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# DIRECTIONS IN LATTICE THEORY

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This paper is about three problems raised by Bjarni Jónsson and their influence on the development and direction of lattice theory.

In lattice theory, like most fields, certain problems have played an important role in its development. R. P. Dilworth's solutions of some of the most difficult problems of the thirties certainly raised the level of lattice theory. His result [17] that every lattice can be embedded into a uniquely complemented lattice surprised most of the lattice theorists of the time. The excellent article [1] by M. Adams details the history of and subsequent work on this problem.

Some of the problems that have had the greatest influence the direction lattice theory has taken were proposed by Jónsson. Many of these problems, as well as many of the problems dating back to the thirties and before, have been solved in the last two decades. In this article, I have chosen three of Jónsson's problems which have had a particularly strong influence on the development of lattice theory. We will also discuss some problems that are still unsolved.

## 1. Free Modular Lattices

My own work in modular lattice theory was sparked by a simple question raised by Jónsson in a preliminary version of [44], namely

- *Are there uncountable distributive sublattices of free modular lattices?*

(The problem was solved quickly enough that it did not appear in the final version of [44].) There is an interesting story behind the solution to this problem.

After two years of graduate school at Caltech, I was joined by J. B. Nation, who had just completed his undergraduate work at Vanderbilt under the supervision of Bjarni Jónsson. I naturally was curious about what Jónsson was doing and J. B. filled me in. One of the results he related to me was the following theorem.

**Theorem 1.** *Suppose that* $\mathbf{M}_3$ *is a sublattice of a modular lattice* $\mathbf{L}$ *and* $a$ *is an atom and* $b$ *is the least element of* $\mathbf{M}_3$*. Then the interval sublattice*

$$a/b = \{x \in L : b \leq x \leq a\}$$

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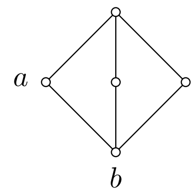


FIGURE 1

*is an Arguesian sublattice of* $\mathbf{L}$.

This result is related to the fact that projective planes which can be embedded into projective three-space are Arguesian and in a certain sense the result is not surprising. (However anyone who has tried to convert these geometric facts into lattice theoretic proofs knows how difficult it can be. The lattice theoretic results are, of course, much more general.) Nevertheless, this result held a particular fascination for me. One reason is the following corollary.

**Corollary 2.** *Every nontrivial free modular lattice contains a proper Arguesian interval. Consequently, there are uncountable Arguesian sublattices of free modular lattices.*

This corollary contrasts sharply with the situation for free lattices, as the following result shows.

**Theorem 3** (Whitman [58]; Galvin and Jónsson [29]). *Every uncountable sublattice of a free lattices has an uncountable free sublattice.*

The next result, which is an unpublished result of the author, clarifies how free lattices of one variety can be embedded into those of another.

**Theorem 4.** *Let* $\mathcal{V}$ *be a variety of lattices and let* $v < u$ *be elements of the sublattice of* $\mathbf{F}_{\mathcal{V}}(X \cup Y)$ *generated by* $X$. *Let* $\mathbf{L}$ *be the interval sublattice* $u/v$ *of* $\mathbf{F}_{\mathcal{V}}(X \cup Y)$ *and* $\mathcal{W} = \boldsymbol{V}(\mathbf{L})$. *If*

$$Y' = \{(y \wedge u) \vee v : y \in Y\}$$

*then* $\mathbf{Sg}(Y') \cong \mathbf{F}_{\mathcal{W}}(Y')$.

*Proof.* Let $f : Y' \to L$ be an arbitrary map. For $y \in Y$, let $y' = (y \wedge u) \vee v$ and define $g : Y \to L$ by $g(y) = f(y')$. Let $h : \mathbf{F}_{\mathcal{V}}(X \cup Y) \to \mathbf{F}_{\mathcal{V}}(X \cup Y)$ be the homomorphism extending the map which sends $x$ to $x$, $x \in X$, and sends $y$ to $g(y)$. Clearly $h(u) = u$ and $h(v) = v$ and, since $v \le y' \le u$, $v \le h(y') \le u$. Thus

$$h(y') = (h(y) \wedge u) \vee v = h(y) = g(y) = f(y').$$

Hence $h|_{\mathbf{Sg}(Y')} : \mathbf{Sg}(Y') \to \mathbf{L}$ is a homomorphism extending $f$. It is elementary (see §4.11 of [48]) that this implies that $\mathbf{Sg}(Y') \cong \mathbf{F}_{\mathcal{W}}(Y')$. $\square$

In the late seventies Jónsson sent me a preprint of [44]. It asked the innocuous sounding question at the beginning of this section.

