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On Jónsson's theorem

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This paper surveys some results which are closely related to Jónsson's famous theorem. The theorem states that every subdirectly irreducible algebra in the variety generated by a class $\mathcal{K}$ of similar algebras is in $HSP_u(\mathcal{K})$ provided $V(\mathcal{K})$ is distributive (i.e. has distributive congruence lattices) [17]. Of course $P_u(\mathcal{K})$ stands for ultraproducts of members of $\mathcal{K}$. This theorem is the main impetus for the resurgence in the study of varieties of algebras, in particular lattice varieties. The theorem is used so often that one often forgets to acknowledge it. There have been several important applications of Jónsson's theorem. Kirby Baker has shown that a finite algebra in a distributive variety has a finite basis for its equational theory [1]. Ralph McKenzie has given several applications to lattice varieties and lattice structure theory in [19].

The first section of this paper shows how Jónsson's theorem is applied to obtain an important but overlooked result of Christian Herrmann and Andras Huhn on the embeddability of modular lattices into complemented modular lattices. In the second section we give what now appears to be the correct generalization of Jónsson's theorem to modular varieties (=varieties of algebras with modular congruence lattices). This result is in terms of the "commutator" – a new binary operation on all congruence lattices of algebras in a modular variety. The definition and important facts about the commutator will be reviewed in §2.

If two finite, subdirectly irreducible algebras generate the same distributive variety, then Jónsson's theorem implies they are isomorphic. This is false for modular varieties. In §3 we investigate under what additional hypotheses it is true. For example, if one of the algebras is simple, it is true. The third section also investigates to what extent other consequences of Jónsson's theorem are true in modular varieties.

§1. Complemented modular lattices

It is easy to see that every lattice can be embedded into a complemented lattice and by Birkhoff's theorem every distributive lattice can be embedded into a

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complemented distributive lattice. The corresponding question for modular lattices was one of the major problems in lattice theory in the 1930's. It was solved in the early 1940's when R. P. Dilworth and M. Hall constructed a counterexample [13], [4]. A proof that not all modular lattices can be embedded into complemented modular lattices using Jónsson's theorem runs as follows. Let $\mathscr{C}$ denote the class of all complemented modular lattices. Let $\mathscr{K}_1$ be the class of modular lattices of dimension (=length) at most three. Let $\mathscr{K}_2$ be the class of arguesian lattices (see p. 103 of [3]). By a result of Jónsson, every subdirectly irreducible lattice in $\mathscr{C}$ lies in $\mathscr{K}_1 \cup \mathscr{K}_2$ ([16] or Theorem 13.5 of [3]). But since ultraproducts preserve first order sentences, $HSP_u(\mathscr{K}_1 \cup \mathscr{K}_2) = \mathscr{K}_1 \cup \mathscr{K}_2$. Thus by Jónsson's theorem every subdirectly irreducible lattice in $V(\mathscr{C})$ lies in $\mathscr{K}_1 \cup \mathscr{K}_2$. The first Hall–Dilworth example, L_1, consists of a nondesarguean projective plane glued over a one dimensional quotient to a copy of M_3. It is schematically represented in Figure 1. This is a nonarguesian, simple modular lattice of length four. Hence by the above $L_1 \notin V(\mathscr{C})$. In particular $L_1 \notin S(\mathscr{C})$. (It is still open whether $S(\mathscr{C}) = HS(\mathscr{C})$.)

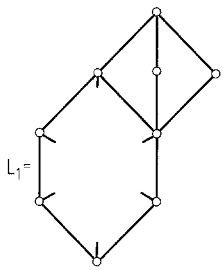

Fig. 1

The result of Jónsson quoted above is relatively easy, not requiring any coordinatization theorems. Nevertheless the above proof cannot really be considered easier than Hall's and Dilworth's but it is more palpable to a modern audience and does prove a stronger result. The technique is also easy to apply to find other counterexamples. Hall and Dilworth produced two other lattices, L_2 and L_3. L_3 is also a simple modular, nonarguesian lattice of length four, and hence not in $V(\mathscr{C})$ by the same argument. With a more involved argument it is possible to show $L_2 \notin V(\mathscr{C})$.

