote the following lattice identity:

$$(\cdots(\bar{a}_0 w(\bar{a}_0, \ldots \bar{a}_{0n}) + \bar{d}_1)\bar{c}_2 \cdots)\bar{c}_{m-1} + \bar{d}_m = \bar{b}_0 w(\bar{b}_0, \ldots, \bar{b}_{0n}).$$

Define special lattice terms w_k in variables z_0, z_1, z_2, z_{01}, z_{02} by induction:

$$w_0 = z_0, \qquad w_{k+1} = ((w_k + z_2)(z_{02} + z_1) + (z_1 + z_2)(z_{01} + z_{02}))(z_0 + z_1)$$

The identities $\gamma_{n,m}(w)$, whose intuitive meaning will be made clearer below, give information on the characteristics of two rings which may be used to coordinatize certain modular lattices. The identity $\gamma_{n,m}(w_k)$ in these lattices says that if k is zero in one of the rings, it is zero in both.

THEOREM. *For all n, odd m, and lattice terms w are the identities* $\gamma_{n,m}(w)$ *valid in all congruence lattices L of algebras in congruence modular classes which are closed under finite subdirect powers. None of the identities* $\gamma_{n,m}(w_k)$ *(m odd, n,* $k \geq 2$*) is a consequence of the modular or even the Arguesian law.*

For the proof let $\psi: W_{n,m} \to L$ be any evaluation in L. Then the elements $a_0 = \psi\bar{a}_0, \ldots, d_n = \psi\bar{d}_n$ satisfy the relations defining $J_{n,m}$. In particular, $\vec{a}$ and $\vec{b}$ are frames of order $n+1$ and $a_0/\Pi\vec{a}$ is projective to $b_0/\Pi\vec{b}$ via the c_i/d_i. Denoting the canonical isomorphism by ϕ the left hand side of $\gamma_{n,m}(w)$ becomes just $\phi(a_0 w(\vec{a}))$. Thus, the proof of the first claim is immediate by the following

PROPOSITION 2. *Let L be as in the Theorem,* $\vec{a}$ *and* $\vec{b}$ *frames of order* $n+1$, *and* $a_0/\Pi\vec{a} \approx_\phi b_0/\Pi\vec{b}$. *Then there is an isomorphism* $\bar{\phi}$ *of* $\Sigma\vec{a}/\Pi\vec{a}$ *onto* $\Sigma\vec{b}/\Pi\vec{b}$ *such that* $\bar{\phi}a_i = b_i$, $\bar{\phi}a_{0i} = b_{0i}$ $(1 \leq i \leq n)$, *and* $\bar{\phi}x = \phi x$ *for* $\Pi\vec{a} \leq x \leq \bar{a}_0$.

In addition, such a lattice is arguesian (Freese and Jónsson [8]) and for any of its frames $\vec{a}$ the interval $\Sigma\vec{a}/\Pi\vec{a}$ can be embedded in the subgroup lattice of an abelian group (see Prop. 12 below). One might ask whether these properties suffice to characterize lattices embeddable in congruence lattices of algebras in congruence modular varieties (lattices of normal subgroups of (abelian) groups). We suspect that is not the case.

To prove the second claim of the Theorem consider a field F and let S be the lattice of F-linear subspaces of an $n+1$-dimensional F-vector space E with basis $e_0, \ldots, e_n$. Let be a_i and a_{0i} the subspaces generated by e_i and $e_i - e_0$, respectively. Then $\vec{a} = (a_0, \ldots, a_{0n})$ is a frame of order $n+1$ in S and $w_k(\vec{a})$ is the subspace generated by $ke_1 - e_0$—as is easily shown by induction. Thus, $a_0 w_k(\vec{a}) = a_0$ if and only if the characteristic of F divides k and $a_0 w_k(\vec{a}) = \Pi\vec{a}$, otherwise.

Now, for given k choose F, S, $\vec{a}$ and F', S', $\vec{b} = \vec{a}'$ such that char (F) divides k and char (F') does not. Let L be the lattice which one obtains identifying $\Sigma\vec{a}$ with b_0 and $a_1 + \cdots + a_n$ with $\Pi\vec{b}$ (s. Fig. 1). This is an argeuesian lattice according to Jónsson [16]. But it does not satisfy $\gamma_{n,m}(w_k)$. To see this, substitute $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ for the variables where $c_i/d_i = b_0/\Pi\vec{b}$ $(1 \leq i \leq m)$. Since $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ satisfy the relations of $J_{n,m}$ $\bar{a}_i$ takes the value a_i etc. whence the left hand side takes the value b_0 and the right hand side the value $\Pi\vec{b}$.

