ON THE TWO KINDS OF PROBABILITY IN ALGEBRA

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In Memoriam Evelyn Nelson

In [6–8] Murskii studied probability in algebra. He showed, for example, that the probability that a random groupoid is simple is 1 and that the probability that it has no idempotent element (one that satisfies $a^2 = a$) is $1/e$. To make these statements precise, define a **groupoid table** on the set $\{0, 1, \dots, n-1\}$ to be an n by n matrix with entries from $\{0, 1, \dots, n-1\}$. Of course, each groupoid table $\mathbf{T}$ defines a groupoid by the rule $i \cdot j = t_{i,j}$, where $t_{i,j}$ is the i, j^{th} entry of $\mathbf{T}$. Let $\mathcal{T}_n$ be the set of groupoid tables on the set $\{0, 1, \dots, n-1\}$. Notice that $|\mathcal{T}_n| = n^{n^2}$. If P is a groupoid property, we define the probability of P by

$$\Pr(\mathrm{P}; \mathcal{T}) = \lim_{n\to\infty} \Pr(\mathrm{P}; \mathcal{T}_n)$$

if this limit exists, where

$$\Pr(\mathrm{P}; \mathcal{T}_n) = \frac{|\{\mathbf{T} \in \mathcal{T}_n : \mathbf{T} \models \mathrm{P}\}|}{n^{n^2}}.$$

This defines a finitely, but not countably, additive probability measure. See [10] for a review of Murskii's work and other related papers.

With this definition, Murskii showed that the probability a finite groupoid will have a finite basis for its identities is 1. More surprisingly, he showed that the fraction of nonfinitely based groupoids has order n^{-6} as $n \to \infty$.

A second, more natural, definition of the probability of P would take the limit as $n \to \infty$ of the number of groupoids (up to isomorphism) of size n satisfying P divided by the total number of groupoids of size n. However the problem of counting the isomorphism classes of groupoids of size n satisfying P is much more difficult than counting the tables satisfying P. Indeed, even counting the number of isomorphism classes of groupoids is very difficult. Following graph theory, we refer to this second kind of probability as **unlabeled probability** and the probability based on the tables as **labeled probability**.

In this note we examine these two kinds of probability. To make this a little more precise let $\mathcal{K}$ be any infinite set of finite algebras and let $\mathcal{K}_n$ be the set of

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those members of $\mathcal{K}$ which have exactly n elements. We assume that $\mathcal{K}_n$ is finite. We define the probability of a property P in $\mathcal{K}$ to be

$$\Pr(\mathrm{P}; \mathcal{K}) = \lim_{n\to\infty} \Pr(\mathrm{P}; \mathcal{K}_n)$$

where

$$\Pr(\mathrm{P}; \mathcal{K}_n) = \frac{|\{\mathbf{A} \in \mathcal{K}_n : A \models \mathrm{P}\}|}{|\mathcal{K}_n|}$$

and the limit is taken over those n for which $\mathcal{K}_n \neq \emptyset$.

These probability concepts can be generalized in an obvious way to any finite similarity type. If τ is a finite similarity type we let $\mathcal{T}(\tau)$ be the set of all finite labeled algebras of type τ and $\mathcal{A}(\tau)$ be the set of all isomorphism classes of finite algebras of type τ. Of course we will only consider properties P such that if one algebra satisfies P then every algebra isomorphic to it does. We call such a property an **algebraic** property. We will show that if the similarity type τ includes at least one operation symbol of arity at least two, or if it contains three unary operation symbols, then the two kinds of probability are the same. That is, if either $\Pr(\mathrm{P}; \mathcal{T}(\tau))$ or $\Pr(\mathrm{P}; \mathcal{A}(\tau))$ exist then the other exists and they are equal, see Theorem 2.

Similar problems have been studied by combinatorists. Polya showed that the labeled and unlabeled probabilities are the same for graphs [9].

We prove our results by showing that if the asymptotic expected size (defined below) of the automorphism group is 1, then the two kinds of probability are the same. Then we show that this is the case if the similarity type includes an operation of arity at least 2, or it includes at least 3 unary operations.

The case of 2-unary algebras is more interesting. In this case the expected size of the automorphism group is $e^{1/(2e^4)} = 1.009\cdots$. As a corollary this means that for 2-unary algebras, although the two kinds of probability can differ, they differ by at most 1%.