I wondered if the approach outlined above for Arguesian lattices could work to produce large distributive sublattices. This led to the question:

- *Is there a modular lattice* **L** *with elements* $a > b$ *such that whenever* **L** *is embedded into a modular lattice,* $a/b$ *is distributive?*

In [19] we were able to construct such a lattice. We give a brief outline.

Let **F** be a field and $\mathbf{L}(\mathbf{F}^n)$ be the lattice of subspaces of an $n$–dimensional vector space over **F**. $\mathbf{L}(\mathbf{F}^2)$ is diagrammed in Figure 2.

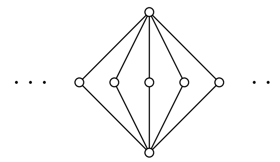


FIGURE 2

The number of atoms is one more that $|F|$. So if $|F| = |K|$, then the two lattices are isomorphic. On the other hand **F** can be recovered from $\mathbf{L}(\mathbf{F}^n)$ if $n \geq 3$.

The desired lattice **L** is obtained by gluing together $\mathbf{L}(\mathbf{F}^3)$ and $\mathbf{L}(\mathbf{K}^3)$ where **F** and **K** are countable fields of characteristics $p$ and $q$, respectively. Figure 3 gives a schematic diagram of this lattice.

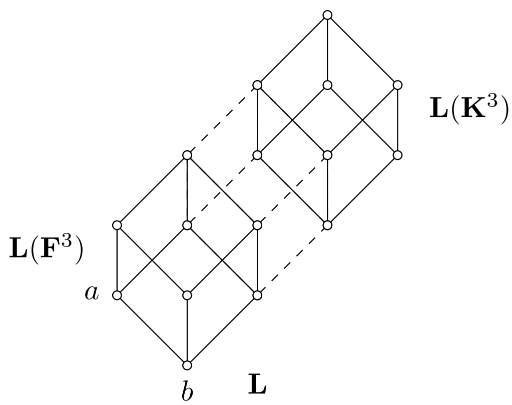


FIGURE 3

**Theorem 5** (Freese [19]). *Whenever* **L** *is embedded into a modular lattice,* $a/b$ *is distributive.*

Although this lattice is not a projective modular lattice, we can show that a certain partial sublattice of it is projective. Using this, it can be shown that:

**Theorem 6.** *There are nontrivial distributive intervals in every free modular lattice. Hence every free distributive lattice can be embedded into a free modular lattice. So every lattice embeddable into a free distributive lattice can be embedded into a free modular lattice.*

The converse is another good question raised by Jónsson and is still open.

- *Can every distributive sublattice of a free modular lattice be embedded into a free distributive lattice?*

Almost immediately after constructing **L** (and before proving the theorems above) I noticed that **L** could solve one of the most important problems in modular lattice theory. Namely, **L** is not in the variety generated by all finite modular lattices.

**Theorem 7** (Freese [19]). *The variety of modular lattices is not generated by its finite members. The variety generated by all finite dimensional modular lattices is not generated by its finite members.*

About a year later I was able to modify the construction of **L** a little so the following theorem could be proved.

**Theorem 8** (Freese [20]). *The word problem for free modular lattices on $n$ generators, $n \geq 5$, is unsolvable.*

In 1900 Dedekind [16] had shown that the free modular lattice on three generators is finite, but the following important question remained: *is the word problem for the free modular lattice on four generators solvable?* This was solved by C. Herrmann.

**Theorem 9** (Herrmann [35]). *The word problem for free modular lattices on four generators is unsolvable.*

Many other important problems in modular lattice theory were solved in Herrmann's important paper [36]. Let $\mathcal{M}_0$ denote the variety generated by all subspaces lattices of vertor spaces over $\mathbb{Q}$.