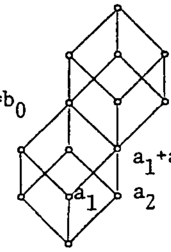

Figure 1

2. The projectivity of $J_{n,m}$

In [5] it is pointed out that our concept of a frame (as a system of generators and relations) is (for modular lattices) equivalent to that of v. Neumann [18]. Thus, it is also equivalent to Huhn's [13] concept of an n-diamond (cf [9; 1.7]) and we have the following (cf [4]).

LEMMA 3. *The modular lattice freely generated by a frame of order $n+1$ is a projective modular lattice.*

From Freese [5] we recall

LEMMA 4. *Let $\vec{a}$ be a frame of order $n+1$ in a modular lattice and $\Pi\vec{a} \leq b_0 \leq a_0$. Define $b_i = (b_0 + a_{0i})a_i$ and $b = b_0 + \cdots + b_n$. Then the following are frames of order $n+1$:*

(a) *$\vec{a}'$ where $a_i' = a_i + b$ and $a_{0i}' = a_{0i} + b$*
(b) *$\vec{a}''$ where $a_i'' = b_i$ and $a_{0i}'' = ba_{0i}$*

LEMMA 5. *Let $\vec{a}$ be a frame of order $n+1$ in a modular lattice and $e_i = \Sigma(a_j \mid j \neq i)$, $e_{0i} = a_{0i} + e_0 e_i$. Then $\vec{e}$ is a dual frame of order $n+1$ with $\Pi\vec{a} = \Pi\vec{e}$, $\Sigma\vec{a} = \Sigma\vec{e}$, $a_i = \Pi(e_j \mid j \neq i)$, and $a_{0i} = e_{0i}(a_0 + a_i)$.*

Proof of Prop. 1. Let ϕ be a homomorphism of M onto $J_{n,m}$, M modular. In several steps we choose inverse images a_i', a_i'' of the a_i etc. such that the relations of $J_{n,m}$ are satisfied, finally, whence there is a retraction map of $J_{n,m}$ into M (cf. [2] [12]). See Fig. 2 for the case $n+1=2$.

Let $\vec{e}$ be the dual frame corresponding to $\vec{a}$ according to Lemma 5. By the dual of Lemma 3 there is a dual frame $\vec{e}'$ in M which is mapped onto $\vec{e}$. Put $c_0' = \Sigma\vec{e}'$,

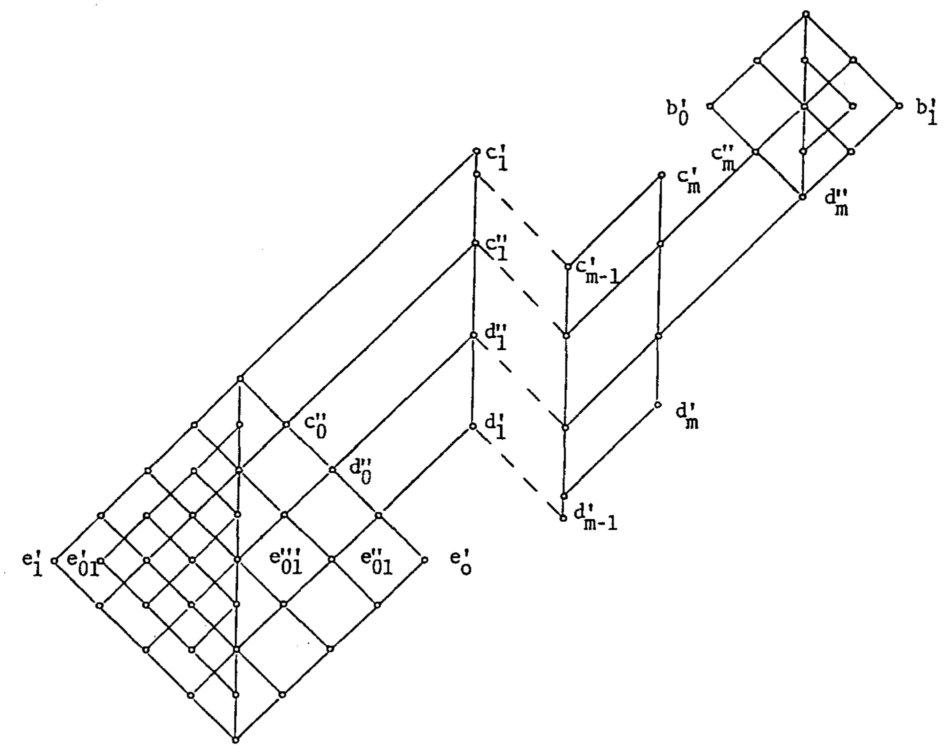

Figure 2

$d'_0=e'_0$, and choose $d'_i\le c'_i$ such that $d'_{i+1}\ge d'_i$, $c'_i=d'_{i+1}+d'_i$ for i even and $c'_{i+1}\le c'_i$, $d'_{i+1}=c'_{i+1}d'_i$ for i odd.

