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FINITELY BASED MODULAR CONGRUENCE VARIETIES ARE DISTRIBUTIVE

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R. Dedekind introduced the modular law, a lattice equation true in most of the lattices associated with classical algebraic systems, see [4]. Although this law is one of the most important tools for working with these lattices, it does not fully describe the equational properties of these lattices. This was made clear in [8] where B. Jónsson and the author showed that if any modular congruence variety actually satisfies the (stronger) Arguesian law. (A *congruence variety* is the variety generated by all the congruence lattices of the members of a variety of algebras.) In this note we show that no finite set of lattice equations is strong enough to describe the equational properties of the lattices associated with classical algebraic systems in the following strong sense: *there is no modular, nondistributive congruence variety which has a finite basis for its equational theory*.

This question was posed by George McNulty in the problem session on lattice theory at the Jónsson symposium. Jónsson had asked a similar question in [17]. He asked, in Problem 9.12, whether there is a nontrivial variety whose congruence variety is neither the variety of all lattices nor the variety of all distributive lattices, but which is finitely based.

Some partial results on this problem had been obtained previously. Let $\mathcal{V}_n$ denote the class of lattices embeddable into the lattice of subgroups of an abelian group of exponent n and let $\mathcal{V}_0$ be those lattices embeddable into the lattice of subspaces of a rational vector space. Thus $\boldsymbol{H}\mathcal{V}_n$ is the congruence variety associated with the variety of abelian groups of exponent n. In [7] it was shown that *if $\mathcal{L}$ is a class of lattices contained in a modular congruence variety and containing $\mathcal{V}_n$ for some n not a prime, then $\mathcal{L}$ cannot be defined by finitely many first order axioms*. It was also shown that every modular, nondistributive congruence variety contains $\boldsymbol{H}\mathcal{V}_n$ for some n.

Mark Haiman, in his proof that there are Arguesian lattices which are not representable as lattices of permuting equivalence relations (solving an old problem of Jónsson), shows that $\boldsymbol{H}\mathcal{V}_p$ is not finitely based for any prime p, [11] and [12].

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These results do not imply the result of this paper, however, since it is not known if a congruence variety properly above $\boldsymbol{H}\mathcal{V}_p$ must contain a $\boldsymbol{H}\mathcal{V}_n$ for some nonprime n.

In recent years the importance of the commutator in modular varieties has become apparent. In the final section, we show that our nonfinite basis results still hold if we expand lattices to include a 'commutation' operation.

Preliminaries. In this section we review some elementary facts about n–frames and about the commutator in modular varieties.

An n–**frame** is a partial lattice $\mathbf{F}$ consisting of a Boolean algebra $\mathbf{B}_n$, isomorphic to $\mathbf{2}^n$ with atoms $a_1, \ldots, a_n$, and additional elements $\{c_{ij} : 1 \leq i, j \leq n, i \neq j\}$, satisfying

$$\begin{aligned} a_i \vee c_{ij} &= a_j \vee c_{ij} = a_i \vee a_j \\ a_i \wedge c_{ij} &= a_j \wedge c_{ij} = a_i \wedge a_j = 0_{\mathbf{F}} \\ c_{ik} &= (c_{ij} \vee c_{jk}) \wedge (a_i \vee a_k) \\ c_{ij} &= c_{ji} \end{aligned}$$

where $0_{\mathbf{F}}$ is the least element of $\mathbf{B}_n$. We say an interval u/v in a lattice contains a **spanning** n–frame if it contains an n–frame $\mathbf{F}$ such that $0_{\mathbf{F}} = v$ and $1_{\mathbf{F}} = u$, where $1_{\mathbf{F}} = a_1 \vee \cdots \vee a_n$. Vector space lattices are important examples of lattices with n–frames. If $e_1, \ldots, e_n$ is a basis of a vector space then the elements $a_i = \langle e_i \rangle$ and $c_{ij} = \langle e_i - e_j \rangle$ form an n–frame.

Huhn [14] showed that n–frames are projective for the class of modular lattices (cf. [6]). We state this more precisely in the following lemma.

