CLARIFICATION TO 'CONGRUENCE LATTICES OF SEMILATTICE'

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This note contains some clarifications for our paper *Congruence lattices of semilattices*, [2]. In that paper we use the term *relatively pseudo-complemented* for the condition:

(1) $\boxed{\begin{array}{l}\text{for each } x \geq z \text{ there is a } y \text{ such that } y \wedge x = \\ z \text{ and } u \wedge x = z \text{ implies } u \leq y.\end{array}}$

i.e., each principal filter is pseudo-complemented. Based on the way relatively complemented lattices are defined (every interval is complemented), this seemed like the natural way to define relatively pseudo-complemented (for a lattice with a greatest element, (1) is equivalent to each each interval being pseudo-complemented). However, the term relatively pseudo-complemented was already in use in algebraic logic. There it was defined as:

(2) $\boxed{\begin{array}{l}\text{for each } x \text{ and } z \text{ there is a } y \text{ such that} \\ y \wedge x \leq z \text{ and } u \wedge x \leq z \text{ implies } u \leq y.\end{array}}$

Unfortunately, (1) and (2) are not the same; in fact, (2) implies distributivity. However, (1) and (2) are equivalent for distributive lattices. Thus (1) could have been used to define relatively pseudo-complemented without changing the meaning for distributive lattices.

Of course, our statement (3) of [2] that the congruence lattice, **Con S**, of a semilattice **S** is relatively pseudo-complemented means that it satisfies (1). It is worth pointing out that for compactly generated lattice condition (1) is equivalent to meet semidistributivity:

($\mathrm{SD}_\wedge$) $\qquad u = x \wedge y = x \wedge z$ implies $u = x \wedge (y \vee z)$.

In statement (4) of [2] we claim that **Con S** is locally distributive. This is true, but our argument stated that a compactly generated lattice **L** is locally distributive if and only if it is semimodular and satisfies (1). We referred to Crawley and Dilworth's book [1]. However, Crawley and Dilworth prove this under the assumption that **L** is strongly atomic. The lattice diagrammed in Figure 1 is compactly generated and locally distributive but is not semimodular. Thus one direction of the above

Date: July 24, 1995.

equivalence does not hold without strong atomicity. However, the following theorem shows that the other direction does hold and thus our statement (4) of [2] is correct.

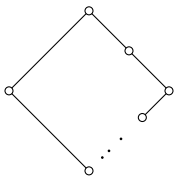

FIGURE 1

Theorem 1. *If a compactly generated, semimodular lattice satisfies* (1) *then it is locally distributive.*[1]

Proof. Let $\mathbf{L}$ be a compactly generated, semimodular lattice satisfying (1). Let $z \in L$ and let u be the join of the covers of z. We need to show the interval sublattice u/z is distributive. Since every interval of $\mathbf{L}$ also is compactly generated, semimodular and satisfies (1), we may assume $z = 0$ and $u = 1$, *i.e.*, 1 is the join of the atoms.

Note that (1) implies that any set of atoms of $\mathbf{L}$ is independent. Let $c \in L$ be a compact element. Since the atoms join to 1, there is a finite set $\{a_1, \ldots, a_n\}$ of atoms which join above c. Then, by semimodularity and independence,

$$0 \prec a_1 \prec a_1 \vee a_2 \prec \cdots \prec a_1 \vee \cdots \vee a_n = 1$$

is a maximal chain in $\bigvee a_i/0$ and so, by (3.8) of [1], every chain is finite. Thus, if we let b be the join of those a_i's lying below c and we suppose $b < c$, then there is an element r with $b \prec r \leq c$. By the definition of b, for no a_i is $a_i \vee b = r$ and obviously $\bigvee(a_i \vee b) = 1$ and by semimodularity each $a_i \vee b$ either equals or covers b. But it is easy to see that this violates (1).

We conclude that every compact element, and thus every element, of $\mathbf{L}$ is the join of atoms. Let $\mathbf{B}$ be the lattice of all subsets of the atoms of $\mathbf{L}$. Map the elements of B to L by mapping each subset to its join in $\mathbf{L}$. By what we have just shown, this map is onto and it clearly preserves joins. Let B and C be sets of atoms. If $\bigvee(B \cap C) < \bigvee B \wedge \bigvee C$ then there is an atom $a \leq \bigvee B \wedge \bigvee C$ with $a \not\leq \bigvee(B \cap C)$. But since sets of atoms are independent, $a \leq \bigvee B$ implies $a \in B$; similarly, $a \in C$. Thus

[1]Actually Crawley and Dilworth define locally distributive only for strongly atomic lattices. We generalize this definition to an arbitrary complete lattice $\mathbf{L}$ in the obvious way: If $u \in L$, let u_a be the join of the covers of u; if there are none, $u_a = u$. $\mathbf{L}$ is *locally distributive* if u_a/u is distributive for each $u \in L$.

$a \leq \bigvee B \cap C$, a contradiction. Hence $\mathbf{L}$ is isomorphic to the lattice of all subsets of its atoms and thus is certainly distributive. □

Most of the above arguments are in Crawley and Dilworth's book. We will take this opportunity to point out a small error in that book. On page 53 they state that if $\mathbf{L}$ is a strongly atomic, compactly generated lattice then the following three conditions are equivalent:

1. $\mathbf{L}$ is locally distributive,
2. $\mathbf{L}$ is semimodular and every modular sublattice is distributive, and
3. for every set of four distinct elements a, p_1, p_2, $p_3 \in L$ for which p_1, p_2, $p_3 \succ a$, the sublattice $p_1 \vee p_2 \vee p_3/a$ is an eight-element Boolean algebra.

Conditions (1) and (2) are equivalent and imply (3), but $\mathbf{N}_5$ shows that (3) is weaker than the other two. If (3) is modified by adding the statement that, if p_1, $p_2 \succ a$ with $p_1 \neq p_2$, then the sublattice $p_1 \vee p_2/a$ is a four-element Boolean algebra, the three conditions become equivalent.

References

1. P. Crawley and R. P. Dilworth, *Algebraic theory of lattices*, Prentice-Hall, Englewood Cliffs, New Jersey, 1973.
2. R. Freese and J. B. Nation, *Congruence lattices of semilattices*, Pacific J. Math. **49** (1973), 51–58.

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