For $\vec{b}$ choose inverse images $\ge d'_m$, first, and pass by Lemma 3 to a frame $\vec{b}'$ with $\Pi b'\ge d'_m$. Put $c''_{m-1}=c'_{m-1}b'_0$, $d''_{m-1}=c'_{m-1}\Pi\vec{b}'$, $d''_m=\Pi\vec{b}'$, $c''_m=c'_{m-1}+d''_m$ and choose c''_i, d''_i such that $d'_i\le d''_i\le c''_i\le c'_i$ and $c''_i/d_i\nearrow c''_{i-1}/d''_{i-1}$ if i is odd and $c''_i/d''_i\searrow c''_{i-1}/d''_{i-1}$ if i is even ($i=m-1,\dots,0$).

Applying Lemma (4b) to $\vec{b}$, and c''_m we get a frame $\vec{b}''$ with $b''_0=c''_m$ and $\Pi\vec{b}''=d''_m$. Applying the dual of Lemma (4b) to $\vec{e}'$ and d''_0 we get a dual frame $\vec{e}''$ with $e''_0=d''_0$ and $\Sigma\vec{e}''=c'_0$. Applying the dual of Lemma (4a) to $\vec{e}''$ and c''_0 we get a dual frame $\vec{e}'''$ with $\Sigma\vec{e}'''/e'''_0\nearrow c''_0/d''_0$. Finally, we choose $\vec{a}''$ as the frame corresponding to $\vec{e}'''$ by Lemma 5. Then $\vec{a}''$, $\vec{b}''$, $\vec{c}''$, $\vec{d}''$ is an inverse image of $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ and satisfies the relations defining $J_{n,m}$.

3. More about frames

For the proof of Prop. 2 we need some results concerning isomorphisms in frames (Lemma 7, 9) and glueing of frames (Lemma 10, 11). The context is modular lattices. The notations $\underset{m}{=}$ ($\underset{i}{=}$, $\underset{f}{=}$) in a proof indicate that modularity (independence, definition of a frame) has been used. The following is well known—cf [18].

LEMMA 6. *Let $\vec{a}$ be a frame of order* 3, $c=(a_1+a_2)(a_{01}+a_{02})$. *Then* a_1, a_2, c *and* a_{01}, a_{02}, c *are frames of order* 2.

LEMMA 7. *Let* $(a_0,\dots,a_n,b_0,\dots,b_n)$ *be independent and for each* i a_i, b_i, c_i *a frame of order* 2. *Then* a, b, c, *is a frame of order* 2 *where* $a=a_0+\cdots+a_n$, $b=b_0+\cdots+b_n$, $c=c_0+\cdots+c_n$.

Proof by induction on n. Write $u=ab=a_ib_i$ and let be $n\geq 1$ and $a'=a_0+\cdots+a_{n-1}$, $b'=b_0+\cdots+b_{n-1}$, $c'=c_0+\cdots+c_{n-1}$. Then by the inductive hypothesis it holds $a'c'=b'c'=a'b'=u$. We conclude

$$ac\leq(a+b')c\underset{m}{=}(a+b')c_n+c'$$
$$=(a+b')(a_n+b_n)c_n+c'\underset{i}{=}a_nc_n+c'=a_nb_n+c'=u+c'=c'.$$

It follows $ac\leq ac'\leq a(a'+b')\underset{m}{=}a'+ab'\underset{i}{=}a'+u=a'$ and $ac\leq a'c'=u$.

COROLLARY 8. *Let* a *be a frame of order at least* $2n+2$. *Then for each* $x\leq a=a_0+\cdots+a_n$ *there are* y *and* z *such* $x+y=x+z=y+z$ *and* $xy=xz=yz=\Pi\vec{a}$.

Proof. In the case $x=a$ let be $b_i=a_{n+i+1}$, $c_0=a_{0n+1}$, and $c_i=(a_i+b_i)(a_{0i}+a_{0n+i+1})$. By Lemma 6 the hypotheses of Lemma 7 are satisfied and we may choose $y=b$ and $z=c$. For arbitrary x take $y=(x+c)b$ $z=(x+b)c$.

LEMMA 9. *Let* $\vec{a}$ *be a frame of order at least* $2n+2$. *Then there is an isomorphism of* $a_0+\cdots+a_n/\Pi\vec{a}$ *onto* $a_0+a_{n+1}+\cdots+a_{2n}/\Pi\vec{a}$ *which maps* a_i *onto* a_{n+i} *and* a_{0i} *onto* a_{0n+i} *for* $1\leq i\leq n$ *and which is the identity on* $a_0/\Pi\vec{a}$.

