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Congruence modularity implies the Arguesian identity

RALPH FREESE AND BJARNI JÓNSSON

DEDICATED TO R. P. DILWORTH

Abstract. It is shown that if $\mathcal{V}$ is any variety of algebras all of whose congruence lattices are modular, then the congruence lattice of every algebra in $\mathcal{V}$ satisfies the Arguesian law.

1. Introduction

For any variety $\mathcal{V}$ of algebras, let Con $(\mathcal{V})$ be the variety of lattices generated by the congruence lattices Con (A) of the algebras $A \in \mathcal{V}$. The purpose of this paper is to show that if Con $(\mathcal{V})$ is modular, then it is Arguesian. It was first proved by J. B. Nation in [4] that certain varieties of lattices are not equal to Con $(\mathcal{V})$ for any variety $\mathcal{V}$ of algebras, and since then his list of excluded varieties has been considerably extended by various investigators. The variety of all modular lattices was added to that list by the first author in a result announced in [1], stating that if Con $(\mathcal{V})$ is modular, then it satisfies a certain identity related to, but apparently weaker than, the Arguesian law. His argument was modified by the second author to yield the full Arguesian law. Actually we prove the dual of the Arguesian law, but as was shown in [2], the two properties are equivalent.

2. The basic construction

The central idea of our argument is to embed Con (A) in two different ways in the congruence lattice of another algebra $B \in \mathcal{V}$. This will allow us to imitate the

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classical proof of Desargues' Law for projective planes embeddable in higher dimensional projective spaces.

Actually the algebra B depends on a fixed congruence relation $\alpha \in \mathrm{Con}(A)$ and is completely determined by α. In fact, we let

$$B = \{\langle a_0, a_1\rangle \in A \times A : a_0 \alpha a_1\}.$$

Thus B is a subalgebra of $A \times A$. Let η_0 and η_1 be the kernels of the projections $\langle c_0, c_1\rangle \to c_0$ and $\langle c_0, c_1\rangle \to c_1$ of B onto A, and let f_0 and f_1 be the induced isomorphisms of $\mathrm{Con}(A)$ onto the intervals $1/\eta_0$ and $1/\eta_1$ in $\mathrm{Con}(B)$. Finally, let $\theta \to \theta'$ be the involutionary automorphism of $\mathrm{Con}(B)$ induced by the automorphism $\langle a_0, a_1\rangle \to \langle a_1, a_0\rangle$ of B. More explicitly, for $\langle a_0, a_1\rangle$, $\langle b_0, b_1\rangle \in B$, $\lambda \in \mathrm{Con}(A)$ and $\theta \in \mathrm{Con}(B)$,

$$\langle a_0, a_1\rangle \eta_i \langle b_0, b_1\rangle \quad \text{iff} \quad a_i = b_i,$$

$$\langle a_0, a_1\rangle f_i(\lambda) \langle b_0, b_1\rangle \quad \text{iff} \quad a_i \lambda b_i,$$

$$\langle a_0, a_1\rangle \theta' \langle b_0, b_1\rangle \quad \text{iff} \quad \langle a_1, a_0\rangle \theta \langle b_1, b_0\rangle.$$

Observe that

$$(f_i(\lambda))' = f_{1-i}(\lambda),$$

$$f_0(\lambda) = f_1(\lambda) \quad \text{whenever} \quad \alpha \subseteq \lambda.$$

Letting $\pi = f_0(\alpha) = f_1(\alpha)$, we thus have

$$\theta' = \theta \quad \text{whenever} \quad \pi \subseteq \theta.$$

We also notice that

$$\pi = \eta_0 + \eta_1,$$

$$\theta + \pi = \theta' + \eta_0 \quad \text{whenever} \quad \eta_0 \subseteq \theta.$$

In fact, if $\langle a_0, a_1\rangle \pi \langle b_0, b_1\rangle$, then a_0, a_1, b_0, b_1 are in the same α-class, so that $\langle a_0, b_1\rangle \in B$ .and $\langle a_0, a_1\rangle \eta_0 \langle a_0, b_1\rangle \eta_1 \langle b_0, b_1\rangle$. This proves that $\pi \subseteq \eta_0 + \eta_1$, and the opposite inclusion is obvious. Since $\pi \subseteq \eta_0 + \eta_1 \subseteq \theta + \eta_1$, we have that $\theta + \pi = \theta + \eta_1$. Hence, since $'$ preserves all congruences which contain π, $\theta + \pi = (\theta + \pi)' = (\theta + \eta_1)' = \theta' + \eta_0$.