Modular lattices are important because most of the lattices associated with the classical algebraic systems are modular. From this point of view the lattices L_1, L_2, and L_3 are esoteric. Indeed, none of them is in any modular congruence variety. (A congruence variety is a variety of lattices generated by the congruence lattices of a variety of algebras.) For L_1 and L_3, which are nonarguesian, this follows from the author's result with Jónsson [7]. L_2 is shown not to be in any modular congruence variety in [6].

Is there a more natural counterexample? The answer is yes. C. Herrmann and A. Huhn have shown that $L = L(\mathbf{Z}_4^3)$, the lattice of subspaces of the direct product of three copies of the cyclic group of order 4, is not in $V(\mathscr{C})$. The idea of the proof is this: with the aid of coordinatization techniques one can show that every subdirectly irreducible complemented modular lattice is in $\mathscr{K}_1$ or $\mathscr{K}_2'$, where $\mathscr{K}_1$ is as before and $\mathscr{K}_2'$ is the class of lattices embedded into a lattice of subspaces of a vector space [16], [3]. Thus by Jónsson's theorem the subdirectly irreducible members of $V(\mathscr{C})$ lie in $HSP_u(\mathscr{K}_1 \cup \mathscr{K}_2') = HSP_u(\mathscr{K}_1) \cup HSP_u(\mathscr{K}_2') = \mathscr{K}_1 \cup HSP_u(\mathscr{K}_2') \subseteq \mathscr{K}_1 \cup V(\mathscr{K}_2')$. Clearly $L(\mathbf{Z}_4^3)$ is not in $\mathscr{K}_1$. There is a lattice equation ε valid in $\mathscr{K}_2'$ but not in $L(\mathbf{Z}_4^3)$, showing $L(\mathbf{Z}_4^3) \notin V(\mathscr{K}_2')$. Indeed, recall that 3-frames of characteristic 4 form a projective configuration in the class of modular lattices (see Theorem 1.6 of [5]). This means that there are lattice words which generate, in any modular lattice, a frame of characteristic 4 (this means the ring associated with the frame satisfies $((1+1)+1)+1=0$). Using these words we make an equation ε which says $1+1=0$. Thus ε will hold in a modular lattice L if and only if every 3-frame satisfying $((1+1)+1)+1=0$ in L actually satisfies $1+1=0$. Thus ε fails in $L(\mathbf{Z}_4^3)$ but holds in $\mathscr{K}_2'$.

§2. Jónsson's theorem in modular varieties

Let $\mathscr{V}$ be a variety of algebras with modular congruence lattices (i.e. a modular variety). J. Hagemann and C. Herrmann [12], following J. D. H. Smith [20], have shown how to define a new binary operation, denoted [,], on the congruence lattices of members of $\mathscr{V}$. This operation has proved useful in obtaining results for modular varieties which seemed impossible before. The author's paper with Ralph McKenzie [9] surveys the basic properties and some of the applications of the commutator. We list some of the properties we require. For $\alpha, \beta, \beta_i \in \Theta(A)$, $A \in \mathscr{V}$ and f a homomorphism from A onto $B \in \mathscr{V}$ with kernel π we have:

(1) $[\alpha, \beta] \leq \alpha \wedge \beta$

(2) $[\alpha, \beta] = [\beta, \alpha]$

(3) $[\alpha, \bigvee \beta_i] = \bigvee [\alpha, \beta_i]$

(4) $[\alpha, \beta] \vee \pi = f^{-1}[f(\alpha \vee \pi), f(\beta \vee \pi)]$.