Proof. We write $b_i=a_{i+n}$, $b_{0i}=a_{0n+i}$, and $c_i=(a_i+b_i)(a_{0i}+b_{0i})$ for $1\leq i\leq n$. Moreover, put $b_0=a_0$, $c_0=a_{2n+1}$, $a=a_0+\cdots+a_n$, $b=b_0+\cdots+b_n$, $c=c_0+\cdots+c_n$, $a'=a_1+\cdots a_n$, $b'=b_1+\cdots+b_n$, and $c'=c_1+\cdots+c_n$. Lemma 6 and 7 give $a'+c'=b'+c'=a'+c'$ and $a'c'=b'c'=a'b'=\Pi\vec{a}$. Thus, we get $a+c=a_0+a'+c'+c_0=a_0+a'+b'+c_0=a+b+c_0$ and

$$ac\underset{i}{=}a'c=a'(a'+b')(c'+c_0)\underset{m}{=}a'(c'+(a'+b')c_0)\underset{i}{=}a'c'=\Pi\vec{a}.$$

Therefore, we have $a/\Pi\vec{a} \nearrow a+b+c_0/c$ and, by symmetry, $a+b+c_0/c \searrow b/\Pi\vec{a}$. By modularity, $\phi x = (x+c)b$ defines an isomorphism of $a/\Pi\vec{a}$ onto $b/\Pi\vec{a}$. We have to check that it has the additional properties. First, if $x \leq a_0 \leq b$ then

$\phi x = (x+c)b \underset{m}{=} x+cb = x+\Pi\vec{a} = x$. Now, let be $1 \leq i \leq n$. Then

$$(a_i+b)\Sigma(c_j \mid j\neq i) = (a_i+b)\Sigma(a_j+b_j \mid j\neq i))(\Sigma(c_j \mid j\neq i))$$
$$\underset{i}{=} (\Sigma(b_j \mid j\neq i))(\Sigma(c_j \mid j\neq i)) \leq bc = \Pi\vec{a}$$

whence

$$(a_i+c)b \leq (a_i+c)(a_i+b) \underset{m}{=} a_i + c(a_i+b) \underset{m}{=} a_i + c_i + (a_i+b)\Sigma(c_j \mid j\neq i) = a_i + c_i$$

and

$$\phi a_i = (a_i+c)b = (a_i+c)(a_i+c_i)b = (a_i+c_i)b = (a_i+b_i)b \underset{i}{=} b_i.$$

Finally, observe that $a_{0i}+c_i = a_{0i}+b_{0i}$ by Lemma 6 whence

$$\phi a_{0i} = (a_{0i}+c)b = (a_{0i}+b_{0i}+\Sigma(c_j \mid j\neq i))b(a_i+b)$$
$$\underset{m}{=} (a_{0i}+b_{0i}+(a_i+b)\Sigma(c_j \mid j\neq i))b$$
$$= (a_{0i}+b_{0i})b \underset{m}{=} a_{0i}b + b_{0i} = a_{0i}(b_0+a_i)b + b_{0i} \underset{i}{=} a_{0i}b_0 + b_{0i} \underset{f}{=} b_{0i}.$$

LEMMA 10. *Let $\vec{a}$ and $\vec{b}$ be frames of order $n+1$ and $m+1$, resp., such that $\Pi\vec{a} = \Pi\vec{b}$ and $(\Sigma\vec{a})(\Sigma\vec{b}) = a_0 = b_0$. Then $\vec{c} = \vec{a} * \vec{b}$ with $c_i = a_i$ $(0 \leq i \leq n)$, $c_{n+i} = b_i$, $c_{0i} = a_{0i}$ $(1 \leq i \leq n)$ and $c_{0n+i} = b_{0i}$ $(1 \leq i \leq m)$ is a frame of order $n+m+1$.*

Proof. It suffices to check the independence of $c_0, \ldots, c_{n+m}$, i.e. $c_k(c_0 + \cdots + c_{k-1}) = \Pi\vec{c} = \Pi\vec{a}$ for $1 \leq k \leq n+m$. For $k \leq n$ this follows from the independence of $a_0, \ldots, a_n$, trivially. Let be $k > n$, $j = k-n$. Then it holds

$$c_k(c_0 + \cdots + c_{k-1}) = c_k(\Sigma\vec{a} + b_1 + \cdots + b_{j-1}) \leq (\Sigma\vec{b})(\Sigma\vec{a} + b_1 \cdots + b_{j-1})$$
$$\underset{m}{=} (\Sigma\vec{a})(\Sigma\vec{b}) + b_1 + \cdots + b_{j-1} = b_0 + \cdots + b_{j-1},$$

whence $c_k(c_0 + \cdots + c_{k-1}) = b_j(b_0 + \cdots + b_{j-1}) \underset{i}{=} \Pi\vec{b} = \Pi\vec{c}$.