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Now we make some of the above concepts precise. A **random variable** X on a class $\mathcal{K}$ of algebras is a function from $\mathcal{K}$ into the real numbers. If X is a random variable on $\mathcal{K}$, then the **expected value** of X is

$$\mathrm{E}(X; \mathcal{K}) = \lim_{n\to\infty} \frac{\sum_{\mathbf{A}\in\mathcal{K}_n} X(\mathbf{A})}{|\mathcal{K}_n|}.$$

Our first theorem shows that if the expected size of the automorphism is 1, then the two kinds of probability are the same. This theorem is known, see [2—4]. We will provide a proof since we need some of the inequalities established in the proof.

Theorem 1. *Let τ be a similarity type and let $X(\mathbf{T}) = |\operatorname{Aut}(\mathbf{T})|$ for $\mathbf{T} \in \mathcal{T}(\tau)$. Let* P *be an algebraic property. If* Theorem ‘palg1’

$$\mathrm{E}(X; \mathcal{T}(\tau)) = 1$$

then if either $\Pr(\mathrm{P}; \mathcal{T}(\tau))$ *or* $\Pr(\mathrm{P}; \mathcal{A}(\tau))$ *exits then both exist and are equal.*

Proof. Let $\mathbf{T} \in \mathcal{T}_n$ and let $\sigma \in \mathbf{S}_n$, the full symmetric group on $\{0, 1, \dots, n-1\}$. For each fundamental operation f of $\mathbf{T}$, we define an operation f^σ on the same set by

$$f^\sigma(x_0, \dots, x_{n-1}) = \sigma^{-1}(f(\sigma(x_0), \dots, \sigma(x_{n-1}))).$$

Let $\mathbf{T}^\sigma$ be the algebra with these operations. Then σ is an isomorphism from $\mathbf{T}$ onto $\mathbf{T}^\sigma$. Notice that $\mathbf{T}^\sigma = \mathbf{T}$ if and only if $\sigma \in \operatorname{Aut} \mathbf{T}$. It follows that, if $\mathbf{T} \in \mathcal{T}_n$, then

$$|\{\mathbf{T}' \in \mathcal{T}_n : \mathbf{T}' \cong \mathbf{T}\}| = [\mathbf{S}_n : \mathbf{Aut\ T}] = \frac{n!}{|\operatorname{Aut} \mathbf{T}|}. \tag{1}$$

Formula 'palgEq0.5'

Now using this we can calculate $\sum_{\mathbf{T} \in \mathcal{T}_n} |\operatorname{Aut} \mathbf{T}|$ by summing $|\operatorname{Aut} \mathbf{A}|$ over $\mathbf{A} \in \mathcal{A}_n$ multiplied by the number of tables corresponding to $\mathbf{A}$ and obtain the following useful formula.

$$\sum_{\mathbf{T} \in \mathcal{T}_n} |\operatorname{Aut} \mathbf{T}| = |\mathcal{A}_n|\, n! \tag{2}$$

Formula 'old3'

For a class of algebras $\mathcal{K}$, let $\mathcal{K}[\mathrm{P}] = \{\mathbf{A} \in \mathcal{K} : \mathbf{A} \models \mathrm{P}\}$. The isomorphism relation divides $\mathcal{T}_n$ into classes and each of these classes contains at most $n!$ elements. Hence $n!|\mathcal{A}_n[\mathrm{P}]| \geq |\mathcal{T}_n[\mathrm{P}]|$. Using this and (2) we see

$$\begin{aligned} \frac{|\mathcal{A}_n[\mathrm{P}]|}{|\mathcal{A}_n|} \geq \frac{|\mathcal{T}_n[\mathrm{P}]|/n!}{|\mathcal{A}_n|} &= \frac{|\mathcal{T}_n[\mathrm{P}]|/n!}{(|\mathcal{T}_n|/n!)(1/|\mathcal{T}_n|)\sum |\operatorname{Aut} \mathbf{T}|} \\ &= \frac{|\mathcal{T}_n[\mathrm{P}]|}{|\mathcal{T}_n|} \frac{1}{\mathrm{E}(X; \mathcal{T}_n)} \end{aligned} \tag{3}$$

Formula 'palgEq1.5'

Thus

$$\Pr(\mathrm{P}; \mathcal{A}_n) \geq \frac{1}{\mathrm{E}(X; \mathcal{T}_n)} \Pr(\mathrm{P}; \mathcal{T}_n) \tag{4}$$

Formula 'old3.1'

This same inequality applies to $\mathrm{Q} = \neg\mathrm{P}$, the logical negation of P. Using this and the fact that $\Pr(\mathrm{Q}) = 1 - \Pr(\mathrm{P})$, we obtain the following formulae, where $E_n = \mathrm{E}(X; \mathcal{T}_n)$.