Lemma 1. *Let $\varphi : \mathbf{L} \twoheadrightarrow \mathbf{M}$ be a epimorphism of modular lattices. If $\mathbf{F}$ is an n–frame in $\mathbf{M}$ then we can find an n–frame in $\mathbf{L}$ mapping under φ to $\mathbf{F}$.*

If $\mathcal{V}$ is a variety of algebras with modular congruences then these lattices have an additional operation known as the commutator. This is denoted $[\alpha, \beta]$. See Freese–McKenzie [9] or Gumm [10] for the definition of this operation. If $\mathbf{A} \in \mathcal{V}$ and α, β, and $\beta_i \in \mathbf{Con\ A}$ then the commutator is monotone and

$$[\alpha, \beta] \leq \alpha \wedge \beta \tag{1}$$

$$[\alpha, \beta] = [\beta, \alpha] \tag{2}$$

$$[\alpha, \bigvee \beta_i] = \bigvee [\alpha, \beta_i] \tag{3}$$

Lemma 2. *Let $\mathbf{F}$ be an n–frame in $\mathbf{Con\ A}$, where $\mathbf{A}$ lies in a congruence modular variety. Then $[1_{\mathbf{F}}, 1_{\mathbf{F}}] \leq 0_{\mathbf{F}}$.*

Proof. Write $1_{\mathbf{F}} = a_1 \vee \cdots \vee a_n$. Using (1), (2), and (3) it follows that $[1_{\mathbf{F}}, 1_{\mathbf{F}}] \leq 0_{\mathbf{F}} \vee \bigvee [a_i, a_i]$. But since $a_i \leq a_j \vee c_{ij}$, we have

$$[a_i, a_i] \leq [a_i, a_j \vee c_{ij}] = [a_i, a_j] \vee [a_i, c_{ij}] \leq (a_i \wedge a_j) \vee (a_i \wedge c_{ij}) = 0_{\mathbf{F}}.$$

Hence $[1_{\mathbf{F}}, 1_{\mathbf{F}}] \leq 0_{\mathbf{F}}$. □

The commutator is related to permutability as the following lemma of Gumm [10] shows. Chapter 6 of [9] also contains a proof.

Lemma 3. *Let α and β be in* **Con A**, *where* **A** *lies in a congruence modular variety. If $[\alpha, \alpha] \leq \beta$, then α and β permute.*

The main result. With these preliminaries we may now begin the proof of our nonfinite basis result.

Theorem 4. *If $\mathcal{L}$ is a class of lattices contained in a modular congruence variety and containing $\mathcal{V}_n$ for some n, then $\mathcal{L}$ cannot be defined by finitely many first order axioms. In particular, no modular, nondistributive congruence variety is finitely based.*

Haiman considered the following lattice equations. The indices are computed modulo n, i.e., $x_n = x_0$ in (D_n).

$$(\mathrm{D}_n) \quad x_0 \wedge [x_0' \vee \bigwedge_{i=1}^{n-1} (x_i \vee x_i')] \leq x_1 \vee \left[(x_0' \vee x_1') \wedge \bigvee_{i=1}^{n-1} (x_i \vee x_{i+1}) \wedge (x_i' \vee x_{i+1}') \right]$$

It is easy to see that this holds in any type I lattice, i.e., a lattice of permuting equivalence relations (a proof will be given below). (D_3) is equivalent to the Arguesian equation. For each $n > 3$, Haiman constructed an Arguesian lattice $\mathbf{H}_n$ in which (D_n) fails,[1] thus answering Jónsson's question. Moreover he showed that every proper sublattice of $\mathbf{H}_n$ does have a type I representation and is in fact embeddable into a vector space lattice. From this it is easy to prove that the class of type I lattices cannot be defined by finitely many first order axioms and that $\boldsymbol{H}\mathcal{V}_p$ is not finitely based, p a prime or 0.

The construction of $\mathbf{H}_n$ is not difficult and clearly presented in [12]. Here we only mention those properties of $\mathbf{H}_n$ that we will need.