4. Congruence amalgamation

The key for the proof of Prop. 2 lies in a special amalgamation property for congruence lattices which has been investigated in Freese and Jónsson [8]. Namely, given an algebra A and a congruence α on A we can consider α as a subalgebra of $A\times A$. Then, the two natural projections of α onto A induce (according to the Isomorphism Theorem) two lattice embeddings of the congruence lattice of A into the congruence lattice of α. Thus, if $\mathcal{A}$ is a class of algebras closed under the formation of finite subdirect powers (which includes such "congruences as algebras") then the class $\mathcal{L}$ of congruence lattices of algebras in $\mathcal{A}$ has the following property (see [8] and [9; 1.3]):

For every L in $\mathcal{L}$ and a in L there exists M in $\mathcal{L}$ and embeddings ϕ_0,ϕ_1 of L in M such that

$$(*)\quad \begin{array}{ll} \phi_0 x=\phi_1 x & \text{for all } x\geq a \text{ in } L\\ \phi_0 x+\phi_1 x=\phi_0 a & \text{for all } x\leq a \text{ in } L\\ \phi_i y+\phi_0 x\phi_1 x=\phi_i x & \text{for all } y\leq x \text{ in } L \text{ and } i=0,1,. \end{array}$$

LEMMA 11. *Let $\mathcal{L}$ have the above property and consist of modular lattices. Let be L in $\mathcal{L}$ and $\vec{a}$ a dual frame of order $n+1$ in L. Then for each $m\geq n$ there is a lattice M in $\mathcal{L}$, an embedding ϕ of L in M, and a dual frame $\vec{b}$ of order $m+1$ in M such that $\phi a_i=b_i$ and $\phi a_{0i}=b_{0i}$ $(0\leq i\leq n)$.*

Proof. For $m=2n$ choose M, ϕ_0, and ϕ_1 as above with $a=a_0$—see Fig. 3. Then $\phi_0\Pi\vec{a}+\phi_1\Pi\vec{a}=a_0$ and $\vec{b}=\phi_0\vec{a}*\phi_1\vec{a}$ yields a dual frame of order $m+1$ by the dual of Lemma 10. Take $\phi=\phi_0$. For arbitrary $m\geq n$ use iteration and restriction.

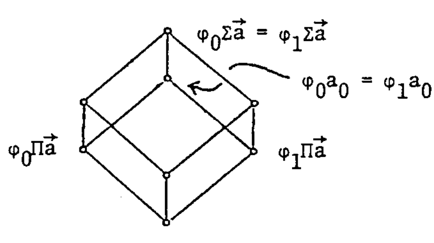

Figure 3

Proof of Prop. 2. We prove the dual statement first: If $\vec{c}$ and $\vec{d}$ are dual frames of order $n+1$ and $\Sigma\vec{c}/c_0\approx_\psi\Sigma\vec{d}/d_0$, then there is an isomorphism $\bar{\psi}$ of $\Sigma\vec{c}/\Pi\vec{c}$ onto $\Sigma\vec{d}/\Pi\vec{d}$ with $\bar{\psi}c_i=d_i$, $\bar{\psi}c_{0i}=d_{0i}$, and $\bar{\psi}x=\psi x$ for $c_0\leq x\leq\Sigma\vec{c}$. We consider the special case $\Sigma\vec{c}=\Sigma\vec{d}$ and $c_0=d_0$. Choose M, ϕ_0, and ϕ_1 such that $(*)$ is satisfied with respect to L and $a=c_0=d_0$. Then $\phi_0\Pi\vec{c}+\phi_1\Pi\vec{d}=\phi_0c_0=\phi_1d_0$ whence

$\vec{e} = \phi_0\vec{c} * \phi_1\vec{d}$ forms a dual frame of order $2n+1$ in M by the dual of Lemma 10. By Lemma 11 there is an extension M' of M and a dual frame $\vec{f}$ of order $2n+2$ in M' which extends $\vec{e}$. Thus, according to Lemma 9 there is an isomorphism χ of $\Sigma\phi_0\vec{c}/\Pi\phi_0\vec{c}$ onto $\Sigma\phi_1\vec{d}/\Pi\phi_1\vec{d}$. Let be χ_0 the restriction of ϕ_0 to $\Sigma\vec{c}/\Pi\vec{c}$ and χ_1 the restriction of ϕ_1 to $\Sigma\vec{d}/\Pi\vec{d}$. Then $\chi_1^{-1}\circ\chi\circ\chi_0$ is an isomorphism with the properties we want.