$$\frac{1}{E_n} \Pr(\mathrm{P}; \mathcal{T}_n) \leq \Pr(\mathrm{P}; \mathcal{A}_n) \leq \frac{1}{E_n}(\Pr(\mathrm{P}; \mathcal{T}_n) + E_n - 1) \tag{5}$$

Formula 'old3.2'

and

$$E_n \Pr(\mathrm{P}; \mathcal{A}_n) - (E_n - 1) \leq \Pr(\mathrm{P}; \mathcal{T}_n) \leq E_n \Pr(\mathrm{P}; \mathcal{A}_n). \tag{6}$$

Formula 'old3.3'

Now the theorem follows easily from these inequalities. □

Notice that (5) and (6) give information about the relation of the labeled and unlabeled probabilities even when $\mathrm{E}(X; \mathcal{T}_n)$ is not 1. The next result shows that for most similarity types $\mathrm{E}(X; \mathcal{T}_n) = 1$.

Theorem 2. *If the similarity type τ contains an operation symbol of rank at least 2, or if it has at least three unary operation symbols, then* Theorem 'palg2'

$$\mathrm{E}(X; \mathcal{T}(\tau)) = 1$$

where $X(\mathbf{T}) = |\operatorname{Aut} \mathbf{T}|$.

Proof. For $\sigma \in \mathbf{S}_n$ let X_σ be the indicator random variable defined by

$$X_\sigma(\mathbf{T}) = \begin{cases} 1 & \text{if } \sigma \in \operatorname{Aut} \mathbf{T} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$X = \sum_{\sigma \in \mathbf{S}_n} X_\sigma.$$

First assume that the similarity type τ contains an operation symbol f of arity k where k is at least 2. In the proof we will now give will assume that τ has only the operation symbol f. The proof in the general case is the same except that the size of all the sets in question gets multiplied by a constant. Of course quotients of such sizes will be the same. If $\sigma \in \mathbf{S}_n$, let $c_i(\sigma)$ be the number of cycles of length i in the decomposition of σ into disjoint cycles and let $c(\sigma)$ be the total number of cycles in this decomposition, counting 1-cycles, i.e., $c(\sigma) = \sum_{i=1}^n c_i(\sigma)$. Notice that if $c(\sigma) = n$ then σ is the identity permutation and that if $c(\sigma) = n-1$ then σ is a transposition, i.e., it interchanges two numbers and fixes the others.

Now suppose that $\sigma \in \operatorname{Aut} \mathbf{T}$ for $\mathbf{T} \in \mathcal{T}_n(\tau)$ and that i and j are in the same orbit of σ. Then the values of $f(i, x_2, \ldots, x_k)$, where $0 \le x_i < n$ for $i = 2, \ldots, k$, determine the values $f(j, y_2, \ldots, y_k)$. From this we see that

$$|\{\mathbf{T} \in \mathcal{T}_n : \sigma \in \operatorname{Aut} \mathbf{T}\}| \le n^{c(\sigma) n^{k-1}}. \tag{7}$$ Formula 'new1'

We can write $\mathcal{T}_n = \mathcal{T}_n' \cup \mathcal{T}_n'' \cup \mathcal{T}_n'''$, where $\mathcal{T}_n'$ is the set of those members of $\mathcal{T}_n$ with trivial automorphism group, $\mathcal{T}_n''$ are those whose automorphism group contains only a transposition and the identity, and $\mathcal{T}_n'''$ are those whose automorphism group contains a nonidentity element which is not a transposition. Now using (7) and this decomposition, we have

$$1 \le \frac{1}{|\mathcal{T}_n|} \sum_{\mathbf{T} \in \mathcal{T}_n} |\operatorname{Aut} \mathbf{T}| \le \frac{|\mathcal{T}_n'|}{n^{n^k}} + \frac{2\binom{n}{2} n^{(n-1)n^{k-1}}}{n^{n^k}} + \frac{n!\, n!\, n^{(n-2)n^{k-1}}}{n^{n^k}}. \tag{8}$$ Formula 'palgEq7a'

The second term corresponds to $\mathcal{T}_n''$ and $\binom{n}{2}$ is the number of transpositions in $\mathbf{S}_n$. In the last term we have we have used $|\operatorname{Aut} \mathbf{T}| \le n!$. Now the first term is at most 1 and it is easy to see that the second and third terms tend to 0 as $n \to \infty$. Hence $\mathrm{E}(X, \mathcal{T}(\tau)) = 1$, as desired. Thus we may assume that τ has only unary operation symbols.

