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A SIMPLE SEMIDISTRIBUTIVE LATTICE

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Under broad finiteness conditions, such as the existence of a greatest or least element, a semidistributive lattice has the two element lattice as a homomorphic image, and so, if it has more than two elements, is not simple. However, the existence of a simple, semidistributive lattice with more than two elements has remained in question. This paper constructs such a lattice.

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1. Introduction

The two element lattice is a simple, semidistributive lattice. *Are there any others?* If, for example, **L** is a nontrivial semidistributive (or even just join semidistributive) lattice with a greatest element 1, then an easy argument using join semidistributivity shows that if I is an ideal of **L** maximal with respect to not containing 1, then I and its complement are the blocks of a congruence of **L**. Hence, *every nontrivial join semidistributive lattice with a* 1 *has the two element lattice as a homomorphic image, and so is not simple unless it is the two element lattice.* See [1,3] for the related notion of **L** having join prime elements; in particular the canonical joinands of 1.

Fred Wehrung observed that the lattice of finite convex subsets of the integers $\mathbb{Z}$ is an infinite, simple, join semidistributive lattice. In this paper we give an example of an infinite, simple semidistributive lattice. We want to thank Ralph McKenzie for suggesting this problem to us.

We use the notation of [2]. If w is join irreducible and has a (necessarily unique) lower cover, we donote this lower cover by w_*. $\mathrm{J}(\mathbf{L})$ denotes the join irreducible elements of $\mathbf{L}$. If X is a subset of a join semilattice, $X^\vee$ denotes its closure under finite joins. If a is an element of an ordered set, $\downarrow a$ denotes $\{x \in S : x \leq a\}$. We use SD, $\mathrm{SD}_\vee$ and $\mathrm{SD}_\wedge$ to denote semidistributivity, join semidistributivity and meet semidistributivity. We say that a set $\{u_1, \ldots, u_k\}$ of join irreducible elements, each with a unique lower cover u_{i*}, is a *minimal join cover of* w if $w \leq u_1 \vee \cdots \vee u_k$ and $w \nleq u_1 \vee \cdots \vee u_{i*} \vee \cdots \vee u_k$ for each i.

In the 1980's and 1990's the authors developed useful programs for doing calculations in lattices, primarily free and finitely presented lattices. We have used these to check the calculations in this paper. These programs, which are written in Lisp, are available from the authors or at `https://github.com/UACalc/LatticeThyPrograms`.

2. The Lattice K

Let

$$X = \{a_i, b_i, c_i, d_i, e_i, f_i : i \in \mathbb{Z}\}.$$

We partially order X by defining the following order relations, for all $i \in \mathbb{Z}$:

- $\cdots < a_{-1} < c_{-1} < a_0 < c_0 < a_1 < c_1 < a_2 < c_2 < \cdots$
- $a_i < e_i < b_i$
- $c_i < f_i < d_i$

We also add the following join dependencies as relations or "rules" for all $i \in \mathbb{Z}$:

(1) $a_i \leq b_{i-1} \vee c_{i-1}$
(2) $b_i \leq e_i \vee a_{i+2}$
(3) $c_i \leq a_i \vee d_{i-1}$
(4) $d_i \leq f_i \vee c_{i+2}$
(5) $e_i \leq a_i \vee e_{i-1}$
(6) $f_i \leq c_i \vee f_{i-1}$
(7) $b_i \leq e_i \vee b_{i+1}$
(8) $d_i \leq f_i \vee d_{i+1}$

Let $\mathbf{K}$ be the (join) semilattice freely generated by X subject to the order and join dependencies described above. We will see below that $\mathbf{K}$ is a lattice. Each of the elements of $u \in X$ is join irreducible and u_* exists. We will use the relations (1)–(6) to show that all of the join irreducibles depend on each other, which implies $\mathrm{con}(u, u_*)$ are all equal. The simplicity of $\mathbf{K}$ follows. The relations (7) and (8) are needed for semidistributivity. Combining these ideas we will prove the following theorem.