Let $\mathbf{K}$ be a field with at least three elements and let $n \geq 3$. Let $\mathbf{V}$ be a vector space over $\mathbf{K}$ of dimension $2n$ with basis $\alpha_0, \ldots, \alpha_{n-1}$, $\beta_0, \ldots, \beta_{n-1}$. Define subspaces

$$\begin{aligned} m &= \langle \alpha_0, \ldots, \alpha_{n-1} \rangle \\ q_i &= \langle \{\alpha_j : j \neq i\} \rangle \\ r_i &= m \vee \langle \beta_i \rangle \\ p_i &= q_i \wedge q_{i+1} \\ s_i &= r_{i-1} \vee r_i \end{aligned} \tag{4}$$

where the indices are computed modulo n. Each interval r_i/p_i and s_i/q_i is three dimensional and $m/0$ and $1/m$ are each n dimensional. It is easy to verify that the union of these intervals is a sublattice, denoted $\widetilde{\mathbf{H}}_n$, of the lattice of subspaces of $\mathbf{V}$. Thus

$$\widetilde{\mathbf{H}}_n = m/0 \cup 1/m \cup \bigcup_i r_i/p_i \cup \bigcup s_i/q_i. \tag{5}$$

The intervals listed above are the maximum complemented intervals of $\widetilde{\mathbf{H}}_n$, and $\widetilde{\mathbf{H}}_n$ may be constructed as the Herrmann gluing [13] of these intervals. The lattice

[1] Haiman used the notation $\mathbf{A}_n$ for these lattices.

$\mathbf{H}_n$ is constructed with Herrmann gluing of these same intervals but with a twist at the connection between two of them. Although not difficult, the details are not important here. $\mathbf{H}_n$ is also the union of the intervals in (5). Haiman gives explicit elements x_i, $x'_i \in r_i/p_i$ which witness the failure of (D_n) in $\mathbf{H}_n$. We record these facts in a lemma.

Lemma 5. *Let $\mathbf{H}_n$ be Haiman's lattice constructed over the field $\mathbf{K}$ and let p be the characteristic of $\mathbf{K}$.*

(1) *$\mathbf{H}_n$ is a union of the intervals displayed in* (5) *above.*
(2) *Each interval r_i/p_i and s_i/q_i is isomorphic to the lattice of subspaces of a three dimensional vector space over $\mathbf{K}$.*
(3) *$m/0$ is isomorphic to the lattice of subspaces of an n dimensional vector space over $\mathbf{K}$.*
(4) *There are elements x_i, $x'_i \in r_i/p_i$ which witness the failure of* (D_n) *in $\mathbf{H}_n$.*
(5) *$r_i \wedge r_j = m$ for $i \neq j$.*
(6) *Any nonprincipal ultraproduct of $\{\mathbf{H}_n : n = 3, 4, \dots\}$ lies in $\mathcal{V}_p$.*

Proof. Most of these facts are obvious from the construction of $\mathbf{H}_n$. (4) is of course proved in [12]. To see (6) first note that $\mathcal{V}_p$ is closed under ultraproducts and (by definition) sublattices. Hence it is a universal class. This is well known, see [15], [19], and [16]. Thus $\mathcal{V}_p$ can be axiomatized by a set Φ of universal sentences. Haiman shows that any generating set of $\mathbf{H}_n$ has at least n elements and every $(n-1)$–generated sublattice of $\mathbf{H}_n$ can be embedded into a subspace lattice of a vector space over $\mathbf{K}$. But every such subspace lattice lies in $\mathcal{V}_p$, since every vector space over $\mathbf{K}$ is a vector space over the prime field of characteristic p. Thus any sentence in Φ holds in $\mathbf{H}_n$ for n sufficiently large. Hence any nonprincipal ultraproduct of $\{\mathbf{H}_n : n = 3, 4, \dots\}$ satisfies such a sentence, showing it lies in $\mathcal{V}_p$. □