Now assume $\Sigma\vec{c}/c_0 \searrow \Sigma\vec{d}/d_0$. Choose M, ϕ_0, ϕ_1 such that ($*$) is satisfied with respect to L and with $a = c_0$. In M, $\phi_0(\Sigma\vec{d}) + \phi_1(\Pi\vec{c}) = \phi_0(\Sigma\vec{c})$. It follows that $\phi_1(\vec{c})\phi_0(\Sigma\vec{d}) = (\phi_1(c_0)\phi_0(\Sigma\vec{d}),\ldots,\phi_1(c_n)\phi_0(\Sigma\vec{d}),\quad \phi_1(c_{01})\phi_0(\Sigma\vec{d}),\ldots,\phi_1(c_{0n})\phi_0(\Sigma\vec{d}))$ is an $n+1$ frame in M and clearly the map $x \to x\phi_0(\Sigma\vec{d})$ induces an isomorphism of $\phi_1(\vec{c})$ to $\phi_1(\vec{c})\phi_0(\Sigma\vec{d})$ of the type we desire. Moreover $\phi_1(c_0)\phi_0(\Sigma\vec{d}) = \phi_0(c_0)\phi_0(\Sigma\vec{d}) = \phi_0(c_0\Sigma\vec{d}) = \phi_0(d_0)$. Now the proof for this case follows by applying the special case to the frames $\phi_0(\vec{d})$ and $\phi_1(\vec{c})\phi_0(\Sigma\vec{d})$. Of course, the case when $\Sigma\vec{c}/c_0 \nearrow \Sigma\vec{d}/d_0$ is handled symmetrically. With the aid of the special case, longer projectivities may be handled by iteration of the above procedure.

Finally, to derive Prop. 2 from its dual consider $\vec{a}$, $\vec{b}$, ϕ and let $\vec{c}$, $\vec{d}$ the dual frames corresponding to $\vec{a}$ and $\vec{b}$ according to Lemma 5. Define an isomorphism ψ of $\Sigma\vec{c}/c_0$ onto $\Sigma\vec{d}/d_0$ by $\psi x = d_0 + \phi a_0 x$ and let $\bar{\psi} = \bar{\phi}$ be the extension of ψ given above. Looking at the statements of Lemma 5 more closely we get $\bar{\phi}a_i = b_i$, $\bar{\phi}a_{0i} + b_{0i}$ and $\bar{\phi}x = \bar{\phi}(a_0(x+c_0)) \underset{m}{=} \bar{\phi}a_0\bar{\phi}(x+c_0) = b_0(\psi(x+c_0) = b_0(d_0+\phi x) \underset{m}{=} \phi x$ for $\Pi\vec{a} \leq x \leq a_0$ since $a_0/\Pi\vec{a} \nearrow \Sigma\vec{c}/c_0$ and $b_0/\Pi\vec{b} \nearrow \Sigma\vec{d}/d_0$.

5. Odds and ends

Let $\mathscr{A}$ be a class of algebras which is closed under finite subdirect powers such that the class $\mathscr{L}$ of congruence lattices of algebras in $\mathscr{A}$ consists of modular lattices.

PROPOSITION 12. *For any frame $\vec{a}$ in a lattice L in $\mathscr{L}$ there is an embedding of the interval sublattice $\Sigma\vec{a}/\Pi\vec{a}$ into the subgroup lattice of an abelian group.*

PROPOSITION 13. *$\mathscr{L}$ consists either of distributive lattices, only, or there is a p, p prime or zero, such that for each m there is an L in $\mathscr{L}$ containing the m-dimensional projective geometry over the prime field of characteristic p as a sublattice.*

Proofs. Let $\vec{a}$ be a frame of order $n+1$ and put $L_n = L$. By Lemma 11 there exist lattices $L_n \subseteq L_{n+1} \subseteq L_{n+2} \subseteq \cdots$ in $\mathscr{L}$ and elements a_{n+i}, a_{0n+i} in L_{n+i} $(i \geq 1)$

such that for $m \geq n$ $(a_0, \ldots, a_{0m})$ is a frame of order $m+1$ in L_m. Let M_m be the interval sublattice $a_0 + \cdots + a_m/\Pi\vec{a}$ of L_m and M the union of the M_m. Due to Corollary 8 the dual of M is an abelian lattice in the sense of Hutchinson [14] and can be embedded into the subgroup lattice of an abelian group. Thus, M itself can be embedded, too, as was shown by Hutchinson [15].

To prove Prop. 13 observe that if $\mathcal{L}$ contains a nondistributive lattice then there is a lattice L_1 in $\mathcal{L}$ containing a nontrivial frame $\vec{a}$ of order 2. Since L_1 is algebraic $a_0/\Pi\vec{a}$ contains a prime quotient. Thus, by Lemma 4 one may assume that $a_0/\Pi\vec{a}$ is a prime quotient, already. Choose L_m, a_m, a_{0m} as above. Then all quotients $a_i/\Pi\vec{a}$ are isomorphic whence $a_0 + \cdots + a_m/\Pi\vec{a}$ is an atomistic modular lattice of length $m+1$. Thus, the sublattice generated by $a_0, \ldots, a_{0m}$ is the m-dimensional projective geometry over a prime field of characteristic p_m—cf [10]. Evidently, $p_m = p_2$ for all $m \geq 2$.