**Theorem 10** (Herrmann [36]). *Every variety of modular lattices which contains $\mathcal{M}_0$ either is not generated by its finite dimensional members or does not have a finite equational basis. Consequently, neither the variety of all modular lattices, nor the variety of Arguesian lattices is generated by its finite dimensional members and the varities generated by the finite, and the finite dimensional, modular lattices are not finitely based.*

## 2. Free Lattices

In [41] Jónsson showed that free lattices are semidistributive, i.e., they satisfy $(\mathrm{SD}_\vee)$ and its dual, $(\mathrm{SD}_\wedge)$.

$$(\mathrm{SD}_\vee) \qquad x \vee y = x \vee z \quad \textit{implies} \quad x \vee y = x \vee (y \wedge z)$$

Clearly sublattices of free lattices inherit these properties and they also satisfy the Whitman condition (W), which states that if $\bigwedge x_i \leq \bigvee y_j$ then, either, for some $i$, $x_i \leq \bigwedge y_j$ or, for some $j$, $\bigvee x_i \leq y_j$. The second Jónsson problem asks if these conditions characterize finite sublattices of a free lattice.

- *If a finite lattice satisfies* $(\mathrm{SD}_\vee)$, $(\mathrm{SD}_\wedge)$, *and* (W), *can it be embedded into a free lattice?*

This problem received the attention of several people over a period of many years. It was eventually shown to be true by J. B. Nation in [52]. This result is one of the three deepest theorems on free lattices, along with Tschantz's theorem (Theorem 14) that every infinite interval of a free lattice contains $\mathbf{FL}(\omega)$ as a sublattice [57], and the theory of covers in free lattices developed in [28].

Jónsson attempted to settle his question and in doing so outlined a general approach. This approach contained some concepts which have played an important role in several branches of lattice theory. Nation used a modification of Jónsson's approach in his solution.

Jónsson's idea is this: A *join cover* of an element $a$ in a lattice is a set $U$ of elements that join above $a$. The join cover $U$ is nontrivial if $a \nleq u$ for all $u \in U$. If $U$ and $V$ are subsets of a lattice, we say $V$ *refines* $U$ if for each $v \in V$ there is a $u \in U$ with $v \leq u$. This is denoted $V \ll U$. We say that $a$ *depends on* $b$ if $b$ is a member of a nontrivial join cover of $a$ which cannot be properly refined.

An element depends on no other element if and only if it is join prime. We say such elements have rank 0. An element is said to have rank $n$ if whenever $a$ depends on $b$, the rank of $b$ is less than $n$, and $a$ does depend on some element of rank $n-1$. It is possible to have cycles of dependencies. Elements in such a cycle do not have a rank. If all of the join irreducible elements of a lattice $\mathbf{L}$ have a rank, we say that $\mathbf{L}$ satisfies $(\mathrm{D}_\vee)$. The dual property is denoted $(\mathrm{D}_\wedge)$ and (D) denotes the conjunction of $(\mathrm{D}_\vee)$ and $(\mathrm{D}_\wedge)$. For a finite lattice, this means that the transitive closure of the dependency relation does not contain any cycles. Sublattices of free lattices satisfy (D) and, if a finite lattice satisfies (W) and (D), it can be embedded into a free lattice. Thus to settle Jónsson's problem, one needs to show that a finite semidistributive lattice satisfying (W), satisfies $(\mathrm{D}_\vee)$.

Jónsson and Nation showed that this dependence relations splits into three cases, one of which cannot occur in a semidistributive lattice. (A more detailed account of this is contained in [45]). The two remaining relations are denoted A and B. Roughly, if $a$ depends on $b$, case A holds if $b < a$ (we write $a \ \mathrm{A} \ b$) and case B holds otherwise. Nation's proof consists of a detailed analysis of the sequence of types of these relations that can occur in a cycle of dependencies, eventually showing that no such cycle can occur. The key breakthrough for the proof is his beautiful duality result: if $a \ \mathrm{A} \ b$ then $\kappa(a) \ \mathrm{B}^{dual} \ \kappa(b)$, where $\kappa(a)$ is the greatest element above $a_*$ (the unique lower cover of $a$), but not above $a$ (which exists by semidistributivity) and a similar fact holds for the B relation. As a simple example of how this is used, note that we cannot have a cycle entirely of A's because $a \ \mathrm{A} \ b$ implies $b < a$. The duality then implies there can be no all B cycle, which is by no means obvious.