Lemma 6. *The lattices $\mathbf{H}_n$ lie in no modular congruence variety.*

Proof. Let $\mathcal{K}$ be a variety with modular congruence lattices and let $\mathcal{V}$ denote the class of lattice which can be embedded into **Con A**, for some $\mathbf{A} \in \mathcal{K}$. Then $\boldsymbol{H}\mathcal{V}$ is closed under $\boldsymbol{P}$ (by Theorem 4.5 of [9]), and hence a variety, the congruence variety associated with $\mathcal{K}$. Suppose $\mathbf{H}_n$ is in this congruence variety. Then there is a homomorphism $\varphi : \mathbf{M} \twoheadrightarrow \mathbf{H}_n$ for some sublattice $\mathbf{M}$ of **Con A** for some $\mathbf{A} \in \mathcal{V}$. Since $m/0$ contains a spanning n–frame, there are elements $\overline{m}$ and $\overline{0}$ in $\mathbf{M}$ which map onto m and 0 and such that $\overline{m}/\overline{0}$ also contains a spanning n–frame, by Lemma 1. Of course the sublattice $1/\overline{0}$ maps onto $\mathbf{H}_n$ under φ, and thus, by Lemma 1, we can find elements $\overline{r}_i$ and $\overline{p}_i \geq \overline{0}$ mapping to r_i and p_i, such that $\overline{r}_i/\overline{p}_i$ contains a spanning 3–frame. Let x_i and $x'_i \in r_i/p_i \subseteq H_n$ be the elements of part (4) of Lemma 5. Choose $\overline{x}_i$ and $\overline{x}'_i$ in $\overline{r}_i/\overline{p}_i$ with $\varphi(\overline{x}_i) = x_i$ and $\varphi(\overline{x}'_i) = x'_i$. Then by the monotonicity of the commutator and Lemmas 2 and 3, *each $\overline{x}_i$ permutes with $\overline{x}'_i$, and each element of $\overline{m}/\overline{0}$ permutes with every element of* $\mathbf{M}$ *above* $\overline{0}$.

Now we need to look at the proof that (D_n) holds in a lattice of permuting equivalence relations. So suppose that x_i, x'_i are equivalence relations in a lattice of permuting equivalence relations, $i < n$, and suppose that $\langle a, b\rangle$ is in the left side of (D_n). Using the permutabitilty it is easy to see that there are elements c, and d_i, $i = 1, \dots, n-1$, such that the relations of Figure 1 hold. To do this we simply take the left side of (D_n) and replace each $\vee$–sign with a $\circ$–sign (a composition sign) and each $\wedge$–sign with a $\cap$–sign and then just write out what it means for $\langle a, b\rangle$ to be in the left side of (D_n).

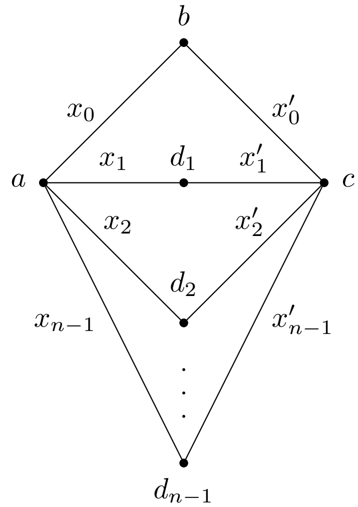

FIGURE 1

Now using the relations of Figure 1 it is easy to verify that $\langle a, b\rangle$ is in the right side of (D_n).

We wish to apply this to the $\overline{x}_i$'s. We know that $\overline{x}_i$ and $\overline{x}_i'$ permute. Since any element of $\mathbf{M}$ in the interval $\overline{m}/\overline{0}$ permutes with every element above $\overline{0}$, $\overline{x}_0'$ permutes with $\overline{m} \wedge \bigwedge(\overline{x}_i \vee \overline{x}_i')$. Using these permutability relations, the same argument as above shows that

$$\overline{x}_0 \wedge [\overline{x}_0' \vee (\overline{m} \wedge \bigwedge_{i=1}^{n-1} (\overline{x}_i \vee \overline{x}_i'))] \leq \overline{x}_1 \vee \left[(\overline{x}_0' \vee \overline{x}_1') \wedge \bigvee_{i=1}^{n-1} (\overline{x}_i \vee \overline{x}_{i+1}) \wedge (\overline{x}_i' \vee \overline{x}_{i+1}') \right] \tag{6}$$

Now by applying φ to this inequality, we see that it holds for the x_i and x_i''s in $\mathbf{H}_n$. It follows from parts (4) and (5) of Lemma 5 that $\bigwedge(x_i \vee x_i') \leq m$. Hence

$$m \wedge \bigwedge(x_i \vee x_i') = \bigwedge(x_i \vee x_i').$$

Thus these x_i's satisfy (D_n), a contradiction. $\square$

Proof of the Theorem 4. Suppose that $\mathcal{L}$ is a class of lattices contained in a modular congruence variety and which contains $\mathcal{V}_n$. If $n \neq 0$, $\mathcal{V}_p \subseteq \mathcal{V}_n$ for any p dividing n. Thus we may assume that $\mathcal{V}_p \subseteq \mathcal{L}$ for some p a prime or 0. First assume $p \neq 2$. Let $\mathbf{H}_n$ be the Haiman lattices constructed using the prime field of characteristic p. By Lemma 6, no $\mathbf{H}_n$ is in $\mathcal{L}$, but, by part (6) of Lemma 5, a nonprincipal ultraproduct of these lattices is in $\mathcal{L}$. So $\mathcal{L}$ cannot be finitely axiomatized.