Theorem 1. $\mathbf{K}$ *is semidistributive, simple lattice.*

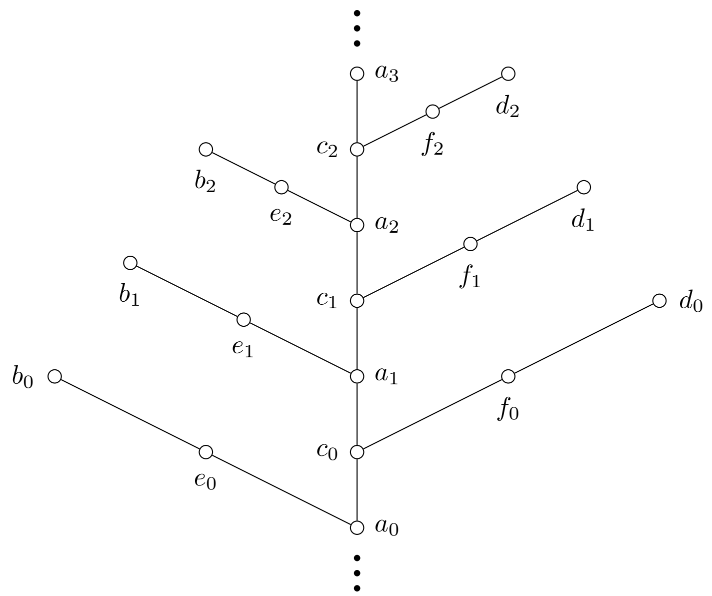

Fig. 1. Schematic of the order on the set X of join irreducible elements of $\mathbf{K}$.

First note there is an order isomorphism from X onto itself respecting the relations which induces an automorphism of $\mathbf{K}$:

$$\begin{aligned} a_i &\mapsto c_i \\ b_i &\mapsto d_i \\ c_i &\mapsto a_{i+1} \\ d_i &\mapsto b_{i+1} \\ e_i &\mapsto f_i \\ f_i &\mapsto e_{i+1} \end{aligned}$$

and we can use this symmetry.

Also observe that in $\mathbf{K} = X^{\vee}$, the elements in Figure 1 are completely join irreducible with their lower covers as indicated in the diagram, that is, $b_{i*} = e_i$ and $e_{i*} = a_i$, etc.

3. K as a Directed Union of Finite Sublattices

We identify some finite subsets of the set X of join irreducibles. Let m, $n \in \mathbb{Z}$ with $m < n$ and $n > 0$.

$$X_n = \{a_i, e_i, b_i, c_i, f_i, d_i : 0 \le i < n\} \cup \{a_n, c_n, e_n, f_n\}$$
$$X_{mn} = \{a_i, e_i, b_i, c_i, f_i, d_i : m \le i < n\} \cup \{a_n, c_n, e_n, f_n\}$$

Then let $\mathbf{S}_n = X_n^{\vee}$, and $\mathbf{S}_{mn} = X_{mn}^{\vee}$. In view of the shift automorphism, we will work with the X_n and $\mathbf{S}_n$.

The next lemma is a straightforward calculation using the order on X and the relations (1)–(8).

Lemma 2. $\downarrow(b_0 \vee \ldots \vee b_{n-1} \vee d_0 \vee \ldots \vee d_{n-1}) \cap X = X_n \cup \{a_j, c_j : j < 0\}$

It turns out that $b_0 \vee \cdots \vee b_{n-1} \vee d_0 \vee \cdots \vee d_{n-1} = b_0 \vee b_{n-1} \vee d_0 \vee d_{n-1}$, but this is not needed.

Corollary 3. *Each $\mathbf{S}_n$ is the interval $[a_0, b_0 \vee \cdots \vee b_{n-1} \vee d_0 \vee \cdots \vee d_{n-1}]$ in $\mathbf{K}$.*

Thus $\mathbf{K} = \bigcup_{m<n} \mathbf{S}_{mn}$ shows that $\mathbf{K}$ is a lattice and expresses it as a directed union of finite interval sublattices. In particular, it is locally finite, and to show that $\mathbf{K}$ is semidistributive it suffices to show that each $\mathbf{S}_n$ is so.

