Algebra Univ. 6 (1976) 69–72 Birkhäuser Verlag, Basel

Planar sublattices of $FM(4)$

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R. McKenzie, A. Kostinsky and B. Jónsson have proved the remarkable result that the class of finite sublattices of a free lattice and the class of finite projective lattices coincide [6]. If we restrict ourselves to modular lattices, the above result is no longer true. Examining the map from the free modular lattice on three generators, $FM(3)$, to the free distributive lattice on three generators, $FD(3)$, we see that $FD(3)$ is not a projective modular lattice (projective in the class of modular lattices. 'Onto' maps are used in the definition of projective). On the other hand any finite distributive lattice is a sublattice of a free modular lattice.

In this paper we show that any finite planar modular lattice which does not contain a sublattice isomorphic to M_4 (the six element two-dimensional lattice) is a sublattice of $FM(4)$. Some of these sublattices which are projective were discovered by Alan Day [4]. However, infinitely many of these lattices are simple, whereas previously no example of a simple sublattice of $FM(4)$ with more than five elements was known. Finally, the results of this paper are used to settle a question raised by E. T. Schmidt in [7].

Following A. Huhn, let P_{n-1} denote a $(n-1)$-diamond. That is, P_{n-1} is a partial lattice consisting of a Boolean algebra of order 2^n with 0 and 1 as least and greatest elements and an element x such that x is a relative complement of the atoms of the Boolean algebra in the quotient $1/0$. Let $FM(P_{n-1})$ denote the modular lattice freely generated by P_{n-1}.

THEOREM 1. *$FM(P_{n-1})$ is a projective modular lattice.*[2])

Before proving this theorem, we prove the following theorem which shows that although the class of finite projective modular lattices is not closed under the formation of sublattices, it is closed under finite direct products.

THEOREM 2. *Let A and B be projective modular lattices both having least and greatest elements. Then $A \times B$ is a projective modular lattice.*

Proof. Let f be a homomorphism from a free modular lattice F onto $A \times B$. Let u

[1]) This research was supported in part by NSF Grant No. GP-37772.

[2]) A proof of this result is contained implicitly in the proof of Satz 2.1 of A. P. Huhn-s *Schwach distributive Verbände* I, Acta Sci. Math. **33** (1972), 297–305.

Presented by R. P. Dilworth. Received August 8, 1974. Accepted for publication in final form January 28, 1975.

be an inverse image of $(1, 0)$ and v an inverse image of $(0, 1)$. Then f restricted to $u/u \wedge v$. maps onto $\{(x, 0): x \in A\}$. Since A is projective, there is a homomorphism g mapping $\{(x, 0): x \in A\}$ into $u/u \wedge v$, such that $f(g(x, 0)) = (x, 0)$. Similarly there is a homomorphism h from $\{(0, y): y \in B\}$ into $v/u \wedge v$ such that $f(h(0, y)) = (0, y)$. Now we let $k: A \times B \to F$ be given by $k(x, y) = g(x, 0) \vee h(0, y)$. Then k is a homomorphism and $f(k(x, y)) = (x, y)$.

An example in [1] shows that the assumption that A and B have a least and greatest element cannot be dropped in this theorem. Moreover, the theorem cannot be extended to infinite direct products [3].

Proceeding with the proof of Theorem 1, we let $X = \{x_1, \ldots, x_{n+1}\}$ and let $f: FM(X) \to FM(P_{n-1})$ be the homomorphism extending the map which sends $x_1, \ldots, x_n$ into the atoms of the Boolean algebra associated with P_{n-1} and sends x_{n+1} into the element x. Since $\mathbf{2}^n$ is projective we can find inverse images $a_1, \ldots, a_n$ of $f(x_1), \ldots, f(x_n)$ such that $a_1, \ldots, a_n$ generate a sublattice isomorphic to $\mathbf{2}^n$. Equivalently, $a_1, \ldots, a_n$ are independent over $a_1 \wedge \ldots \wedge a_n$. Let $y \in FM(X)$ be an inverse image of x such that $a_1 \wedge \ldots \wedge a_n \leqslant y \leqslant a_1 \vee \ldots \vee a_n$. Let $b = \bigvee_{i=1}^{n} (a_i \wedge y)$. Clearly $a_1 \vee b, \ldots, a_n \vee b$ are inverse images of $f(x_1), \ldots, f(x_n)$. Moreover $(a_i \vee b) \wedge y = b \vee (a_i \wedge y) = b$. The following calculation shows that $a_1 \vee b, \ldots, a_n \vee b$ are independent over b.

$$
\begin{aligned}
&(a_1 \vee b) \wedge (a_2 \vee \ldots \vee a_n \vee b) \\
&\quad = b \vee (a_1 \wedge (a_2 \vee \ldots \vee a_n \vee b)) \\
&\quad = b \vee (a_1 \wedge (a_2 \vee \ldots \vee a_n \vee \textstyle\bigvee (a_i \wedge y))) \\
&\quad = b \vee (a_1 \wedge ((a_1 \wedge y) \vee a_2 \vee \ldots \vee a_n)) \\
&\quad = b \vee (a_1 \wedge y) \vee (a_1 \wedge (a_2 \vee \ldots \vee a_n)) \\
&\quad = b \vee (a_1 \wedge y) = b.
\end{aligned}
$$