When $p = 2$ we construct $\mathbf{H}_n$ over the field $\mathbf{K}$ with four elements. The class of sublattices of vector space lattices over $\mathbf{K}$ is equal to $\mathcal{V}_2$. Using these facts we can argue as above that $\mathcal{L}$ is not finitely axiomatizable.

If $\mathcal{V}$ is a modular, nondistributive congruence variety, then $\mathcal{V}_p \subseteq \mathcal{V}$ for some p a prime or 0 by [7].[2] Hence $\mathcal{V}$ is not finitely based by the above argument. □

The commutator. Since the commutator plays an important part in the study of varieties with modular congruence lattices, it is natural to ask if our nonfinite basis result holds if we expand lattices to include a third binary operation, commutation, denoted $[x, y]$. In this section we show how to extend our result to this class of structures. We also add the constants 0 and 1 to the language, since the results are a little deeper in this context.

Let $\mathcal{K}$ be a congruence modular variety. For each $\mathbf{A} \in \mathcal{K}$, we let $\mathbf{Con}^c\,\mathbf{A}$ denote the algrebraic structure with three binary operations (join, meet, and the commutator) and two nullary operations (0 and 1). In this paper we will use the term *commutation lattice* for an algebra $\mathbf{L} = \langle L, \vee, \wedge, [\ ,\ ], 0, 1\rangle$ such that $\langle L, \vee, \wedge, 0, 1\rangle$ is a 0–1–lattice and $\mathbf{L}$ satisfies (1), (2), and (the finite version of) (3).[3]Let $\mathcal{V}$ denote the class of all lattices (in the ordinary sense) which can be embedded into $\mathbf{Con}\,\mathbf{A}$ for some $\mathbf{A} \in \mathcal{K}$, and let $\mathcal{W}$ denote the class of those commutation lattices which can be embedded into $\mathbf{Con}^c\,\mathbf{A}$ for some $\mathbf{A} \in \mathcal{K}$. In particular, we let $\mathcal{W}_n$ denote the class of commutation lattices which can be embedded into $\mathbf{Con}^c\,\mathbf{A}$ for some abelian group $\mathbf{A}$ of exponent n and $\mathcal{W}_0$ the class of commutation lattices which can be embedded into $\mathbf{Con}^c\,\mathbf{A}$ for some vector space $\mathbf{A}$ over $\mathbb{Q}$. Of course for the members of $\mathcal{W}_n$, the commutator is identically 0. We call the variety generated by $\mathcal{W}$ the *commutation congruence variety* associated with $\mathcal{K}$.

Let $\mathbf{H}^c_n$ be the commutation lattice $\mathbf{H}_n$ with $[x, y] = 0$ and let $\mathbf{J}^c_n$ be $\mathbf{H}^c_n$ with a new 1 adjoined and the commutation operation defined by

$$\begin{aligned} [1, x] &= x, &&\text{for all } x, \\ [x, y] &= 0, &&\text{if } x \neq 1 \neq y. \end{aligned} \tag{7}$$

We wish to show that the $\mathbf{H}^c_n$ and $\mathbf{J}^c_n$ lie in no modular commutation congruence variety. This can be done with a simple modification of Lemma 6. Let $\mathcal{K}$ be a congruence modular variety and suppose that $\mathbf{H}^c_n$ is in the commutation congruence variety associated with $\mathcal{K}$. Then there is a homomorphism $\varphi : \mathbf{M} \twoheadrightarrow \mathbf{H}^c_n$ for some commutation lattice $\mathbf{M}$ embedded in $\prod_{j\in J} \mathbf{Con}^c\,\mathbf{A}_j$, where $\mathbf{A}_j \in \mathcal{K}$.