Assume that the similarity type contains r unary operation symbols, $r \ge 3$, and no other operation symbols. Suppose that $\sigma \in \operatorname{Aut} \mathbf{T}$ and that f is a basic unary operation of $\mathbf{T}$. Let $x \in T = \{0, 1, \ldots, n-1\}$ lie in a cycle of length k in the unique cycle decomposition of σ into disjoint cycles. Then $f(x)$ must lie in a cycle of lenth i, for some $i \mid k$ (i dividing k). The values of f on the elements of the cycle containing x are determined by $f(x)$. Moreover, the values of f on distinct cycles are independent. It follows from this that if $c_i(\sigma)$ is the number of cycles of σ of length i, then

$$\mathrm{E}(X_\sigma; \mathcal{T}_n(\tau)) = \frac{1}{n^{rn}} \prod_{k=1}^{n} \left(\sum_{i|k} i\, c_i(\sigma) \right)^{rc_k(\sigma)}.$$

The cycle structure of σ is determined by the vector of the cycle lengths,

$$\langle c_1(\sigma), c_2(\sigma), \dots, c_n(\sigma) \rangle.$$

The entries of this vector are nonnegative integers and satisfy $\sum ic_i(\sigma) = n$. If $\overline{\jmath} = \langle j_1, j_2, \dots, j_n \rangle$ is a vector of nonnegative integers satisfying $\sum ij_i = n$, then the number of elements of S_n with this cycle structure is

$$\frac{n!}{j_1! j_2! \dots j_n! 1^{j_1} 2^{j_2} \dots n^{j_n}}.$$

Hence

$$\mathrm{E}(X; \mathcal{T}_n(\tau)) = \frac{1}{n^{rn}} \sum_{\overline{\jmath}} \frac{n!}{j_1! j_2! \dots j_n! 1^{j_1} 2^{j_2} \dots n^{j_n}} \prod_{k=1}^{n} \left(\sum_{i|k} ij_i \right)^{rj_k}. \tag{9}$$

Formula 'bigsum'

(We take $0^0 = 1$.)

The term in this sum corresponding to $\overline{\jmath} = \langle n, 0, \dots, 0 \rangle$ is 1, reflecting the fact that the identity permutation is an automorphism of every algebra. The term corresponding to $\overline{\jmath} = \langle n-2, 1, 0, \dots, 0 \rangle$, i.e., to a permutation which is a single transposition, is

$$\frac{1}{n^{rn}} \frac{n!}{(n-2)!2} (n-2)^{r(n-2)} n^r \le \frac{1}{n^{rn}} \frac{n^2}{2} n^{r(n-2)} n^r = \frac{1}{2n^{r-2}}.$$

Since $r \ge 3$, this term goes to 0 as $n \to \infty$.

In each of the remaining terms j_1 has the form $j_1 = n - t$, for $t = 3, \dots, n$. The number of $\sigma \in \mathbf{S}_n$ with $c_1(\sigma) = n - t$ is at most $\binom{n}{n-t} t!$. Moreover, $\sum_{i|k} ij_i \le n$ and $\sum_{i=2}^{n} j_i \le t/2$. From this it follows that the sum of the remaining terms is bounded above by

$$\frac{1}{n^{rn}} \sum_{t=3}^{n} \binom{n}{n-t} t! (n-t)^{r(n-t)} n^{rt/2}. \tag{10}$$

Formula 'restofterms'

Now the t^{th} term of this sum is

$$\begin{aligned} \frac{1}{n^{rn}} \frac{n!}{(n-t)!} (n-t)^{r(n-t)} n^{rt/2} &\le \frac{1}{n^{rn}} n^t n^{r(n-t)} n^{rt/2} \\ &= \frac{1}{n^{t(r/2-1)}} \\ &\le \frac{1}{n^{3/2}}, \end{aligned}$$

since $t \ge 3$ and $r \ge 3$. Thus (10) is at most $n(1/n^{3/2}) = 1/n^{1/2} \to 0$ as $n \to \infty$. Thus $\mathrm{E}(X; \mathcal{T}(\tau)) = 1$, as desired. □

Theorem 3. *If the similarity type τ consists of two unary operation symbols and nothing else, then* Theorem 'palg3'

$$\mathrm{E}(X; \mathcal{T}(\tau)) = e^{1/(2e^4)} = 1.009\cdots$$

where $X(\mathbf{T}) = |\operatorname{Aut}\ \mathbf{T}|$.