3. Central and axial perspectivity

The following lemma is a slight modification of Lemma 2 in Grätzer, Jónsson and Lakser [2]. The proof of that lemma carries over essentially unchanged, and is therefore omitted.

LEMMA A. *Suppose* L *is a modular lattice and* $a=\langle a_0, a_1, a_2\rangle$ *and* $b=\langle b_0, b_1, b_2\rangle$ *are centrally perspective triangles in* L, *whose center of perspectivity* p *satisfies the conditions*

$$p+a_i=p+b_i=a_i+b_i \quad \text{for} \quad i=0,1.$$

Let $u=a_0+a_1+a_2+b_0+b_1+b_2$. *If there exist* q, $r\in L$ *such that*

$$p+q=p+r=q+r, \qquad uq=pa_0, \qquad ur=pb_0,$$

then a *and* b *are axially perspective.*

4. The main theorem

We now apply the construction from Section 2 to obtain a situation to which Lemma A can be applied.

LEMMA B. *Suppose* $\mathcal{V}$ *is a variety of algebras, and let* $\mathcal{K}$ *be the class of all lattices* L *such that* L *is embeddable in the dual of* Con (A) *for some* $A\in\mathcal{V}$. *For any* $L\in\mathcal{K}$ *and* p, s, t, $u\in L$, *if* $p+s=p+t=s+t\leq u$, *then* L *has an extension* $L'\in\mathcal{K}$ *such that for some* q, $r\in L'$,

$$p+q=p+r=q+r, \qquad qu=ps, \qquad ru=pt.$$

Proof. Choose a dual embedding $g:L\to \mathrm{Con}\,(A)$ with $A\in\mathcal{V}$, and let α, σ, τ and ν be the images of p, s, t and u under g. Construct B, η_0, η_1, π, f_0 and f_1 as in Section 2, and let $\sigma_i=f_i(\sigma)$, $\tau_i=f_i(\tau)$ and $\nu_i=f_i(\nu)$. The dual embedding $f_0g:L\to 1/\eta_0$ can be extended to a dual isomorphism h of an extension L' of L onto Con (B), and we take q and r to be the members of L' such that $h(q)=\sigma_1$ and $h(r)=\tau_1$. Since $\sigma_1=\sigma_0'$ and $\eta_0\subseteq\sigma_0$, we have $\sigma_1+\eta_0=\sigma_0+\pi$, and using the fact that $\eta_0\subseteq\nu_0\subseteq\sigma_0$ (because $1\geq u\geq s$), we infer that $\sigma_1+\nu_0=\sigma_0+\pi$. Applying h^{-1} to both sides of this equation we find that $qu=sp$. Similarly, $ru=tp$. Finally, applying the dual embedding $f_1g:L\to 1/\eta_1$ to the equations $p+s=p+t=s+t$, we obtain $\pi \cap \sigma_1 = \pi \cap \tau_1 = \sigma_1 \cap \tau_1$, and taking images under h^{-1} we conclude that $p+q=p+r=q+r$.

THEOREM. *For any variety* $\mathcal{V}$ *of algebras, if* Con $(\mathcal{V})$ *is modular, then it is Arguesian.*

Proof. By Jónsson [3], the Arguesian identity is selfdual, and it is therefore sufficient to show that, for $\mathcal{K}$ as in Lemma B, every lattice L in $\mathcal{K}$ is Arguesian. By Lemma 1 of Grätzer, Jónsson and Lakser [2], a modular lattice L is Arguesian if any two triangles a and b in L that satisfy the conditions of Lemma A are axially perspective. To prove that this property holds, we apply Lemma B with $s=a_0$ and $t=b_0$, and then apply Lemma A with L replaced by L'.

It is interesting to note that the algebra B in our construction is simply the congruence relation α itself, viewed as a subalgebra of $A \times A$. In fact, our proof yields the following stronger local version of the main theorem.

COROLLARY 1. *For any algebra* A, *if* Con (α) *is modular for every* $\alpha \in$ Con (A), *then* Con (A) *is Arguesian.*

COROLLARY 2. *If* $\mathcal{V}$ *is an equational class such that* Con (A) *is modular for every finite* $A \in \mathcal{V}$, *then* Con (A) *is Arguesian for every finite* $A \in \mathcal{V}$.

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University of Hawaii
Honolulu, Hawaii
U.S.A.

Vanderbilt University
Nashville, Tennessee
U.S.A.