In the late 1960's, McKenzie began his fundamental study of lattice varieties, which appeared in [47]. In this paper he formlated the concept of a bounded homomorphism. An epimorphism $f : \mathbf{L} \twoheadrightarrow \mathbf{M}$ is *upper bounded* if each $b \in M$ has a greatest preimage in $\mathbf{L}$, denoted $\alpha(b)$. *Lower bounded* is defined dually and $\beta(b)$ denotes the least preimage, and an epimorphism is *bounded* if it is both upper and lower bounded.

This concept has played an extremely important role in several parts of lattice theory. McKenzie showed, among other things, that if $a \succ b$ in a free lattice $\mathbf{FL}(n)$, and $\psi(a,b)$ is the unique largest congruence not containing $\langle a, b\rangle$ (which exists by Dilworth's characterization of lattice congruences [18]) then the natural homomorphism from $\mathbf{FL}(n)$ to $\mathbf{FL}(n)/\psi(a,b)$ is bounded. McKenzie used this to show that the predicate $a \succ b$ in a free lattice is recursive, along with several other results, some of which are described below. Lattices of the form $\mathbf{FL}(n)/\psi(a,b)$ can be characterized as finite, subdirectly irreducible, bounded images of free lattices. Such lattice are called *splitting* lattices. Splitting lattices are important in the theory of lattice varieties because if

$\mathbf{L} = \mathbf{FL}(n)/\psi(a,b)$ is a splitting lattice then every variety either contains $\mathbf{L}$ or satisfies the equation $a \approx b$.

Although the property (D) and the concept of a bounded homomorphism appear quite different, they are actually closely related: *a finitely generated lattice satisfies* (D) *if and only if it is a bounded epimorphic image of a free lattice.* Several important results in lattice theory are based on one or both of these concepts. The applications to varieties and to the solution of Jónsson's problem have already been mentioned. We now give a sampling of some of the other results related to these concepts. Credit for the germination of these ideas should be shared by Jónsson and McKenzie.

**Projective lattices.** Using these two concepts, Jónsson, Kostinsky, and McKenzie proved the following result, see [46] and [47].

**Theorem 11** (Jónsson, Kostinsky, and McKenzie). *A finitely generated lattice is projective if and only it satisfies* (D) *(equivalently, it is a bounded homomorphic image of a free lattice) and satisfies* (W)*.*

Freese and Nation were able to characterize arbitrary projective lattices as follows.

**Theorem 12** (Freese and Nation [27]). *A lattice* $\mathbf{L}$ *is projective if and only if it satisfies all of the following.*

(1) (W)
(2) (D)
(3) *For each* $a \in L$ *there is a finite set* $\mathrm{S}(a)$ *of join covers of* $a$ *such that any join cover of* $a$ *can be refined to one in* $\mathrm{S}(a)$*, and the dual property holds.*
(4) *For each* $a \in L$ *there are two finite subsets* $\mathrm{A}(a) \subseteq \{c \in L : c \geq a\}$ *and* $\mathrm{B}(a) \subseteq \{c \in L : c \leq a\}$ *such that if* $a < b$ *then* $\mathrm{A}(a) \cap \mathrm{B}(a) \neq \emptyset$*.*

The most difficult and most surprising part of this theorem is showing that an arbitrary projective lattice satisfies (D).

**Covers in free and finitely presented lattices.** The connection between coverings in free lattices and splitting lattices is clear. A thorough study of covers in free lattice is contained in [28]. For $w$ a join irreducible element in a free lattice, let $\mathrm{J}(w)$ be the smallest set $S$ containing $w$ such that if $u$ depends on $v$ and $u \in S$ then $v \in S$. $\mathrm{J}(w)$ is essentially the set of join irreducible subterms of the canonical form of $w$. Let $\mathrm{L}(w)$ be the join closure of $\mathrm{J}(w)$ in the free lattice. Then, since $\mathrm{L}(w)$ is finite and join closed, it is a lattice.