Proof. As in the last theorem,

$$\mathrm{E}(X; \mathcal{T}_n(\tau)) = \frac{1}{n^{2n}} \sum_{\bar{\jmath}} \frac{n!}{j_1!j_2!\dots j_n!1^{j_1}2^{j_2}\dots n^{j_n}} \prod_{k=1}^{n} \Bigl(\sum_{i|k} ij_i\Bigr)^{2j_k}. \tag{11}$$

Formula 'bigsum2'

summed over all $\bar{\jmath} = \langle j_1, \dots, j_n\rangle$ with $\sum_{i=1}^n ij_i = n$.

Consider the contribution to the sum of terms of the form $j_1 = n - t$, $j_2 = t/2$, $j_i = 0$, for $i > 2$, where $0 \le t < n$ and t is even. These correspond to permutations of order 2 consisting of exactly $t/2$ transpositions. (It is easy to see that the term for $t = n$ gives a contribution which goes to 0 as $n \to \infty$.) These terms give

$$\begin{aligned}
&\sum_{\substack{t=0\\ t\text{ even}}}^{n-1} \frac{1}{n^{2n}} \frac{n!}{(n-t)!} \frac{1}{(t/2)!2^{t/2}} (n-t)^{2(n-t)} n^t \\
&= \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \frac{1}{n^{2n}} \frac{n!}{(n-2k)!} \frac{1}{k!2^k} (n-2k)^{2(n-2k)} n^{2k} \\
&= \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \frac{1}{k!2^k} \left[\left(\frac{n-2k}{n}\right)^{2n} \frac{n^{2k}}{(n-2k)^{2k}} \frac{n(n-1)\dots(n-2k+1)}{(n-2k)^{2k}} \right] \qquad (12)
\end{aligned}$$

Formula 'bigbracket'

Let $a_k(n)$ be the term in the square brackets in (12) provided that $k \le (n-1)/2$ and let $a_k(n) = 0$ otherwise. Thus we wish to evaluate

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{1}{k!2^k}\, a_k(n) \tag{13}$$

Formula 'akn'

Notice that $a_k(n) \le 1$, for all n and k. Thus the above sum is bounded by $e^{1/2}$. Hence we can interchange the limit and the sum. Now clearly, for k fixed,

$$\lim_{n\to\infty} \frac{n^{2k}}{(n-2k)^{2k}} \frac{n(n-1)\dots(n-2k+1)}{(n-2k)^{2k}} = 1.$$

Thus $\lim_{n\to\infty} a_k(n) = (1/e^4)^k$. From this it follows that (13) evaluates to $e^{1/(2e^4)}$. Thus to complete the proof we need to show that the contribution of the other terms goes to 0 as $n \to \infty$.

Consider $\overline{\jmath}$ with $j_1 = n - t$, $j_2 = \frac{t-3}{2}$, $j_3 = 1$, and $j_i = 0$ for $i > 3$, where $t \geq 3$ and odd. The contribution of these terms to (11) is

$$\begin{aligned}
&\sum_{\substack{t=3\\ t \text{ odd}}}^{n} \frac{1}{n^{2n}} \frac{n!}{(n-t)!\left(\frac{t-3}{2}\right)!\,2^{\frac{t-3}{2}}\,3}(n-t)^{2n-2t}(n-3)^{t-3}(n-t+3)^2 \\
&\quad \leq \sum_{\substack{t=3\\ t \text{ odd}}}^{n} \frac{1}{\left(\frac{t-3}{2}\right)!\,2^{\frac{t-3}{2}}\,3} \frac{n^t}{n^t} \frac{(n-t)^{2n-2t}}{n^{2n-2t}} \frac{(n-3)^{t-3}}{n^t}(n-t+3)^2 \\
&\quad \leq \frac{1}{3n} \sum_{\substack{t=3\\ t \text{ odd}}}^{n} \frac{1}{\left(\frac{t-3}{2}\right)!2^{\frac{t-3}{2}}} \\
&\quad \leq \frac{1}{3n} \sum_{k=0}^{\infty} \frac{1}{k!2^k} \\
&\quad = \frac{1}{3n} e^{1/2}.
\end{aligned}$$

Thus the contribution of these terms tends to 0 as $n \to \infty$.