Changing notation, we now assume that $a_1, \ldots, a_n$ are independent over $a_1 \wedge \ldots \wedge a_n$ and that $y \wedge a_i = a_1 \wedge \ldots \wedge a_n$, $i = 1, \ldots, n$. Let $c = \bigwedge (a_i \vee y)$. Clearly, $a_1 \wedge c, \ldots, a_n \wedge c$ are independent over $a_1 \wedge \ldots \wedge a_n$ and are inverse images of $f(x_1), \ldots, f(x_n)$. Let $d = (a_1 \wedge c) \vee \ldots \vee (a_n \wedge c)$. Then $a_i \wedge c = a_i \wedge d$ and $y \wedge d$ is an inverse image of x. Moreover,

$$
\begin{aligned}
(y \wedge d) \vee (a_i \wedge c) &= (y \wedge d) \wedge (a_i \wedge d) \\
&= d \wedge (y \vee (a_i \wedge d)) \\
&= d \wedge (y \vee (a_i \wedge c)) \\
&= d \wedge (y \vee a_i) \wedge c \\
&= d
\end{aligned}
$$

Consequently, $a_1 \wedge c, \ldots, a_n \wedge c$ generate $\mathbf{2}^n$, which, together with $y \wedge d$, forms a copy of P_{n-1}. Theorem 1 now follows.

Notice that this proof shows that if f is a homomorphism from L_1 onto L_2 and

P_{n-1} is a weak sublattice of L_2 then there is a weak sublattice P'_{n-1} of L_1 isomorphic to P_{n-1} and such that f takes P'_{n-1} onto P_{n+1}.

THEOREM 3. *Let L be a finite planar modular lattice which does not have a sublattice isomorphic to* M_4. *Then L is a sublattice of* $FM(4)$.

Proof. We may take L to be the direct product of two copies of the chain $0<1<\cdots<n$ together with elements z_{ij} $1\leqslant i,j\leqslant n$ such that z_{ij} is a relative complement of $(i,j-1)$ and of $(i-1,j)$ in the quotient $(i,j)/(i-1,j-1)$.

This claim may be easily proved by induction using that fact that L must have an element which is both join and meet irreducible [2], [8].

Let A be the lattice of subspaces of a vector space of dimension $2n$ over the field with two elements. Then A is four-generated [5] and has a weak sublattice isomorphic to P_{2n-1}. Let f be a homomorphism from $FM(4)$ onto A. Then, as remarked above, we can find a copy of P_{2n-1} as a weak sublattice of $FM(4)$. Thus there exist elemens $a_1,\ldots,a_{2n}, y$ in $FM(4)$ such that $a_1,\ldots,a_{2n}$ are the atoms of a copy of $\mathbf{2}^{2n}$ and y is a relative complement of a_i in the quotient sublattice $a_1\vee\ldots\vee a_{2n}/a_1\wedge\ldots\wedge a_{2n}$. Let $b_i=a_1\vee\ldots\vee a_i$, $i=1,\ldots,n$, $c_i=a_{n+1}\vee\ldots\vee a_{n+i}$, $i=1,\ldots,n$, $b_0=c_0=a_1\wedge\ldots\wedge a_{2n}$. Let $d_{ij}=(y\wedge(a_i\vee a_{n+j}))\vee b_{i-1}\vee c_{j-1}$, i, $j=1,\ldots,n$. The elements $b_i\vee c_j$, d_{ij}, i, $j=1,\ldots,n$ form a sublattice of $FM(4)$ isomorphic to L. This follows easily from the fact that a_i, a_{n+j}, $y\wedge(a_i\vee a_{n+j})$, $a_i\wedge a_{n+j}$, $a_i\vee a_{n+j}$ form a copy of M_3 and that $a_i\vee a_{n+j}/a_i\wedge a_{n+j}$ transposes up to $b_i\vee c_j/b_{i-1}\vee c_{j-1}$.

E. T. Schmidt raises the following problem in [7]. Let $\mathscr{V}$ be a variety of lattices and suppose that P is a weak sublattice of a free $\mathscr{V}$-lattice, such that P is a primitive subset. Is it true that if φ is a homomorphism of M onto L and P is a weak sublattice of L, one can find a weak sublattice P' of M, isomorphic to P, such that φ maps P' onto P? We can answer this question in the negative by taking $\mathscr{V}$ to be the variety of modular lattices and M, P and L to be the lattices in Figure 1.

By Theorem 3 P is a primitive sublattice of $FM(4)$. However, the obvious homomorphism from M onto L shows that P does not have the properties described above.

We close with a question of obvious importance. Is M_4 a sublattice of a free modular lattice?

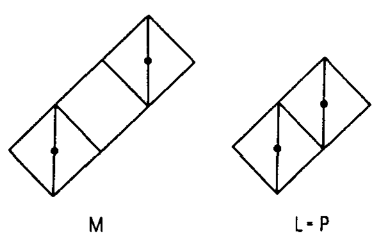

Figure 1

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