4. Semidistributivity

To see that $\mathbf{S}_n$ is semidistributive, we can use the following characterization [2]. For a finite lattice $\mathbf{L}$ and $p \in \mathrm{J}(\mathbf{L})$, let $\mathrm{K}(p) = \{x \in L : p \not\leq p_* \vee x\}$. We say that $\mathbf{L}$ *has κ's* if each $\kappa(p) := \bigvee \mathrm{K}(p)$ is in $\mathrm{K}(p)$.

Theorem 4. *Let $\mathbf{L}$ be a finite lattice.*

(1) $\mathbf{L}$ satisfies $\mathrm{SD}_\wedge$ iff $\mathbf{L}$ has κ's.
(2) $\mathbf{L}$ satisfies $\mathrm{SD}_\wedge$ and $\mathrm{SD}_\vee$ iff it has κ's and the map $p \mapsto \kappa(p)$ is one-to-one.

Proof. (1) is Theorem 2.56 of [2]. One can adopt the argument for Theorem 11.20(2) of [2] to prove (2). □

Lemma 5. *Every element of X_n has a κ in $\mathbf{S}_n$, and the map $p \mapsto \kappa(p)$ is one-to-one.*

Proof. Recall $X_n = \{a_i, e_i, b_i, c_i, f_i, d_i : 0 \leq i < n\} \cup \{a_n, c_n, e_n, f_n\}$.

Ad $\kappa(a_i)$: note a_0 is the least element of $\mathbf{S}_n$. For other i's, $a_{i*} = c_{i-1}$. $\kappa(a_1) = e_0 \vee d_0$ and, for $i > 0$, $\kappa(a_i) = b_0 \vee \ldots \vee b_{i-2} \vee e_{i-1} \vee d_0 \vee \ldots \vee d_{i-1}$.

Ad $\kappa(c_i)$: note $\kappa(c_0) = b_0$. By symmetry, with adjustments for missing elements at the top and bottom of $\mathbf{S}_n$, for $0 < i < n$ we have $c_{i*} = a_i$ and $\kappa(c_i) = b_0 \vee \ldots \vee b_i \vee d_0 \vee \ldots \vee d_{i-2} \vee f_{i-1}$. For $i = n$, this becomes $c_{n*} = a_n$ and $\kappa(c_n) = b_0 \vee \ldots \vee b_{n-1} \vee e_n \vee d_0 \vee \ldots \vee d_{n-2} \vee f_{n-1}$.

Ad $\kappa(b_i)$: note $b_{i*} = e_i$. For $i = 0$ we have $\kappa(b_0) = d_0 \vee d_1 \vee e_0 \vee e_1$. For $0 < i < n-1$ we get $\kappa(b_i) = b_0 \vee \ldots \vee b_{i-1} \vee d_0 \vee \ldots \vee d_{i+1} \vee e_i \vee e_{i+1}$. For $i = n-1$ it is $\kappa(b_{n-1}) = b_0 \vee \ldots \vee b_{n-2} \vee d_0 \vee \ldots \vee d_{n-1} \vee f_n \vee e_{n-1} \vee e_n$.

Ad $\kappa(d_1)$: note $d_{i*} = f_i$ and, by symmetry for $i < n-1$, we have $\kappa(d_i) = b_0 \vee \ldots \vee b_{i+2} \vee d_0 \vee \ldots \vee d_{i-1} \vee f_i \vee f_{i+1}$. For $i = n-2$ this becomes $\kappa(d_{n-2}) = b_0 \vee \ldots \vee b_{n-1} \vee e_n \vee d_0 \vee \ldots \vee d_{n-3} \vee f_{n-2} \vee f_{n-1}$. A similar adjustment gives $\kappa(d_{n-1}) = b_0 \vee \ldots \vee b_{n-1} \vee e_n \vee d_0 \vee \ldots \vee d_{n-2} \vee f_{n-1} \vee f_n$.