**Theorem 13** (Freese and Nation [28]). *If* $w$ *is a join irreducible element of a free lattice, it has a lower cover if and only if* $\mathbf{L}(w)$ *satisfies* (D)*.*

In [22] a similar theorem is proved for finitely presented lattices. Theorem 13 is the starting point of the characterization of finite intervals in free lattices obtained in [28] and of all the connected components of the covering relation in [21]. It is also the starting point of S. Tschantz's theorem mentioned at the beginning of this section.

**Theorem 14** (Tschantz [57]). *Every infinite interval of a free lattice has a sublattice isomorphic to* $\mathbf{FL}(\omega)$*.*

Tschantz raises the following interesting question.

- *Can a free lattice have elements $v < w < u$ such that the intervals $u/w$ and $w/v$ are both infinite and $u/v = u/w \cup w/v$?*

**Transferable lattices.** A lattice $\mathbf{L}$ is *transferable* if whenever it is embedded into the ideal lattice, $\mathcal{I}(\mathbf{K})$ for a lattice $\mathbf{K}$, it can be embedded into $\mathbf{K}$. $\mathbf{L}$ is *sharply transferable* if whenever $f : \mathbf{L} \rightarrowtail \mathcal{I}(\mathbf{K})$ is an embedding, there is an embedding $g : \mathbf{L} \rightarrowtail \mathbf{K}$ such that if $x \leq y$ in $\mathbf{L}$ then $g(x) \in f(y)$. Grätzer and Platt characterized sharply transferable lattices by conditions closely realted to (D). A lattice satisfies the condition ($\mathrm{R}_\vee$) if there is a mapping $\rho : L \to \omega$ such that if $a$ depends on $b$, then $\rho(a) < \rho(b)$. Note that this is a reverse ranking from the one used to define ($\mathrm{D}_\vee$).

**Theorem 15** (Grätzer and Platt [32]). *A lattice* $\mathbf{L}$ *is sharply transferable if and only if it satisfies* (W), ($\mathrm{R}_\vee$), ($\mathrm{D}_\wedge$), *and the complement of any principal filter is finite.*

Using his duality result mentioned above, J. B. Nation has shown in [51] that *sharply transferable lattices are projective*. He also gives an example showing that the converse is false.

**Congruence Varieties.** A variety of lattices which can be generated by all the congruence lattices of some variety of algebras is called a *congruence variety*. Splitting lattices are one of the main tools in this area. For example, all of the papers [5], [6], [7], [26], [30], [42], [43], [49], and [50] make use of splitting lattices.

S. V. Polin [56] constructed a variety $\mathcal{P}$ of algebras whose congruences are not modular, but do satisfy some nontrivial lattice equations, solving an old problem in this area. Day and Freese showed that *every nonmodular congruence variety contains Polin's congruence variety*, [8]. They used this to characterize congruence modularity. In [4] it is shown that this characterization is effective. The following theorem, which is crucial for these results, is an example of the role of splitting lattices in this area.

**Theorem 16** (Day and Freese [8]). *For each $n \geq 0$, the congruence lattice of the free Polin algebra on $n$ generators,* $\mathbf{Con}\ \mathbf{F}_{\mathcal{P}}(n)$*, is a splitting lattice. A variety is congruence modular if and only if it satisfies one of the equations associated with these splitting lattices.*

## 3. Lattices of Permuting Equivalence Relations and Arguesian Lattices

A lattice which can be represented with permuting equivalence relations is said to be *type* I. Dedekind's proof in [15] that the lattice of normal subgroups of a group is modular shows that any lattice of permuting equivalence relations is modular, see also [2] and [54]. Jónsson proved in [39] that type I lattices satisfy a stronger equation, which is called the Arguesian equation since it is a lattice theoretic analogue of Desargues Theorem in projective geometry. The third Jónsson problem is the following.