Just as in the proof of Lemma 6, we can find inverse images $\overline{m}$, $\overline{r}_i$, $\overline{p}_i$, $\overline{x}_i$, $\overline{x}'_i$, and $\overline{0} \in M$ of m, r_i, p_i, x_i, x'_i, and 0, such that $\overline{m}/\overline{0}$ contains a spanning n–frame, $\overline{r}_i/\overline{p}_i$,

[2]An outline of the proof will be given in the next section.

[3]The study of lattices with a multiplication goes back to Dedekind [5] and was extensively developed by Dilworth, Ward, and others, see Chapter 6 of [2]. The term 'commutation lattice' was coined by Birkhoff in [1]. His definition was less restrictive than ours since it modelled the multiplication of subgroups which are not necessarily normal.

The nonfinite basis result which we prove does not depend on the exact definition of commutation lattice. The axioms on the commutator operation in such a lattice could be weakened (or even omitted altogether) or strengthened without affecting our result. Because of this, we do not consider the definition of commutation lattice set in stone. Other authors may prefer a different definition.

contains a spanning 3–frame, $\overline{p}_i \geq \overline{0}$, and $\overline{x}_i$ and $\overline{x}'_i \in \overline{r}_i/\overline{p}_i$. Of course the images of these elements under the projection from $\mathbf{M}$ to $\mathbf{Con}^c\, \mathbf{A}_j$ also have these same properties. Thus the proof of Lemma 6 shows that (6) holds in each projection, and hence it holds in $\mathbf{M}$. Again, just as in the proof of Lemma 6, this shows that the x_i and x'_i's satisfy (D_n), a contradiction.

The same argument works for $\mathbf{J}_n^c$. We record these facts as a lemma.

Lemma 7. *The commutation lattices $\mathbf{H}_n^c$ and $\mathbf{J}_n^c$ lie in no modular commutation congruence variety.* □

Let $\mathcal{K}$ be a congruence modular, but nondistributive, variety. We will show that the commutation congruence variety associated with $\mathcal{K}$ is not finitely based. First suppose that $\mathcal{K}$ has a nontrivial abelian algebra, i.e., one that satisfies $[1,1] = 0$. The abelian algebras of $\mathcal{K}$ form a subvariety which, by Magari's Theorem [18], has a simple algebra $\mathbf{A}$. We may as well assume that $\mathcal{K} = \boldsymbol{V}\mathbf{A}$ so that it is abelian. It follows by modularity that $\mathbf{Con}\ \mathbf{A}^n$ has length n. The kernels of the n projections of $\mathbf{A}^n$ onto $\mathbf{A}$ are coatoms whose meet is 0. This means $\mathbf{Con}\ \mathbf{A}^n$ is a complemented modular lattice of length n, see 4.3 of [3]. Since $\mathbf{A}$ is abelian, $\mathbf{A}^2$ contains $\mathbf{M}_3$ as a 0–1–sublattice, see Exercise 5.2 of [9]. Thus the filter above the meet of any two projections onto $\mathbf{A}$ contains $\mathbf{M}_3$ and it follows from this that $\mathbf{Con}\ \mathbf{A}^n$ is simple. By classical coordinatization results, for example 13.4 and 13.5 of [3], $\mathbf{Con}\ \mathbf{A}^n$ is isomorphic to the lattice of subspaces of an n–dimensional vector space over some field $\mathbf{K}$. Of course it also satisfies $[1,1] = 0$ so this isomorphism also respects the commutator.

Now the (ordinary) congruence variety associated with $\mathcal{K}$ is generated by the lattices $\mathbf{Con}\ \mathbf{F}_{\mathcal{K}}(n)$, $n \in \omega$. (This is true for any variety of algebras. It is explicit in Nation's thesis [20]. It is also proved in proved in [21] and [16].) Applying this to the variety of vector spaces over $\mathbf{K}$, we see that the ordinary congruence variety associated with $\mathcal{K}$ contains all lattices of subspaces of vector spaces over $\mathbf{K}$. Of course it then contains all subspace lattices of vector spaces over the prime subfield of $\mathbf{K}$. Thus the ordinary congruence variety associated with $\mathcal{K}$ contains $\mathcal{V}_p$ for some p a prime or 0. Since all members of $\mathcal{K}$ satisfy $[1,1] = 0$, all of the members of the commutation congruence variety associated with $\mathcal{K}$ satisfy this law. From this it is easy to see that $\mathcal{W}_p$ is a subclass of the commutation congruence variety associated with $\mathcal{K}$.