In fact the commutator can be defined as *the largest binary operation defined on the congruence lattices of members of* $\mathscr{V}$ *satisfying* (1) *and* (4). Note by (2) and (3) the commutator is monotone in each variable. If $\mathcal{V}$ is the variety of groups, the commutator is just the usual commutator of normal subgroups. If $\mathcal{V}$ is the variety of commutative rings, then the commutator is just ideal multiplication. The connection between distributivity and the commutator is this: *a modular variety is distributive if and only if* $[\alpha, \beta] = \alpha \wedge \beta$ *holds identically.*

If $\beta \in \Theta(B)$, there is, by (3) a largest congruence α satisfying $[\alpha, \beta] = 0$. We call α the *annihilator* of β. The *monolith* of a subdirectly irreducible algebra is the unique atom of the congruence lattice. With this terminology the generalization of Jónsson's theorem can be stated.

THEOREM 1 (Jónsson, Hagemann, Herrmann, Freese, McKenzie, Hrushovskii). *Let* $\mathcal{K}$ *be a class of algebras with* $V(\mathcal{K})$ *modular. If* $B \in V(\mathcal{K})$ *is subdirectly irreducible and* α *is the annihilator of the monolith of* B, *then* $B/\alpha \in HSP_u(\mathcal{K})$. *If* $V(\mathcal{K})$ *is locally finite, then* $B/\alpha \in SP_uHS(\mathcal{K})$.

If $V(\mathcal{K})$ is distributive, then by the above remarks $\alpha = 0$ so Jónsson's theorem is a corollary of this theorem. If $\alpha \neq 0$, then $\alpha \geq \mu$, where μ is the monolith of B. In this case we get $[\mu, \mu] = 0$. The corollary that either $[\mu, \mu] = 0$ or $B \in HSP_u(\mathcal{K})$ is the Hagemenn–Herrmann result [12]. The last sentence of the theorem is the Freese–McKenzie result [9]. An example showing that local finiteness is necessary is given there. Quite recently Udi Hrushovskii proved the theorem as stated. The proof we present here (which was construted with William Lampe) is quite close to Jónsson's proof of his theorem. However Hrushovskii's proof is not too different. Jónsson's proof depends on the fact that in a distributive lattice a meet irreducible element is meet prime. The present theorem depends on a weakened version of this statement for modular lattices given in (*) below.

By Birkhoff's theorem we may assume $B \cong C/\theta$ where $C \subseteq \prod_{i \in I} A_i$, $A_i \in \mathcal{K}$. If $J \subseteq I$, we let η_J be the kernel of the natural map from C to $\prod_{i \in J} A_i$. Recall that $\eta_J \wedge \eta_K = \eta_{J \cup K}$ and that $J \subseteq K$ implies $\eta_J \geq \eta_K$. Since B is subdirectly irredicible, θ is uniquely covered by a congruence ψ. If we let $\varphi \in \Theta(C)$ be the largest congruence such that $[\psi, \varphi] \leq \theta$, then by (4) φ corresponds to α in the isomorphism of $1/\theta$ and $\Theta(B)$. In particular, $B/\alpha \cong C/\varphi$.

If $\beta, \gamma \in \Theta(C)$, then

$$\beta \wedge \gamma \leq \theta \text{ implies either } \beta \leq \theta \text{ or } \gamma \leq \theta \text{ or both } \beta \leq \varphi \text{ and } \gamma \leq \varphi. \qquad (*)$$

To see this suppose $\beta \wedge \gamma \leq \theta$, $\beta \not\leq \theta$ and $\gamma \not\leq \theta$. Then $\beta \vee \theta \geq \psi$ and hence $[\gamma, \psi] \leq [\gamma, \beta \vee \theta] = [\gamma, \beta] \vee [\gamma, \theta] \leq (\gamma \wedge \beta) \vee \theta = \theta$. Thus by definition $\gamma \leq \varphi$. Similarly $\beta \leq \varphi$.