Now let $\overline{\jmath}$ be arbitrary with $j_1 = n - t$. As noted above, there are at most $n!/(n-t)!$ such $\overline{\jmath}$. Since $\sum_{i=1}^{n} i j_i = n$ and $\sum_{i=2}^{n} i j_i = t$, the contribution of all such terms to the sum (11) is bounded by

$$\frac{1}{n^{2n}} \frac{n!}{(n-t)!}(n-t)^{2(n-t)} n^{(2\sum_{i=2}^{n} j_i)}. \tag{14}$$

Formula 'term'

Clearly, for $\overline{\jmath}$ satisfying $t = \sum_{t=2}^{n} i j_i$, the maximum value of $\sum_{i=2}^{n} j_i$ is $t/2$, and this can only be obtained when $j_2 = t/2$ and $j_i = 0$ for $i > 2$. Since we have already condsidered this case, we may assume that

$$\sum_{i=2}^{n} j_i \leq \frac{t-1}{2}. \tag{15}$$

Formula 'assumption'

If equality obtains in (15), then from the fact that $t = \sum_{i=2}^{n} i j_i$ we have that $\sum_{i=2}^{n} (i-2) j_i = 1$. It follows from this that equality in (15) obtains only if $j_2 = \frac{t-3}{2}$, $j_3 = 1$, and $j_i = 0$, for $i > 3$. Since we have already shown that the contribution from these terms tends to 0, we may assume that $\sum_{i=2}^{n} j_i \leq \frac{t-2}{2}$. In this case (14) is bounded above by

$$\frac{1}{n^{2n}} \frac{n!}{(n-t)!}(n-t)^{2n-2t} n^{t-2} \leq \frac{n^t}{n^t} \frac{(n-t)^{2n-2t}}{n^{2n-2t}} \frac{n^{t-2}}{n^t} \leq \frac{1}{n^2}.$$

Since there are at most n such terms, there total contribution is at most $n(1/n^2) = 1/n$. Thus as $n \to \infty$ the contribution of these terms goes to 0. $\square$

This last result, together with (5) and (6), give the following corollary.

Corollary 4. *For 2-unary algebras, the labeled and the unlabeled probabilities can differ by at most* .01 *when they both exist.* □ Theorem 'palg4'

We present an example showing that the labeled and unlabeled probabilities can differ for 2-unary algebras. Let P be the property that the automorphism group of an algebra contains a transposition. Let P_τ be the property that τ is in the automorphism group of an algebra, τ a transposition. Then $\mathcal{T}_n[\mathrm{P}] = \bigcup_\tau \mathcal{T}_n[\mathrm{P}_\tau]$, where the union is over transpositions in $\mathbf{S}_n$. By the use of inclusion and exclusion and arguments similar to the ones above, we calculate that $\Pr(\mathrm{P};\mathcal{T}) = 1-e^{-1/(2e^4)}$. Using the fact the automorphism group of an algebra satisfying P has at least 2 elements, we can derive the following relation from (1) in a manner similar to the derivation of (3).

$$\begin{aligned}\Pr(\mathrm{P};\mathcal{A}) &\geq 2\Pr(\mathrm{P};\mathcal{T})e^{-1/(2e^4)}\\ &= 2(1-e^{-1/(2e^4)})e^{-1/(2e^4)}\\ &> 1-e^{-1/(2e^4)}\\ &= \Pr(\mathrm{P};\mathcal{T})\end{aligned}$$

This particular example comes quite close to achiving equality in (5) and (6).

Quite recently, Murskii showed that if, instead of groupoid tables one considers two algebras on the same set to be equivalent if they have the same clone of term operations, then the limit infimum of the fraction of finitely based algebras and of the nonfinitely based algebras are both positive [8]. Notice that this again is a labeled property: isomorphic algebras may fail to be equivalent in this sense.

Looking closely at the proof of Theorem 2 and of (8), one can see that if the type contains an operation of arity at least two, then for any fixed m,

$$\lim_{n\to\infty} n^m(E_n - 1) = 0,$$

where, as above, E_n is the expected size of the automorphism group of table based on n elements. Now using this fact and the arguments of Theorem 1 and of (5) and (6), one can show that Murskii's result cited above and his result mentioned in the introduction also hold for unlabeled algebras as well.

Lynch [5] has shown that for unary algebras (the limit defining) the labeled probability of any property expressible in first order logic must exist. On the other hand, Compton, Henson, and Shelah have given an example showing that this is not the case for algebras with a binary operation. The results of this paper show that both of these facts apply to unlabeled probabilities as well, provided, in the unary case, there are at least 3 operations.

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