- *Does every Arguesian lattice have a type* I *representation?*

It is very tempting to conjecture that this is true. To quote from Crawley and Dilworth [3], "It would be a pleasant occurence if the class of all Arguesian lattices were identical with the class of all those lattice having a representation of type I. We (optimistically) conjecture that this is so." This problem has had a great influence on the direction of lattice theory. It was recently solved by M. Haiman. Jónsson had shown that type I lattice could be characterized by an infinite class of Horn sentences. Haiman not only showed that there is an Arguesian lattice which is not type I, but that type I lattices could not be finitely axiomatized.

**Theorem 17** (Haiman [33] and [34]). *The class of type* I *lattices cannot be defined by any finite set of first order axioms.*

The following problem is still open.

- *Can the class of type* I *lattices be defined by equations?*

It is easy to see that this class of lattices is closed under sublattices and products, so the above problem is equivalent to the question *is the class of type* I *lattices closed under homomorphic images?*

Of course Jónsson began the field of Arguesian lattices. This area is a major branch of lattice theory and too large to do justice to here. We will simply mention some important theorems. Nation's article [53] has more information.

Jónsson was able to show that, for the class of complemented lattices, type I is equivalent to the Arguesian equation [40]. This is important for the representation of such lattices by classical algebraic structures. For example, every complemented Arguesian lattice can be embedded into the the lattice of subgroups of some abelian group, and conversely. A clear account of this is given in [3].

Grätzer, Jónsson, and Lakser, using Arguesian lattice techniques, were able to show that there there are no modular, nondistributive lattice varieties with the amalgamation property. Later this result was extended to all varieties by A. Day and J. Ježek.

**Theorem 18** (Grätzer, Jónsson, and Lakser [31]; Day and Ježek [10]). *The variety of distributive lattices and the two trivial varieties are only lattice varieties with the amalgamation property.*

Arguesian lattices are important in the study of congruence varieties, as the following theorem shows.

**Theorem 19** (Freese and Jónsson [25]). *Every modular congruence variety is Arguesian.*

This showed that Dedekind's modular law is not strong enough to capture the lattices arising from classical algebraic structures. In [24], Herrmann, Huhn, and I showed that *there are equations stronger than the Arguesian law which hold in every modular congruence variety and there are type* I *lattices which lie in no modular congruence variety*. Very recently I have been able to prove the following theorem, which answers a question posed by G. McNulty at the Jónsson symposium. This question had been raised earlier by Jónsson in [44].

**Theorem 20** (Freese [23]). *No modular, nondistributive congruence variety can be finitely based.*

The next two problems are open. Although there certainly are varieties of algebras with modular conguences which do not have permutable congruences, e.g., lattices, it is still possible that all the congruence lattices of algebras in such a variety have a type I representation. This question was raised by W. A. Lampe in his lecture at the Jónsson symposium. Let $\mathcal{M}$ denote the variety with one ternary operation $p$ in its language satisfying the Maltsev equations $p(x, x, y) = y = p(y, x, x)$. Note that even if the first question has a positive answer, it does not logically follow that the second does.

- *If* **A** *lies in a congruence modular variety, does* **Con A** *necessarily have a type* I *representation?*

- *Is the congruence variety associated with* $\mathcal{M}$ *the unique largest modular congruence variety?*

Much more is known when **A** is a finite algebra in a congruence modular variety. Hobby and McKenzie have shown that the congruence lattice of such an algebra does have a type I representation and in fact lies in the congruence variety associated with loops, see Theorem 8.7 of [37].

The last few years have seen a more detailed study of the structure of Arguesian lattices. Several results were obtained on minimal non-Arguesian lattices. The paper [9] by Day, Herrmann, Jónsson, Nation, and Pickering gives a detailed account of this work. The next theorem guarantees that these problems will be difficult.

**Theorem 21** (Pickering [55]). *For each $n$ there is a minimal non-Arguesian lattice of length $6n$. There is a minimal non-Arguesian variety all of whose members of finite length are Arguesian.*

Our understanding of Arguesian lattices was particularly enhanced by the work of Day and Jónsson on the nature of failures of the Arguesian equation, [11], [12], [13], and [14]. This work brings a great deal more symmetry and duality to Arguesian configurations, and shows that these configurations are much more closely connected to the projective geometry from which the Arguesian law arose than had been previously thought.

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