Ad $\kappa(e_i)$: note $e_{i*} = a_i$ and $\kappa(e_i) = \bigvee(X_n \setminus \{b_j, e_j : j \leq i\})$.
Ad $\kappa(f_i)$: similarly, $f_{i*} = c_i$ and $\kappa(f_i) = \bigvee(X_n \setminus \{d_j, f_j : j \leq i\})$. $\square$

Corollary 6. **K** *is semidistributive.*

5. Simplicity

The join dependency relation D is defined on join irreducibles by $x \mathrel{\mathrm{D}} y$ whenever y is a member of some minimal nontrivial join cover of x. Originally defined for finite lattices, this makes sense whenever **L** has the minimal join cover refinement property, which **K** does. When that property holds, the transitive closure of D is a quasi-order on $\mathrm{J}(\mathbf{L})$, and if $x \mathrel{\mathrm{D}} y$, then $\mathrm{con}(x, x_*) \leq \mathrm{con}(y, y_*)$. Every proper congruence collapses some join irreducible to its lower cover, and indeed the congruence lattice Con **L** is isomorphic to the lattice of order filters of (X, D). See [2,4].

Lemma 7. *Rules* (1)–(6) *are minimal nontrivial join covers.*

Certainly these are not all of them: besides (7) and (8), there are other join covers obtained by composing the given ones.

Proof. By symmetry we check (1), (2) and (5). For clarity, let us use subscripts $0, 1, 2$.

Ad(1): $a_1 \leq b_0 \vee c_0$. Note $a_1 \nleq e_0 \vee c_0$ (no rules apply) and $a_1 \nleq b_0 \vee a_0 = b_0$.

Ad (2): $b_0 \leq e_0 \vee a_2$. Now $e_{0*} = a_0$ so e_0 cannot be lowered, while $a_{2*} = c_1$; then $e_0 \vee c_1 \geq e_1$ but after that no rule applies.

Ad (5): $e_1 \leq a_1 \vee e_0$. Now $a_{1*} = c_0$ and no rule applies to $e_0 \vee c_0$, while $e_{0*} = a_0 \leq a_1$. $\square$

Lemma 8. *For any two distinct* x, $y \in \mathrm{J}(\mathbf{K})$, $x \mathrel{\mathrm{D}^n} y$ *for some* n.

Proof. Using Lemma 7 we see that $a_i \mathrel{\mathrm{D}} b_{i-1} \mathrel{\mathrm{D}} a_{i+1}$ by (1) and (2). On the other hand, $a_i \mathrel{\mathrm{D}} c_{i-1} \mathrel{\mathrm{D}} a_{i-1}$ by (1) and (3). So we need to show that the other join irreducibles are connected to some a_i.

Now $a_{i+1} \mathrel{\mathrm{D}} b_i \mathrel{\mathrm{D}} e_i$ by (1) and (2), while $a_{i+1} \mathrel{\mathrm{D}} c_i \mathrel{\mathrm{D}} d_{i-1} \mathrel{\mathrm{D}} f_{i-1}$ by (1), (3) and (4). Meanwhile b_{i-2}, c_i, $e_i \mathrel{\mathrm{D}} a_i$ directly by (2), (3) and (5), and d_{i-2}, $f_i \mathrel{\mathrm{D}} c_i \mathrel{\mathrm{D}} a_i$ by (4), (6) and (3). $\square$

Corollary 9. **K** *is simple.*

Proof. Every nontrivial conguence of **K** collapses some (x, x_*) with $x \in X$, whence by Lemma 8 every nontrivial congruence collapses every (x, x_*), including (b_i, e_i) and (e_i, a_i) and (a_i, c_{i-1}) and (c_{i-1}, a_{i-1}), and symmetrically. So **K** is simple. $\square$

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