A modular congruence variety is a lattice variety generated by a class $\mathcal{L}$ as above. Let $\mathcal{V}_0$ and $\mathcal{V}_n$ $(n \neq 0)$ denote the quasivariety of all lattices embeddable in rational projective geometries and subgroup lattices of abelian groups of exponent n, respectively. Then $H\mathcal{V}_n$ is the congruence variety generated by $\mathcal{V}_n$ and for $n \neq 0$ $H\mathcal{V}_m \subseteq H\mathcal{V}_n (\mathcal{V}_m \subseteq \mathcal{V}_n)$ if and only if m divides n.

The variety $H\mathcal{V}_n$ is generated by the subgroup lattices of the finite powers of the cyclic group of order n if $n \neq 0$ and by the finite dimensional rational projective geometries if $n = 0$. This is due to the fact that every algebraic lattice is in the variety generated by its sections $c/0$ with c compact. From Prop. 13 we get (cf Freese [6]).

COROLLARY 14. *Every nondistributive modular congruence variety contains one of the varieties $H\mathcal{V}_p$, p prime or zero.*

PROPOSITION 15. *Let $\mathcal{K}$ be any class of lattices which is contained in a modular congruence variety and contains a class $\mathcal{V}_n$, n not prime. Then $\mathcal{K}$ cannot be defined by finitely many first order axioms.*

In the proof we construct a sequence of lattices not in $\mathcal{K}$ an ultraproduct of which belongs to $\mathcal{K}$. To show the first we use the following immediate consequence of the Theorem.

COROLLARY 16. *Let L be lattice in a modular congruence variety and $\vec{a}$ and $\vec{b}$ frames of the same order in L such that $a_0/\Pi\vec{a}$ and $b_0/\Pi\vec{b}$ are projective. Then the sublattices generated by $\vec{a}$ and $\vec{b}$, respectively, are isomorphic.*

Proof of Prop. 15. $n = 0$: Use the lattices referred to in the introduction—cf. [11].

Let n have different prime factors p and q. Let L_p and L_q be arguesian planes of order p and q respectively. For $m \geq 1$ let M_m be the lattice which arises from L_p, L_q, and S_m (see Fig. 4) by identifying a line of L_p with u, the top of L_p with x_1, the bottom of L_q with y_m, and a point of L_q with v. Then $x_1/u \approx v/y_m$ whence M_m is not in $\mathcal{K}$. Let M be a nontrivial ultraproduct of the M_m. Then M can be decomposed into a linear sum in the following way: At the bottom (top) there is a plane over a field of characteristic $p(q)$ and an upward (downward) infinite "snake" glued to an upper (lower) edge of the plane and in the middle there are several upward and downward infinite "snakes" one at top of the other. Since the "snake" parts are in $\mathcal{V}_k$ for any k the lower plane together with the attached "snake" is in $\mathcal{V}_p$ and the remainder in $\mathcal{V}_q$. Since M is a subdirect product of these two parts M is in $\mathcal{V}_{pq}$ and in $\mathcal{V}_n$.

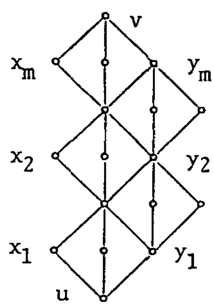

Figure 4. S_m $(m = 3)$

Let n be a proper power of the prime p. Then let U be the subgroup lattice of the abelian group $C_{p^2} \oplus C_{p^2} \oplus C_{p^2}$ where C_{p^2} is the cyclic group of order p^2. Let V be the lattice of submodules of the R module $R \oplus R \oplus R$ where $R = F_p[x]/(x^2)$ with F_p the field of order p. Observe that U is generated by its canonical frame of order 3—[cf. 10; 3.2]—while the sublattice generated by the canonical frame of V is isomorphic to a plane of order p since V is in $\mathcal{V}_p$—see [10; 2.3]. Now, let M_m be the lattice which arises from U, V and T_m (see Fig. 5) by identifying u with $0 \oplus C_{p^2} \oplus C_{p^2}$, z with ${}_pC_{p^2} \oplus C_{p^2} \oplus C_{p^2}$, x_1 with the top of U, y_m with the bottom of V, w with $xR \oplus 0 \oplus 0$, and v with $R \oplus 0 \oplus 0$. Again, M_m is not in $\mathcal{K}$ since $x_1/u \approx v/y_m$ but an nontrivial ultraproduct is in $\mathcal{V}_n$ since it is a subdirect product of its bottom belonging to $\mathcal{V}_{p^2}$ and the remainder belonging to $\mathcal{V}_p$.

We say that a class $\mathcal{K}$ of lattices has exponent p, p prime, if every lattice in $\mathcal{K}$ which is generated by a frame of order $n+1 \geq 3$ is an n-dimensional projective geometry of order p.