Now we can use the proof of part (6) of Lemma 5 to show that any nontrivial ultraproduct of $\{\mathbf{H}_n^c : n = 3, 4, \dots\}$ (constructed using the prime subfield of $\mathbf{K}$), lies in $\mathcal{W}_p$ and hence in the commutation congruence variety associated with $\mathcal{K}$. As was pointed out in that proof, every $(n-1)$–generated sublattice of $\mathbf{H}_n$ can be embedded into a vector space over $\mathbf{K}$, and hence over the prime subfield of $\mathbf{K}$. The commutation sublattice of $\mathbf{H}_n^c$ generated by $n-1$ elements is just the ordinary sublattice generated by these elements union $\{0,1\}$ (with the commutator identically 0, of course). Thus every $(n-1)$–generated subcommutation lattice of $\mathbf{H}_n^c$ lies in $\mathcal{W}_p$. With this it is easy to see that proof of part (6) of Lemma 5 applies to the present situation.

In the remaining case, $\mathcal{K}$ has no abelian algebra. This means that $\mathcal{K}$ satisfies the commutator law $[1,1] = 1$, which implies the law $[1,x] = x$, see Theorem 8.5 of [9]. Since $\mathcal{K}$ is not congruence distributive, there is an $\mathbf{A} \in \mathcal{K}$ having $\mathbf{M}_3$ as a sublattice. Since congruence lattices are weakly atomic by 2.2 of [3], we may assume that the embedding of $\mathbf{M}_3$ into this congruence lattice is cover-preserving. By taking a homomorphic image of $\mathbf{A}$ we may also assume that this embedding preserves 0. Let α, β, and γ be the images of the three atoms of $\mathbf{M}_3$ under this embedding and let δ be the image of 1. Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}^n$ with universe

$$\{\langle a_1, \dots, a_n\rangle : a_i \ \gamma \ a_j \text{ for all } i \text{ and } j\}.$$

If $\theta \in \mathbf{Con}\,\mathbf{A}$, let $\theta_i \in \mathbf{Con}\,\mathbf{B}$ be $\{\langle a, b\rangle : a_i \ \theta \ b_i\}$. Then $\gamma_1 = \cdots = \gamma_n$ and $\delta_1 = \cdots = \delta_n$ and we will denote these again by γ and δ, respectively. Straightforward calculations with elements show that $\gamma \wedge \alpha_1 \wedge \cdots \wedge \alpha_n = 0$. Of course the interval $\delta/(\gamma \wedge \alpha_i)$ contains a copy of $\mathbf{M}_3$ with atoms α_i, β_i, and γ. From this it follows that $\delta/0$ is a simple, complemented lattice of length n and that $[\delta, \delta] = 0$. As above, this implies that $\delta/0$ contains a sublattice isomorphic to the lattice of subspaces of a vector space over some field $\mathbf{K}$. Moreover, this sublattice together $1_{\mathbf{B}}$ forms a subcommutation lattice of $\mathbf{Con}^c\,\mathbf{B}$ whose commutator satisfies (and is determined by) (7). As above, this means that the commutation congruence variety associated with $\mathcal{K}$ will contain all the commutation lattices consisting of a lattice of subspaces of a vector space over $\mathbf{K}$ with a new 1 adjoined and whose commutator is defined by (7).

Now part (6) of Lemma 5 shows that a nonprincipal ultraproduct of $\{\mathbf{H}_n : n = 3, 4, \dots\}$ (constructed using the field $\mathbf{K}$ from above) can be embedded into a vector space lattice over $\mathbf{K}$. Thus (the lattice retract of) a nonprincipal ultraproduct of $\{\mathbf{J}_n^c : n = 3, 4, \dots\}$ consists of a lattice which can be embedded into a vector space together with a new 1. Of course the commutator of this ultraproduct is determined by (7), since this is first order. Thus this ultraproduct lies in the commutation congruence variety associated with $\mathcal{K}$. Hence we have proved the following theorem.

Theorem 8. *Let $\mathcal{K}$ be a variety of algebras with modular, nondistributive congruences. Then the variety generated by the class $\{\mathbf{Con}^c\,\mathbf{A} : \mathbf{A} \in \mathcal{K}\}$ of commutation lattices associated with $\mathcal{K}$ does not have a finite equational basis.*

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