Clarification to 'Congruence Lattices of Semilattice'

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This note contains some clarifications for our paper *Congruence lattices of semilattices*, [2]. In that paper we use the term *relatively pseudo-complemented* for the condition:

$$\boxed{\begin{array}{l}\text{for each } x \geq z \text{ there is a } y \text{ such that } y \wedge x = z\\ \text{and } u \wedge x = z \text{ implies } u \leq y.\end{array}} \tag{1}$$

i.e., each principal filter is pseudo-complemented. Based on the way relatively complemented lattices are defined (every interval is complemented), this seemed like the natural way to define relatively pseudo-complemented (for a lattice with a greatest element, (1) is equivalent to each each interval being pseudo-complemented). However, the term relatively pseudo-complemented was already in use in algebraic logic. There it was defined as:

$$\boxed{\begin{array}{l}\text{for each } x \text{ and } z \text{ there is a } y \text{ such that } y \wedge x \leq\\ z \text{ and } u \wedge x \leq z \text{ implies } u \leq y.\end{array}} \tag{2}$$

Unfortunately, (1) and (2) are not the same; in fact, (2) implies distributivity. However, (1) and (2) are equivalent for distributive lattices. Thus (1) could have been used to define relatively pseudo-complemented without changing the meaning for distributive lattices.

Of course, our statement (3) of [2] that the congruence lattice, **Con S**, of a semilattice **S** is relatively pseudo-complemented means that it satisfies (1). It is worth pointing out that for compactly generated lattice condition (1) is equivalent to meet semidistributivity:

$$u = x \wedge y = x \wedge z \text{ implies } u = x \wedge (y \vee z). \tag{$\mathrm{SD}_\wedge$}$$

In statement (4) of [2] we claim that **Con S** is locally distributive. This is true, but our argument stated that a compactly generated lattice **L** is locally distributive if and only if it is semimodular and satisfies (1). We referred to Crawley and Dilworth's book [1]. However, Crawley and Dilworth prove this under the assumption that **L** is strongly atomic. The lattice diagrammed in Figure 1 is compactly generated and locally distributive but is not semimodular. Thus one direction of the above equivalence does not hold without strong atomicity. However, the following theorem shows that the other direction does hold and thus our statement (4) of [2] is correct.

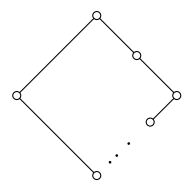

Figure 1:

Theorem 1. *If a compactly generated, semimodular lattice satisfies* (1) *then it is locally distributive.*[1]

Proof. Let **L** be a compactly generated, semimodular lattice satisfying (1). Let $z \in L$ and let u be the join of the covers of z. We need to show the interval sublattice u/z is distributive. Since every interval of **L** also is compactly generated, semimodular and satisfies (1), we may assume $z = 0$ and $u = 1$, *i.e.*, 1 is the join of the atoms.

Note that (1) implies that any set of atoms of **L** is independent. Let $c \in L$ be a compact element. Since the atoms join to 1, there is a finite set $\{a_1, \dots, a_n\}$ of atoms which join above c. Let $t =\prec a_1 \vee \cdots \vee a_n$. Then, by semimodularity and independence,

$$0 \prec a_1 \prec a_1 \vee a_2 \prec \cdots \prec a_1 \vee \cdots \vee a_n = t$$

is a maximal chain in $t/0$ and so, by (3.8) of [1], every chain is finite. Thus, if we let b be the join of those a_i's lying below c and we suppose $b < c$,

[1] Actually Crawley and Dilworth define locally distributive only for strongly atomic lattices. We generalize this definition to an arbitrary complete lattice **L** in the obvious way: If $u \in L$, let u_a be the join of the covers of u; if there are none, $u_a = u$. **L** is *locally distributive* if u_a/u is distributive for each $u \in L$.

then there is an element r with $b \prec r \leq c$. By the definition of b, for no a_i is $a_i \vee b = r$ and obviously $\bigvee_{i=1}^{n}(a_i \vee b) = t$ and by semimodularity each $a_i \vee b$ either equals or covers b. But it is easy to see that this violates (1).

We conclude that every compact element, and thus every element, of **L** is the join of atoms. Let **B** be the lattice of all subsets of the atoms of **L**. Map the elements of B to L by mapping each subset to its join in **L**. By what we have just shown, this map is onto and it clearly preserves joins. Let B and C be sets of atoms. If $\bigvee(B \cap C) < \bigvee B \wedge \bigvee C$ then there is an atom $a \leq \bigvee B \wedge \bigvee C$ with $a \not\leq \bigvee(B \cap C)$. But since sets of atoms are independent, $a \leq \bigvee B$ implies $a \in B$; similarly, $a \in C$. Thus $a \leq \bigvee(B \cap C)$, a contradiction. Hence **L** is isomorphic to the lattice of all subsets of its atoms and thus is certainly distributive. □

Most of the above arguments are in Crawley and Dilworth's book. We will take this opportunity to point out a small error in that book. On page 53 they state that if **L** is a strongly atomic, compactly generated lattice then the following three conditions are equivalent:

1. **L** is locally distributive,

2. **L** is semimodular and every modular sublattice is distributive, and

3. for every set of four distinct elements a, p_1, p_2, $p_3 \in L$ for which p_1, p_2, $p_3 \succ a$, the sublattice $p_1 \vee p_2 \vee p_3/a$ is an eight-element Boolean algebra.

Conditions (1) and (2) are equivalent and imply (3), but $\mathbf{N}_5$ shows that (3) is weaker than the other two. If (3) is modified by adding the statement that, if p_1, $p_2 \succ a$ with $p_1 \neq p_2$, then the sublattice $p_1 \vee p_2/a$ is a four-element Boolean algebra, the three conditions become equivalent.

References

[1] P. Crawley and R. P. Dilworth, *Algebraic theory of lattices*, Prentice-Hall, Englewood Cliffs, New Jersey, 1973.

[2] R. Freese and J. B. Nation, *Congruence lattices of semilattices*, Pacific J. Math. **49** (1973), 51–58.