Let $\mathcal{F}$ be a filter on I maximal with respect to the property $J \in \mathcal{F}$ implies

$\eta_J \leq \theta$. Let $\mathscr{U}$ be an ultrafilter extending $\mathscr{F}$. We claim that $J \in \mathscr{U}$ implies $\eta_J \leq \varphi$. This is clear if $J \in \mathscr{F}$; so assume $J \in \mathscr{U} - \mathscr{F}$. By the maximality of $\mathscr{F}$ neither J nor J' can be adjoined to $\mathscr{F}$. It is easy to see that this implies there is a $K \in \mathscr{F}$ such that $\eta_{J \cap K} \not\leq \theta$ and $\eta_{J' \cap K} \not\leq \theta$. But then

$$\eta_{J \cap K} \wedge \eta_{J' \cap K} = \eta_K \leq \theta.$$

Hence by (*) $\eta_{J \cap K} \leq \varphi$. But $\eta_J \leq \eta_{J \cap K}$, so the claim is proved.

Let $\theta_{\mathscr{U}} \in \Theta(C)$ be the restriction of the ultraproduct congruence on $\prod_I A_i$ to C. Then $C/\theta_{\mathscr{U}}$ is a subalgebra of the ultraproduct of $\prod_I A_i$ by $\mathscr{U}$. It follows from the claim that $\theta_{\mathscr{U}} \leq \varphi$ so that $B/\alpha \cong C/\varphi \in HSP_u(\mathscr{K})$. $\square$

For an example of the structural implications of Theorem 1 let B be a subdirectly irredicible algebra with monolith μ, satisfying $[\mu, \mu] = 0$, then each μ-block is in a natural way a module over a certain ring [11], [9]. By a theorem of Gumm if two μ-blocks lie in the same α-block, then they are isomorphic [10], [9]. The above theorem then tells us that if $B \in V(A)$, A finite, and $|B| > |A|$, then $[\mu, \mu] = 0$ and there are at most $|A|$ isomorphisms types of modules of μ-blocks. Moreover by a result of [9], each μ-block has cardinality at most $|A|$.

§3. Other Jónsson-type theorems

If two finite subdirectly irreducible algebras generate the same distributive variety, then Jónsson's theorem implies that they are isomorphic. This is false for modular varieties. In [8] we constructed subdirectly irreducible modules of sizes 8 and 16 which generate the same variety. The two nonabelian groups of order 8 generate the same variety. For modular varieties we have the following result.

THEOREM 2. *Let B be a subdirectly irreducible algebra and assume that A is either subdirectly irreducible with $[\mu_A, \mu_A] = \mu_A$, or is simple. Also assume A and B generate the same modular variety. If either A or B is finite, then $A \cong B$.*

This theorem is proved in [9]. The proof is a nice application of the commutator theory and Theorem 1.

Another consequence of Jónsson's theorem is that in a finitely generated, distributive variety the subdirectly irreducible algebras can be no larger than the generating algebra. This is false in the modular case; in fact a finitely generated modular variety can have arbitrarily large subdirectly irreducible algebras. We do know that either there are subdirectly irreducible algebras of arbitrarily high cardinality or there is an $n<\omega$ such that all subdirectly irreducible algebras have size at most n [8]. Even in the latter case it is possible to have subdirectly irreducible algebras larger than the generating algebra (see [8], page 428). For simple algebras we do have the following result.

THEOREM 3. *Let A be finite and assume $V(A)$ is modular. Let $B \in V(A)$ be simple. Then $|B| \leq |A|$.*

Jónsson's theorem also implies that a finitely generated distributive variety can have only finitely many subvarieties. Sheila Oates and Mike Vaughan-Lee have constructed a finitely generated modular variety with a descending chain of subvarieties [18]. They use this example to refute certain possible conjectures concerning critical algebras and finitely based algebras. However the following question, first suggested to the author by Ralph McKenzie, is open: *does the lattice of subvarieties of a finitely generated modular variety satisfy the ascending chain condition?* Or equivalently, *is every subvariety of a finitely generated modular variety finitely generated?*

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