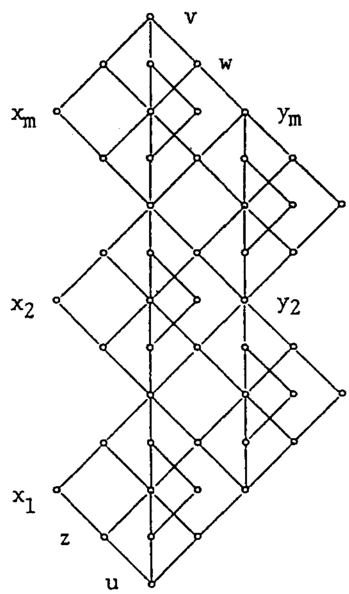

Figure 5. T_m ($m = 3$)

COROLLARY 17. *A nondistributive modular congruence variety not having a prime exponent cannot have a finite equational base.*

Proof. Let $\mathcal{K}$ be generated by $\mathcal{L}$ as above. Clearly, if $\mathcal{L}$ would have exponent p so would have $\mathcal{K}$. Now, look at the lattices in $\mathcal{K}$ which are generated by a frame of order 3. By Corollary 12 the subdirectly irreducibles have to be in the list given in [10]. Therefore, as sublattices of lattices in $\mathcal{L}$ have to occur either a rational projective plane or two planes of different prime orders p and q or a subgroup lattice of a group $C^3_{p^2}$. In the first case let $\vec{a}$ be a generating frame of order 3 for the plane. As in the proof of Prop. 12 for each $m \geq 2$ there is a lattice L_m in $\mathcal{L}$ and a frame $\vec{b}$ of order $m+1$ in L_m which extends $\vec{a}$. Let S be the sublattice generated by $\vec{b}$. Since S can be embedded in the subgroup lattice of an abelian group the subdirectly irreducible factors S_i appear in the list given in [10]. But in S_i the image of a generates the rational plane (since this is a simple lattice). Thus, S_i has to be the m-dimensional projective geometry for every i and S has to be so, too. One concludes that $H\mathcal{V}_0$ is contained in $\mathcal{K}$. By similar arguments one gets that $H\mathcal{V}_p$ and $H\mathcal{V}_q$ (whence $H\mathcal{V}_{pq}$) is contained in $\mathcal{K}$ in the second case and $H\mathcal{V}_{p^2}$ in the third. Thus, in any case Prop. 16 implies.

REFERENCES

[1] P. Crawley and R. P. Dilworth, *Algebraic theory of lattices*, Prentice-Hall, Englewood Cliffs, N.J. 1973.
[2] A. Day, *Splitting algebras and a weak notion of projectivity*, Algebra Universalis *5* (1975), 153–162.
[3] R. P. Dilworth and M. Hall, *The embedding theorem for modular lattices*, Ann. Math. *45* (1944), 450–456.
[4] R. Freese, *Planar sublattices of FM*(4), Algebra Universalis *6* (1976), 69–72.
[5] R. Freese, *The variety of modular lattices is not generated by its finite members*, Transactions Amer. Math. Soc., *255* (1979), 277–300.
[6] R. Freese, *Minimal modular congruence varieties*, Notices Amer. Math. Soc. *23* (1976), #76T-A14.
[7] R. Freese, *The class of Arguesian lattices is not a congruence variety*, Notices Amer. Math. Soc. *23* (1976), #76T-A181.
[8] R. Freese and B. Jónsson, *Congruence modularity implies the Arguesian identity*, Algebra Universalis *6* (1976), 225–228.
[9] J. Hagemann and C. Hermann, *A concrete ideal multiplication for algebraic systems and its relation with congruence distributivity*, Arch. Math. *32* (1979), 234–245.
[10] C. Herrmann and A. Huhn, *Lattices of normal subgroups which are generated by frames*, Coll. Math. Soc. J. Bolyai *14* (1974), 97–136.
[11] C. Hermann and W. Poguntke, *The class of sublattices of normal subgroup lattices is not elementary*, Algebra Universalis *4* (1974), 280–286.
[12] D. X. Hong, *Covering relations among lattice varieties*, Pac. J. Math. *40* (1972), 575–603.
[13] A. P. Huhn, *Schwach distributive Verbände I*, Acta Sci. Math. *23* (1972), 297–305.
[14] G. Hutchinson, *Modular lattices and abelian categories*, J. Alg. *19* (1971), 156–184
[15] G. Hutchinson, *A duality principle for lattices representable by modules and corresponding abelian categories*, J. Pure Appl. Algebra, *10* (1977/78), 115–119.
[16] B. Jónsson, *Modular lattices and Desargues' Theorem*, Math. Scand. *2* (1954), 295–314.
[17] B. Jónsson, *Varieties of algebras and their congruence varieties*, Proc. Int. Congr. of Math., Vancouver 1974.
[18] J. von Neumann, *Continuous geometries*, Princeton Univ. Press, Princeton N.J. 1960.

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